Advanced Mathematical Methods February 23, 2021

1) $f(x,y) = \dfrac{x^2 y^3}{1+y^2} \in C^1(\mathbb{R}^2)$

$\Rightarrow$ $\exists!$ local solution of the Cauchy Probl.

$f_y(x,y) = x^2 \dfrac{3y^2+y^4}{(1+y^2)^2}$

$|f_y| \le \frac{3}{2} a^2 \quad \forall (x,y) \in [-a,a] \times \mathbb{R}$
$\forall a > 0$

because

$0 \le \dfrac{3y^2+y^4}{(1+y^2)^2} = \dfrac{y^2}{1+2y^2+y^4} + \dfrac{2y^2+y^4}{1+2y^2+y^4}$

$\le \frac{1}{2} + 1 = \frac{3}{2}$

and $x^2 \le a^2$ on $[-a,a]$.

Hence $\exists!$ global solution $y: \mathbb{R} \to \mathbb{R}$ of the Cauchy Problem $\forall (x_0, y_0) \in \mathbb{R}^2$.

$y \equiv 0$ is a constant solution

And if one supposes $y \not\equiv 0$ and separate the variables:

$$\frac{y'}{y^3}(1+y^2) = x^2$$

one gets the solutions:

$$-\frac{1}{2y^2} + \log|y| = \frac{x^3}{3} + c, \quad c \in \mathbb{R}$$

of the ODE

Being $y \equiv 0$ a constant solution of the ODE the other solutions cannot intersect it. So, if $y_0 > 0 \Rightarrow y > 0 \ \forall x$

if $y_0 < 0 \Rightarrow y < 0 \ \forall x$

The y axis $(x=0)$ is the set of stationary points and $y' > 0$ for $y>0$, $x \neq 0$

and so $x=0$ is the line of flexes.

By monotonicity we can deduce that

$$\exists \lim_{x \to -\infty} y(x) = \ell$$

and for the positive solutions we have $\ell \geq 0$, $\ell < +\infty$.

It cannot be $\ell > 0$ otherwise we would have $\lim_{x \to -\infty} y'(x) = \lim_{x \to -\infty} x^2 \frac{\ell^3}{1+\ell^2} =$

$= +\infty$

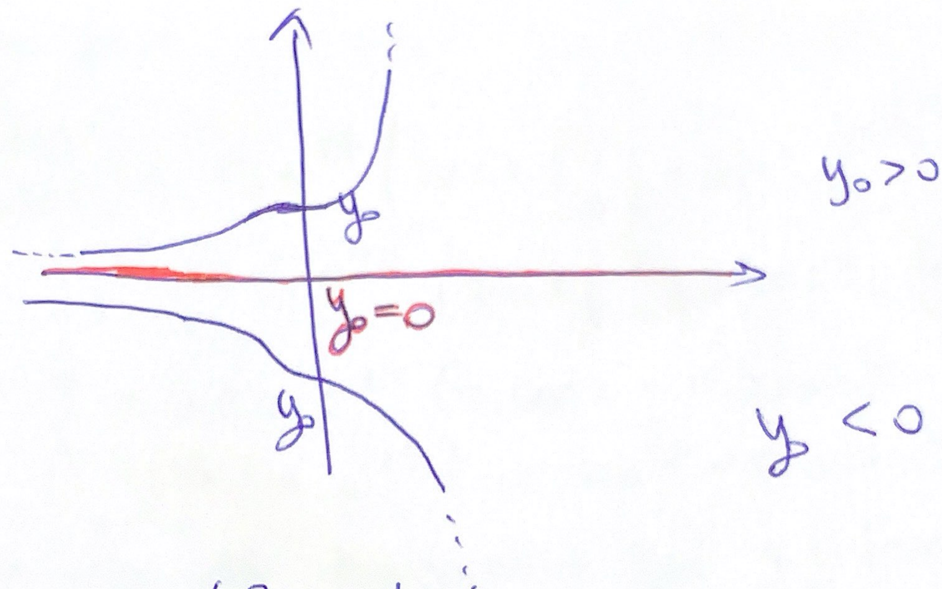
and so $\lim_{x \to -\infty} y'(x) = 0$ if $y_0 > 0$

Also by monotonicity we get

$$\exists \lim_{x \to +\infty} y(x) = \ell \text{ and}$$

$\ell = +\infty$, arguing as before.

By simmetry we can deduce the case $y_0 < 0$.



2) $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

$0 = \det A = (2-\lambda)^2 + 1 = 0 \iff$

$(2-\lambda)^2 = -1 \iff 2-\lambda = \pm i \iff$

$\lambda_{1,2} = 2 \pm i \implies$ the solutions are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t}\left[\cos t \cdot \underline{u} - \operatorname{sen} t\, \underline{v}\right] + c_2 e^{2t}\left[\operatorname{sen} t \cdot \underline{u} + \cos t\, \underline{v}\right]$$

when $\underline{u} + i\underline{v}$ is eigenvector associated to $\lambda = 2+i$ i.e.

$$(A-(2+i)I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{i.e.}$$

$-ix + y = 0 \implies$ we get $(x=1)$

$$\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\implies \underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and we get

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t}\left(\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \operatorname{sen} t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$
$$+ c_2 e^{2t}\left(\operatorname{sen} t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

i.e.
$$\begin{cases} x(t) = e^{2t}(c_1 \cos t + c_2 \operatorname{sen} t) \\ y(t) = e^{2t}(-c_1 \operatorname{sen} t + c_2 \cos t) \end{cases}$$

which are <u>all</u> bounded for $t \in (-\infty, 0]$.

---

3) Using the fact that
$(\arctan(x) + \arctan(\frac{1}{x}))' = 0$ on
$(-\infty, 0) \cup (0, +\infty)$ we get
$\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2} \operatorname{Sign}(x)$
$\forall x \neq 0$.

a) So for $x \in (0,1)$ $f_n(x) \leq \pi^2/4$
and for $x \geq 1$ $f_n(x) \leq 1/n^2 x^2$
and so $f_n \in L^1(0, +\infty)$ $\forall n$.

b) $\lim\limits_{n \to \infty} \left(\frac{\pi}{2} - \arctan(nx)\right)^2 = \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 0$

$\Rightarrow f \equiv 0$

c) $\left| \int_0^{+\infty} f_n(x)dx \right| \leq \frac{1}{n^2} \underbrace{\left| \int_1^{+\infty} \frac{1}{x^2} dx \right|}_{\text{finite because } 2>1} \xrightarrow{n \to \infty} 0$

$\Rightarrow \int_0^{+\infty} f_n(x) \underset{n \to \infty}{\rightarrow} \int_0^{+\infty} f(x)dx = 0.$

4) V is a closed (because finite dimensional) subspace of H. So we have to minimize $\int_0^1 (t^3 - at^2 - bt - c)^2 dt$

By the projection theorem, setting $P_V t^3 = At^2 + Bt + C$ and $g(t) = t^3 - P_V t^3$ we have the condition:

$$\int_0^1 g(t)(at^2 + bt + c)\,dt = 0 \quad \forall a, b, c$$

In particular, we get that is holds true if

$$0 = \int_0^1 g(t) = 0 = \int_0^1 (t^3 - At^2 - Bt - C)\,dt$$
$$= \frac{1}{4} - \frac{A}{3} - \frac{B}{2} - C$$

$$0 = \int_0^1 g(t)t\,dt = \int_0^1 (t^4 - At^3 - Bt^2 - Ct)\,dt$$
$$= \frac{1}{5} - \frac{A}{4} - \frac{B}{3} - \frac{C}{2}$$

$$0 = \int_0^1 g(t)t^2\,dt = \int_0^1 (t^5 - At^4 - Bt^3 - Ct^2)\,dt$$
$$= \frac{1}{6} - \frac{A}{5} - \frac{B}{4} - \frac{C}{3}$$

which gives the system:

$$\begin{cases} 4A + 6B + 12C = 3 \\ 15A + 20B + 30C = 12 \\ 12A + 15B + 20C = 10 \end{cases}$$

$$\Rightarrow A = 3/2, \quad B = -3/5, \quad C = 1/20$$

whence we get

$$P_V f = \frac{3}{2} t^2 - \frac{3}{5} t + \frac{1}{20}.$$