Solutions 31.01.2017

1) $f(x,y) = 2y - y^2 \in C^1(\mathbb{R}^2;\mathbb{R})$

$\Rightarrow \exists!$ local solution (on $I \subset \mathbb{R}$)

$y = 0$ and $y = 2$ ore solutions if $y_0 = 0$ and $y_0 = 2$ respectively

If $y_0 \in (0,2) \Rightarrow y \in (0,2)$ on I

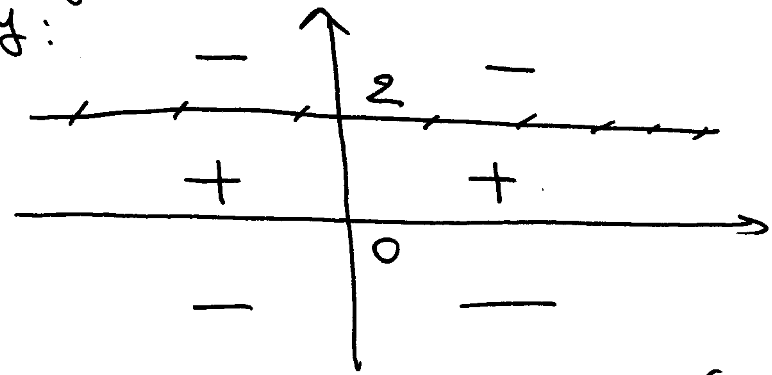
If $y_0 < 0 \Rightarrow y < 0$ on I

If $y_0 > 2 \Rightarrow y > 2$ on I

in order not to contradolict the local uniqueness of solutions.

Moreover if $y_0 \in (0,2) \Rightarrow I \equiv \mathbb{R}$ because if y would have had bdd domain we could always prolongate the solution in a neighborhood of the exhema of I. And this is not possible.

<u>Monotonicity</u>:



y is increasing in case $y_0 \in (0,2)$ and decreasing otherwise

If $y_0 \in (0,2)$

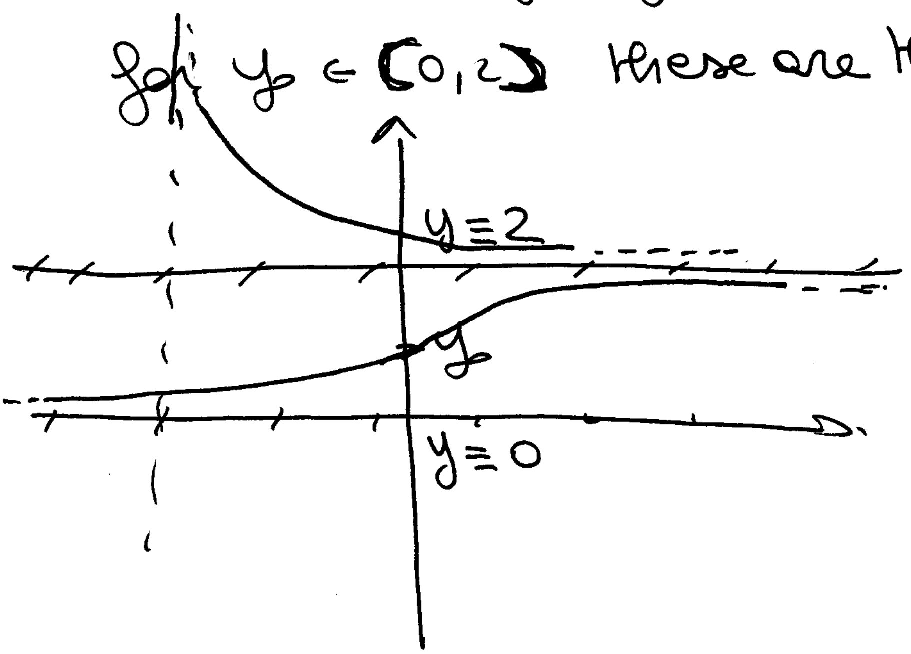
We can also explicitly find the solution by separation of variables if $y_0 \neq 0, 2$ ②

$$\frac{y'}{y(y-2)} = -1 \quad \Rightarrow$$

$$\log\left(\frac{y}{2-y}\right) - \log\left(\frac{y_0}{2-y_0}\right) = 2x$$

$$\Rightarrow y(x) = \frac{2y_0 e^{2x}}{2-y_0+y_0 e^{2x}}$$

for $y_0 \in [0,2]$ these are the solutions



If $y_0 > 2 \Rightarrow \dfrac{1}{y(y-2)}$ is integrable in $\mathcal{U}(+\infty) \Rightarrow \mathrm{dom}\, y = (\bar{x}, +\infty)$

and $\lim_{x \to \bar{x}^+} y(x) = +\infty$ while

$\lim_{x \to +\infty} y(x) = 2$

2) $A = \begin{pmatrix} 1 & \alpha^2 \\ \frac{1}{\alpha} & \alpha \end{pmatrix}$ $\alpha \neq 0$

The characteristic polynomia is

$\lambda^2 - (1+\alpha)\lambda = 0$ whose

solutions are $\lambda_1 = 0$ $\lambda_2 = 1+\alpha$

1) If $\lambda_2 \neq 0$ and $\underline{v_1}, \underline{v_2}$ are the eigenvector associated to λ_1, λ_2 $\Rightarrow$ the solutions are $c_1 e^{\lambda_1 t}\underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2}$

$= c_1 \underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2}$

$c_1, c_2 \in \mathbb{R}$

$\Rightarrow$ they are bounded if $\lambda_2 = 1+\alpha < 0$

i.e. $\alpha < -1$

2) If $\alpha = -1 \Rightarrow \lambda = 0$ is double eigenvalue and the solutions are of the type

$c_1 \underline{v_1} + c_2 (\underline{\omega_1} + t\underline{v_1})$, $c_1, c_2 \in \mathbb{R}$

are bounded if $c_2 = 0$

$\Rightarrow$

a) $\alpha < -1$

b) $\alpha \leq -1$ and $\alpha > -1, \alpha \neq 0$

$\Rightarrow \forall \alpha \neq 0$

3) a) $f_n \in C^0(0,+\infty) \Rightarrow$ it's sufficient to study $|f_n|$ in $\mathcal{U}(0^+)$ and $\mathcal{U}(+\infty)$ (4)

in $\mathcal{U}(0^+)$ $|f_n(x)| \sim \dfrac{|\log x|}{\sqrt[n]{x}}$

and for $n \geq 2$

$\frac{1}{n} < 1 \Rightarrow$

$f_n \in L^1(\mathcal{U}(0^+))$

in $\mathcal{U}(+\infty)$ $|f_n(x)| \sim \dfrac{|\log x|}{x^n}$

which for $n \geq 2$ is integrable

b) if $0 < x < 1 \Rightarrow x^{1/n} \to 1$ and $x^n \xrightarrow[n \to +\infty]{} 0 \Rightarrow$

$$\lim_{n\to\infty} f_n(x) = \frac{4 \log x}{\pi} \in L^1(0,1)$$

and

if $x > 1 \Rightarrow x^n \xrightarrow[n \to +\infty]{} +\infty$

$$\lim_{n\to\infty} f_n(x) = 0$$

$f_n(1) = 0 \Rightarrow$

$$f(x) = \begin{cases} \dfrac{4 \log x}{\pi}, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

c) if $x \in (0,1)$

$$|f_n(x)| \leq \frac{|\log x|}{\operatorname{arctg}(x^{1/2})} \overset{\mathcal{U}(0^+)}{\sim} \frac{|\log x|}{x^{1/2}} \in L^1(\mathcal{U}(0))$$

if $x \geq 1 \Rightarrow |f_n(x)| \leq \frac{\log x}{x^2} \in L^1(1,+\infty)$

$\Rightarrow \quad |f_n(x)| \leq g(x) \quad \forall x \in (0,+\infty)$

where $g(x) = \begin{cases} -\frac{\log x}{\operatorname{arctg}\sqrt{x}} & x \in (0,1) \\ \frac{\log x}{x^2} & x \geq 1 \end{cases}$

$g \in L^1(0,+\infty) \underset{\text{Leb. thm.}}{\Longrightarrow} \lim_{n\to\infty} \int_0^{+\infty} |f_n - f| = 0$

d) and $\lim_{n\to\infty} \int_0^{+\infty} f_n(x) = \int_0^{+\infty} f(x)$

$= \int_0^1 \frac{4}{\pi} \log x = -\frac{4}{\pi}$

4) First we need the compatibility conditions:

$f(0) = u(0,0) = 0$

$f(e) = u(e,0) = 0$

Use the separation of variables:

$u(x,t) = X(x)T(t) \Rightarrow \frac{T'}{T} = \frac{X''}{X} = \lambda$

$\begin{cases} X'' = \lambda X & \lambda \in \mathbb{R} \\ X(0) = X(e) = 0 \end{cases} \qquad \lambda = -\left(\frac{n\pi}{e}\right)^2$

$\Rightarrow$

$X_n(x) = B_n \operatorname{sen}\left(\frac{n\pi}{e}x\right)$, B_n arbitrary

$\Rightarrow T' = -\frac{n^2\pi^2}{e^2} T \Rightarrow T(t) = c_n e^{-\frac{n^2\pi^2}{e^2}t}$

$\Rightarrow$ superposition principle $u(x,t) = \sum B_n e^{-\frac{n^2\pi^2}{e^2}t} \operatorname{sen}\left(\frac{n\pi}{e}x\right)$

⑥

$u(x,0) = f(x) = \sum B_n \operatorname{sen}\left(\frac{n\pi}{\ell}x\right) \Rightarrow$

$$B_n = b_n = \frac{2}{\ell}\int_0^\ell f(x) \operatorname{sen}\left(\frac{n\pi}{\ell}x\right) dx$$

<u>Uniqueness</u> : test by $w = u_1 - u_2$ the difference of two equation written for u_1 and u_2 :

$$\frac{1}{2}\frac{d}{dt}\int_0^\ell |w|^2 dx + \int_0^\ell |w_x|^2 dx = 0$$

integrate by parts and use the boundary conditions

$$\Rightarrow \quad \frac{d}{dt}\int_0^\ell |w|^2 dx = 0 \Rightarrow$$

$$\left(\int_0^\ell |w|^2 dx\right)(t) = \int_0^\ell |w(0)|^2 dx$$

$$= 0$$

initial conditions

$\Rightarrow \quad w \equiv 0$ in $(0,\ell)$ $\forall t > 0 \Rightarrow$

$u_1 \equiv u_2$ in $(0,\ell)$ $\forall t > 0$