A nonlocal phase-field model with nonconstant specific heat

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Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitzian boundary, N being an arbitrary integer.

The classical GINZBURG-LANDAU FREE ENERGY:

$$\mathcal{GL}[\vartheta,\chi] = \int_{\Omega} \left(GL(\vartheta,\chi) + \frac{\nu}{2} |\nabla \chi|^2 \right) dx$$

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is substituted by a NONLOCAL alternative (cf. [van der Waals, 1893 and Bates, Fife, Gajewski, Giacomin and Lebowitz, '97–2004]:

$$\mathcal{NL}[\vartheta,\chi] = \int_{\Omega} \left(\mathsf{NL}(\vartheta,\chi) + \int_{\Omega} \mathsf{k}(\mathsf{x},\mathsf{y}) |\chi(\mathsf{x}) - \chi(\mathsf{y})|^2 \, d\mathsf{y} \right) \, d\mathsf{x}$$

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with k(x, y) a given symmetric KERNEL accounting for NONLOCAL INTERACTIONS

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$$\mathcal{NL}[\vartheta,\chi] = \int_{\Omega} \left(NL(\vartheta,\chi) + \int_{\Omega} k(x,y) |\chi(x) - \chi(y)|^2 \, dy \right) \, dx$$

with k(x, y) a given symmetric KERNEL accounting for NONLOCAL INTERACTIONS.

EXAMPLE. k= the Newton potential $\longrightarrow k(|x|) = \kappa |x|^{-1}$ in 3D.

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The classical GINZBURG-LANDAU FREE ENERGY can be obtained as a formal limit as $m \to \infty$ from the nonlocal one with the choice $K(x,y) = m^{N+2}k(|m(x-y)|^2)$, where k is a nonnegative function with support in [0,1].

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$$\int_{\Omega} m^{N+2} k(|m(x-y)|^2) |\chi(x) - \chi(y)|^2 dy$$

$$= \int_{\Omega_m(x)} k(|z|^2) \left| \frac{\chi(x + \frac{z}{m}) - \chi(x)}{\frac{1}{m}} \right|^2 dz$$

$$\xrightarrow{m \to \infty} \int_{\mathbb{R}^N} k(|z|^2) \langle \nabla \chi(x), z \rangle^2 dz = \frac{\nu}{2} |\nabla \chi(x)|^2$$

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$$\xrightarrow{m \to \infty} \int_{\mathbb{R}^N} k(|z|^2) \langle \nabla \chi(x), z \rangle^2 dz = \frac{\nu}{2} |\nabla \chi(x)|^2$$

for a sufficiently regular χ , where $\nu = 2 \int_{\mathbb{R}^N} k(|z|^2)|z|^2 dz$ and $\Omega_m(x) = m(\Omega - x)$.

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$$\mathcal{F}[\vartheta, \chi] = \int_{\Omega} \left(c_{V}(\chi) \vartheta(\mathbf{x}) (1 - \log \vartheta(\mathbf{x})) + \vartheta(\mathbf{x}) \sigma(\chi(\mathbf{x})) \right) \\ + \lambda(\chi(\mathbf{x})) + (\beta + \vartheta(\mathbf{x})) \varphi(\chi(\mathbf{x})) \\ + \int_{\Omega} K(\mathbf{x}, \mathbf{y}) G(\chi(\mathbf{x}) - \chi(\mathbf{y})) d\mathbf{y} \right) d\mathbf{x},$$

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▶ $|c_V(\chi) \ge c_0 > 0$ is the specific heat, $\beta > 0$;

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• e.g., in solid-liquid phase transitions it may be $c_V(\chi) = c_V^0 + \chi(c_V^1 - c_V^0) , \text{ where } c_V^0 \text{ is the specific heat in the solid and } c_V^1 \text{ in the liquid;}$

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- e.g., in solid-liquid phase transitions it may be $\boxed{c_V(\chi) = c_V^0 + \chi(c_V^1 c_V^0)}, \text{ where } c_V^0 \text{ is the specific heat in the solid and } c_V^1 \text{ in the liquid;}$
- σ and λ are smooth functions;
- $ightharpoonup \varphi: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and l.s.c.;
- ▶ $K : \Omega \times \Omega \to \mathbb{R}$ describes nonlocal interactions;

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- e.g., in solid-liquid phase transitions it may be $c_V(\chi) = c_V^0 + \chi(c_V^1 c_V^0)$, where c_V^0 is the specific heat in the solid and c_V^1 in the liquid;
- \blacktriangleright σ and λ are smooth functions;
- $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and l.s.c.;
- ▶ $K : \Omega \times \Omega \to \mathbb{R}$ describes nonlocal interactions;
- ▶ *G* is an even function bounded on $\mathcal{D}(\varphi)$

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$$+ \lambda(\chi(\mathbf{x})) + (\beta + \vartheta(\mathbf{x})) \varphi(\chi(\mathbf{x}))$$
$$+ \int_{\Omega} K(\mathbf{x}, \mathbf{y}) G(\chi(\mathbf{x}) - \chi(\mathbf{y})) d\mathbf{y} d\mathbf{x},$$

- ▶ e.g., in solid-liquid phase transitions it may be $c_V(\chi) = c_V^0 + \chi(c_V^1 - c_V^0)$, where c_V^0 is the specific heat in the solid and c_V^1 in the liquid;
- \triangleright σ and λ are smooth functions:
- $\blacktriangleright \varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and l.s.c.;
- ▶ $K : \Omega \times \Omega \to \mathbb{R}$ describes nonlocal interactions;
- ▶ e.g., $G(\chi(x) \chi(y)) = |\chi(x) \chi(y)|^2$ if $\varphi = I_{I-1.11}$.

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Possible choices of $\partial \varphi$

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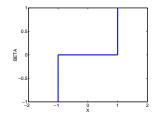
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Possible choices of $\partial \varphi$

Subdifferential case: $\partial \varphi := \partial I_{[-1,1]} = \mathcal{H}^{-1}$:



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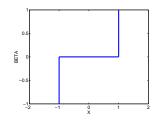
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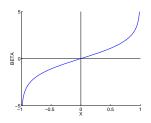
existence

Possible choices of $\partial \varphi$

Subdifferential case: $\partial \varphi := \partial I_{[-1,1]} = \mathcal{H}^{-1}$:



Logarithmic case: $\partial \varphi = \log(1 + \chi) - \log(1 - \chi)$:



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 The internal energy balance over an arbitrary control volume Ω' is

$$\frac{\textit{d}}{\textit{d}t} \int_{\Omega'} \textit{E}[\vartheta, \chi] \, \textit{d}x + \int_{\partial \Omega'} \langle \mathbf{q}, \mathbf{n} \rangle \, \textit{d}s = \Psi(\Omega'),$$

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► The internal energy balance over an arbitrary control volume Ω' is

$$\frac{\textit{d}}{\textit{d}t} \int_{\Omega'} \textit{E}[\vartheta, \chi] \, \textit{d}\textbf{x} + \int_{\partial\Omega'} \langle \textbf{q}, \textbf{n} \rangle \, \textit{d}\textbf{s} = \Psi(\Omega'),$$

where
$$E = F - \vartheta \frac{\partial F}{\partial \vartheta}$$
 and

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 The internal energy balance over an arbitrary control volume Ω' is

$$\frac{\textit{d}}{\textit{d}t} \int_{\Omega'} \textit{E}[\vartheta, \chi] \, \textit{d}\textbf{x} + \int_{\partial\Omega'} \langle \textbf{q}, \textbf{n} \rangle \, \textit{d}\textbf{s} = \Psi(\Omega'),$$

where
$$\pmb{E} = \pmb{F} - \vartheta \frac{\partial \pmb{F}}{\partial \vartheta}$$
 and

 $\Psi(\Omega')$ is the energy exchange through $\partial\Omega'$

due to nonlocal interactions;

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 The internal energy balance over an arbitrary control volume Ω' is

$$rac{d}{dt}\int_{\Omega'} m{E}[artheta, \chi] \, d\mathbf{x} + \int_{\partial\Omega'} \langle \mathbf{q}, \mathbf{n}
angle \, d\mathbf{s} = \Psi(\Omega'),$$

where
$$\pmb{E} = \pmb{F} - \vartheta \frac{\partial \pmb{F}}{\partial \vartheta}$$
 and

 $\Psi(\Omega')$ is the energy exchange through $\partial\Omega'$

due to nonlocal interactions;

• the phase equation, which follows from the tendency of the system to move towards local minima of the free energy with speed proportional to $1/\mu(\vartheta)$

$$\mu(\vartheta)\frac{\partial \chi}{\partial t} \in -\delta\chi \mathcal{F}[\vartheta,\chi]$$
.

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The model is compatible with the Second Principle of Thermodynamics.

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The model is compatible with the

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The model is compatible with the

Second Principle of Thermodynamics . Assuming that

 $S = -\partial F/\partial \vartheta$,

 $\vartheta > 0$, using the expression of F into the entropy

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Second Principle of Thermodynamics. Assuming that

 $\vartheta > 0$, using the expression of F into the entropy

 $S = -\partial F/\partial \vartheta$, and choosing the Fourier heat flux

 $\mathbf{q}:=-\kappa\nabla\vartheta,$ with $\kappa>0$,

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Second Principle of Thermodynamics . Assuming that $\theta > 0$, using the expression of F into the entropy $S = -\partial F/\partial \vartheta$, and choosing the Fourier heat flux $\mathbf{q} := -\kappa \nabla \vartheta$, with $\kappa > 0$, we obtain that the pointwise

Clausius-Duhem inequality
$$\left|S_t + \operatorname{div}\left(rac{\mathbf{q}}{artheta}
ight) \geq 0
ight|$$
 holds true

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Second Principle of Thermodynamics . Assuming that $\vartheta>0$, using the expression of F into the entropy $S=-\partial F/\partial \vartheta$, and choosing the Fourier heat flux $\mathbf{q}:=-\kappa\nabla\vartheta$, with $\kappa>0$, we obtain that the pointwise Clausius-Duhem inequality $S_t+\operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right)\geq 0$ holds true provided that we choose

$$\Psi(\Omega') = \int_{\Omega'} (-b[\chi]\chi_t + B[\chi]_t) \, dx, \quad \text{where}$$

$$B[\chi](x,t) := \int_{\Omega} K(x,y) \, G(\chi(x,t) - \chi(y,t)) \, dy,$$

$$b[\chi](x,t) := 2 \int_{\Omega} K(x,y) \, G'(\chi(x,t) - \chi(y,t)) \, dy$$

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The model is compatible with the

Second Principle of Thermodynamics . Assuming that $\vartheta>0$, using the expression of F into the entropy $S=-\partial F/\partial \vartheta$, and choosing the Fourier heat flux $\mathbf{q}:=-\kappa\nabla\vartheta$, with $\kappa>0$, we obtain that the pointwise Clausius-Duhem inequality $S_t+\operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right)\geq 0$ holds true provided that we choose

$$\Psi(\Omega') = \int_{\Omega'} (-b[\chi]\chi_t + B[\chi]_t) \, dx, \quad \text{where}$$

$$B[\chi](x,t) := \int_{\Omega} K(x,y) \, G(\chi(x,t) - \chi(y,t)) \, dy,$$

$$b[\chi](x,t) := 2 \int_{\Omega} K(x,y) \, G'(\chi(x,t) - \chi(y,t)) \, dy$$

(⇒ the entropy does not decrease along the solution paths).

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With these choices the internal energy balance becomes

$$(c_{V}(\chi(x))\vartheta(x) + \lambda(\chi(x)) + \beta\varphi(\chi(x)))_{t} + 2\chi_{t} \int_{\Omega} K(x,y)G'(\chi(x) - \chi(y)) dy - \kappa\Delta\vartheta(x) = 0$$

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With these choices the internal energy balance becomes

$$\begin{split} & \left(c_V(\chi(x)) \vartheta(x) + \lambda(\chi(x)) + \beta \varphi(\chi(x)) \right)_t \\ & + 2\chi_t \int_{\Omega} K(x, y) G'(\chi(x) - \chi(y)) \, dy - \kappa \Delta \vartheta(x) = 0 \end{split}$$

coupled with the initial-boundary conditions

$$egin{aligned} & \vartheta(\textbf{\textit{x}},0) = \vartheta_0(\textbf{\textit{x}}) & \text{in } \Omega, \ & \partial_{\mathbf{n}} \vartheta(\textbf{\textit{s}},t) = \gamma(\textbf{\textit{s}})(\vartheta_{\Gamma}(\textbf{\textit{s}},t) - \vartheta(\textbf{\textit{s}},t)) & \text{on } \Sigma_{\infty} := \partial \Omega \times (0,\infty), \end{aligned}$$

where Δ is the Laplace operator, γ , $\vartheta_{\Gamma} \geq 0$ bounded. Let $\kappa = 1$ in the following.

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$$\mu(\vartheta(\mathbf{x}))\chi_{t}(\mathbf{x}) + \vartheta(\mathbf{x})\sigma'(\chi(\mathbf{x})) + \lambda'(\chi(\mathbf{x}))$$

$$+ 2 \int_{\Omega} K(\mathbf{x}, \mathbf{y})G'(\chi(\mathbf{x}) - \chi(\mathbf{y})) d\mathbf{y} + (\beta + \vartheta(\mathbf{x}))\partial\varphi(\chi(\mathbf{x}))$$

$$\ni c'_{V}(\chi(\mathbf{x}))\vartheta(\mathbf{x}) (\log \vartheta(\mathbf{x}) - 1),$$

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$$\mu(\vartheta(\mathbf{x}))\chi_{t}(\mathbf{x}) + \vartheta(\mathbf{x})\sigma'(\chi(\mathbf{x})) + \lambda'(\chi(\mathbf{x}))$$

$$+ 2 \int_{\Omega} K(\mathbf{x}, \mathbf{y})G'(\chi(\mathbf{x}) - \chi(\mathbf{y})) d\mathbf{y} + (\beta + \vartheta(\mathbf{x}))\partial\varphi(\chi(\mathbf{x}))$$

$$\ni c'_{V}(\chi(\mathbf{x}))\vartheta(\mathbf{x}) (\log \vartheta(\mathbf{x}) - 1),$$

which we couple with the initial condition

$$\chi(x,0) = \chi_0(x)$$
 in Ω

where χ_0 is a given initial configuration.

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Suppose for a moment to have a relaxed Stefan problem, where thermodynamic equilibria are on the curve

$$c_V'(\boldsymbol{\chi})\vartheta(\log\vartheta-1)+\tfrac{L}{\vartheta_c}(\vartheta-\vartheta_c)\in\partial \emph{\textbf{I}}_{[-1,1]}(\boldsymbol{\chi})$$

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Suppose for a moment to have a relaxed Stefan problem, where thermodynamic equilibria are on the curve

$$c'_{V}(\chi)\vartheta(\log\vartheta-1)+\frac{L}{\vartheta_{c}}(\vartheta-\vartheta_{c})\in\mathcal{H}^{-1}(\chi)$$

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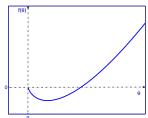
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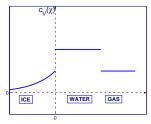
Suppose for a moment to have a relaxed Stefan problem, where thermodynamic equilibria are on the curve

$$\chi \in \mathcal{H}(c_V'(\chi)f(\vartheta) + \frac{L}{\vartheta_c}(\vartheta - \vartheta_c))$$
 with $f(\vartheta) := \vartheta(\log \vartheta - 1)$

.

Then, $c_V'(\chi) := c_V^1 - c_V^0$ must have a positive sign, otherwise $\chi = -1$ would be the only stable equilibrium both for low and high ϑ .





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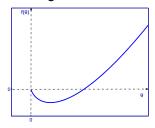
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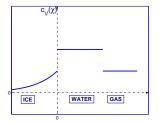
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Suppose for a moment to have a relaxed Stefan problem, where thermodynamic equilibria are on the curve

$$X \in \mathcal{H}(c_V'(X)f(\vartheta) + \frac{L}{\vartheta_c}(\vartheta - \vartheta_c))$$
 with $f(\vartheta) := \vartheta(\log \vartheta - 1)$

Then, $c_V'(\chi) := c_V^1 - c_V^0$ must have a positive sign, otherwise $\chi = -1$ would be the only stable equilibrium both for low and high ϑ .





Hence, the model is appropriate for ice-water phase transitions (and also phase changes in alloys), but not, e.g., for water-vapour phase changes (where $c_V^1 < c_V^0$).

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Assume that there exist c_0 , $\bar{\vartheta}_{\Gamma}$, $\mu_* > 0$ such that

- (i) $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and l.s.c.;
- (ii) $\sigma, \lambda \in W^{2,\infty}(\mathcal{D}(\varphi));$

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- (ii) $\sigma, \lambda \in W^{2,\infty}(\mathcal{D}(\varphi));$
- (iii) $G \in W^{2,\infty}(\mathcal{D}(\varphi) \mathcal{D}(\varphi))$, G(z) = G(-z) for all $z \in (\mathcal{D}(\varphi) \mathcal{D}(\varphi))$, and $K \in L^{\infty}(\Omega \times \Omega)$: K(x,y) = K(y,x) a.e. in $\Omega \times \Omega$;

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- (i) $\varphi:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is proper, convex, and l.s.c.;
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- (iv) $c_V \in W^{2,\infty}(\mathcal{D}(\varphi))$ such that $c_V(z) \ge c_0 > 0$ for all $z \in \mathcal{D}(\varphi)$;

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- (iv) $c_V \in W^{2,\infty}(\mathcal{D}(\varphi))$ such that $c_V(z) \geq c_0 > 0$ for all $z \in \mathcal{D}(\varphi)$;
- (v) $\gamma \in L^{\infty}(\partial\Omega)$ is non-negative and $\vartheta_{\Gamma} \in L^{\infty}(\Sigma_{\infty})$, $(\vartheta_{\Gamma})_{t} \in L^{2}_{loc}(\Sigma_{\infty})$, $\vartheta_{\Gamma} \geq \bar{\vartheta}_{\Gamma} > 0$ a.e. in Σ_{∞} ;

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Assume that there exist c_0 , $\bar{\vartheta}_{\Gamma}$, $\mu_* > 0$ such that

- (i) $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and l.s.c.;
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- (iii) $G \in W^{2,\infty}(\mathcal{D}(\varphi) \mathcal{D}(\varphi))$, G(z) = G(-z) for all $z \in (\mathcal{D}(\varphi) \mathcal{D}(\varphi))$, and $K \in L^{\infty}(\Omega \times \Omega)$: K(x,y) = K(y,x) a.e. in $\Omega \times \Omega$;
- (iv) $c_V \in W^{2,\infty}(\mathcal{D}(\varphi))$ such that $c_V(z) \ge c_0 > 0$ for all $z \in \mathcal{D}(\varphi)$;
- (v) $\gamma \in L^{\infty}(\partial\Omega)$ is non-negative and $\vartheta_{\Gamma} \in L^{\infty}(\Sigma_{\infty})$, $(\vartheta_{\Gamma})_{t} \in L^{2}_{loc}(\Sigma_{\infty})$, $\vartheta_{\Gamma} \geq \bar{\vartheta}_{\Gamma} > 0$ a.e. in Σ_{∞} ;
- (vi) There exist constants $\psi^* > \psi_* > 0$ such that $\psi^* \geq \psi_1(x) \geq \psi_*$ a. e., where $\psi_1 \in H^1$ is the eigenfunction with unit H-norm corresponding to the smallest eigenvalue $\lambda_1 \geq 0$ of

$$-\Delta\psi_1 = \lambda_1\psi_1$$
 in Ω , $\partial_{\mathbf{n}}\psi_1 + \gamma\psi_1 = 0$ on $\partial\Omega$.

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Assume that there exist $c_0, \bar{\vartheta}_{\Gamma}, \mu_* > 0$ such that

- (i) $\varphi: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and l.s.c.;
- (ii) $\sigma, \lambda \in W^{2,\infty}(\mathcal{D}(\varphi));$
- (iii) $G \in W^{2,\infty}(\mathcal{D}(\varphi) \mathcal{D}(\varphi))$, G(z) = G(-z) for all $z \in (\mathcal{D}(\varphi) \mathcal{D}(\varphi))$, and $K \in L^{\infty}(\Omega \times \Omega)$: K(x,y) = K(y,x) a.e. in $\Omega \times \Omega$;
- (iv) $c_V \in W^{2,\infty}(\mathcal{D}(\varphi))$ such that $c_V(z) \ge c_0 > 0$ for all $z \in \mathcal{D}(\varphi)$;
- (v) $\gamma \in L^{\infty}(\partial\Omega)$ is non-negative and $\vartheta_{\Gamma} \in L^{\infty}(\Sigma_{\infty})$, $(\vartheta_{\Gamma})_{t} \in L^{2}_{loc}(\Sigma_{\infty}), \ \vartheta_{\Gamma} \geq \bar{\vartheta}_{\Gamma} > 0$ a.e. in Σ_{∞} ;
- (vi) There exist constants $\psi^* > \psi_* > 0$ such that $\psi^* \geq \psi_1(x) \geq \psi_*$ a. e., where $\psi_1 \in H^1$ is the eigenfunction with unit H-norm corresponding to the smallest eigenvalue $\lambda_1 \geq 0$ of
 - $-\Delta\psi_1 = \lambda_1\psi_1 \quad \text{in } \Omega, \quad \partial_{\mathbf{n}}\psi_1 + \gamma\psi_1 = \mathbf{0} \quad \text{on } \partial\Omega.$
- (vii) μ is locally Lipschitz, $\mu_*(1+\tau) \leq \mu(\tau) \ \forall \tau \in \mathbb{R}^+$.

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Let the initial data satisfy the condition

$$\begin{split} &\vartheta_0 \in H^1(\Omega) \cap L^\infty(\Omega)\,, \ \exists \, \vartheta_* \, > 0 \, : \, \vartheta_0(x) \geq \vartheta_* \ \text{ a. e. in } \Omega\,, \\ &\chi_0 \in L^\infty(\Omega)\,, \ \exists \, C_0 > 0 \, : \, \chi_0(x) \in \mathcal{D}_{C_0}(\varphi) \ \text{ a. e. in } \Omega\,, \\ &\text{where } \mathcal{D}_{C}(\varphi) = \{\chi \in \mathcal{D}(\varphi)\,; \, \partial \varphi(\chi) \cap [-C,C] \neq \emptyset\}. \end{split}$$

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$$\begin{split} \vartheta_0 \in H^1(\Omega) \cap L^\infty(\Omega) \,, \ \exists \, \vartheta_* \, > 0 \, : \, \vartheta_0(x) \geq \vartheta_* \ \text{a. e. in } \Omega \,, \\ \chi_0 \in L^\infty(\Omega) \,, \ \exists \, C_0 > 0 \, : \, \chi_0(x) \in \mathcal{D}_{C_0}(\varphi) \ \text{a. e. in } \Omega \,, \\ \text{where } \mathcal{D}_C(\varphi) = \{ \chi \in \mathcal{D}(\varphi) \,; \, \partial \varphi(\chi) \cap [-C,C] \neq \emptyset \}. \text{Then } \\ \text{there exists at least one pair } (\vartheta,\chi) \text{ solving our problem} \end{split}$$

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Let the initial data satisfy the condition

$$\begin{split} \vartheta_0 \in H^1(\Omega) \cap L^\infty(\Omega) \,, \ \exists \, \vartheta_* \, > 0 \, : \, \vartheta_0(x) \geq \vartheta_* \ \text{ a. e. in } \Omega \,, \\ \chi_0 \in L^\infty(\Omega) \,, \ \exists \, C_0 > 0 \, : \, \chi_0(x) \in \mathcal{D}_{C_0}(\varphi) \ \text{ a. e. in } \Omega \,, \\ \text{where } \mathcal{D}_C(\varphi) = \{\chi \in \mathcal{D}(\varphi) \,; \, \partial \varphi(\chi) \cap [-C,C] \neq \emptyset \}. \text{Then } \\ \text{there exists at least one pair } (\vartheta,\chi) \text{ solving our problem } \\ \text{and such that} \end{split}$$

$$egin{aligned} \vartheta \in L^{\infty}(\mathsf{Q}_{\infty}) \,, \; artheta_t, \Delta artheta \in L^2_{\mathrm{loc}}(0,\infty\,;\; L^2(\Omega)) \,, \ artheta(x,t) > 0 \quad ext{a. e. in } \mathsf{Q}_{\infty} := \Omega imes (0,\infty) \,, \ artheta \in L^{\infty}_{\mathrm{loc}}(\mathsf{Q}_{\infty}), \; arkappa_t \in L^{\infty}(\mathsf{Q}_{\infty}); \ \exists \; C > 0 \,:\; arkappa(x,t) \in \mathcal{D}_{C}(arphi) \quad ext{a. e. in } \mathsf{Q}_{\infty} \,. \end{aligned}$$

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Let the initial data satisfy the condition

$$egin{aligned} artheta_0 &\in H^1(\Omega) \cap L^\infty(\Omega)\,, \ \exists \, artheta_* > 0 \,:\, artheta_0(x) \geq artheta_* \ ext{a. e. in } \Omega\,, \ \chi_0 &\in L^\infty(\Omega)\,, \ \exists \, C_0 > 0 \,:\, \chi_0(x) \in \mathcal{D}_{C_0}(arphi) \ ext{ a. e. in } \Omega\,, \end{aligned}$$

where $\mathcal{D}_{\mathcal{C}}(\varphi) = \{\chi \in \mathcal{D}(\varphi) : \partial \varphi(\chi) \cap [-C, C] \neq \emptyset\}$. Then there exists at least one pair (ϑ, χ) solving our problem and such that

$$egin{aligned} \vartheta \in L^{\infty}(\mathsf{Q}_{\infty}) \,, \; \vartheta_t, \Delta \vartheta \in L^2_{\mathrm{loc}}(0,\infty\,;\; L^2(\Omega)) \,, \ & \vartheta(x,t) > 0 \quad \text{a. e. in } \mathsf{Q}_{\infty} := \Omega imes (0,\infty) \,, \ & \chi \in L^{\infty}_{\mathrm{loc}}(\mathsf{Q}_{\infty}), \; \chi_t \in L^{\infty}(\mathsf{Q}_{\infty}); \ & \exists \; C > 0 \,:\, \chi(x,t) \in \mathcal{D}_C(\varphi) \quad \text{a. e. in } \mathsf{Q}_{\infty} \,. \end{aligned}$$

Moreover, there exist two positive constants $\overline{\vartheta}$ and $\underline{\vartheta}$ (independent of t) such that:

$$\underline{\vartheta} < \vartheta(\mathbf{x},t) < \overline{\vartheta} \quad \text{for a. e. } (\mathbf{x},t) \in \mathsf{Q}_{\infty}.$$

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Let the same Hypotheses hold, and let (ϑ_1, χ_1) , (ϑ_2, χ_2) be solutions to our problem associated with respective boundary and initial data $\vartheta_{\Gamma 1}, \vartheta_{01}, \chi_{01}$ and $\vartheta_{\Gamma 2}, \vartheta_{02}, \chi_{02}$.

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Let the same Hypotheses hold, and let (ϑ_1, χ_1) , (ϑ_2, χ_2) be solutions to our problem associated with respective boundary and initial data $\vartheta_{\Gamma 1}, \vartheta_{01}, \chi_{01}$ and $\vartheta_{\Gamma 2}, \vartheta_{02}, \chi_{02}$. Put $\hat{\theta} = \vartheta_1 - \vartheta_2$, $\hat{\chi} = \chi_1 - \chi_2$, $\hat{\theta}_{\Gamma} = \vartheta_{\Gamma 1} - \vartheta_{\Gamma 2}$, $\hat{\chi}_0 = \chi_{01} - \chi_{02}$, $\hat{\theta}_0 = \vartheta_{01} - \vartheta_{02}$.

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Let the same Hypotheses hold, and let (ϑ_1, χ_1) , (ϑ_2, χ_2) be solutions to our problem associated with respective boundary and initial data $\vartheta_{\Gamma 1}, \vartheta_{01}, \chi_{01}$ and $\vartheta_{\Gamma 2}, \vartheta_{02}, \chi_{02}$. Put $\hat{\theta} = \vartheta_1 - \vartheta_2$, $\hat{\chi} = \chi_1 - \chi_2$, $\hat{\theta}_{\Gamma} = \vartheta_{\Gamma 1} - \vartheta_{\Gamma 2}$.

Put
$$\theta = \theta_1 - \theta_2$$
, $\chi = \chi_1 - \chi_2$, θ_1
 $\hat{\chi}_0 = \chi_{01} - \chi_{02}$, $\hat{\theta}_0 = \theta_{01} - \theta_{02}$.

Then for every T > 0 there exists a constant

 $C_T > 0$ such that

$$\int_0^T \int_{\Omega} |\hat{\theta}(x,t)|^2 dx dt + \max_{t \in [0,T]} \int_{\Omega} |\hat{\chi}(x,t)|^2 dx$$

$$\leq C_T \left(|\hat{\theta}_0|_H^2 + |\hat{\chi}_0|_H^2 + \int_0^T \int_{\partial \Omega} \gamma \hat{\theta}_{\Gamma}^2(x,t) ds dt \right).$$

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▶ Truncate our PDE's system in the logarithmic functions in ϑ

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- Truncate our PDE's system in the logarithmic functions in θ
- Prove existence of solutions to the approximating system by a Faedo-Galerkin scheme and a fixed point argument
- ► Find uniform estimates (w.r.t. the truncation parameter and to time) on such a solutions using the properties of differential inclusions

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- Truncate our PDE's system in the logarithmic functions in θ
- Prove existence of solutions to the approximating system by a Faedo-Galerkin scheme and a fixed point argument
- ► Find uniform estimates (w.r.t. the truncation parameter and to time) on such a solutions using the properties of differential inclusions and a Moser iteration scheme

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- ► Prove the positivity of temperature and a uniform in time lower bound by a maximum principle result

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- Truncate our PDE's system in the logarithmic functions in θ
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- ► Find uniform estimates (w.r.t. the truncation parameter and to time) on such a solutions using the properties of differential inclusions and a Moser iteration scheme
- ► Prove the positivity of temperature and a uniform in time lower bound by a maximum principle result
- ▶ Prove uniqueness using a L¹-Lipschitz continuity of the solution operator associated to the differential inclusion related to the phase variable

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PROPOSITION (DI). Let
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, $Q_T := \Omega \times (0, T)$, $\vartheta \in L^1(Q_T)$ and $\chi_0 \in L^\infty(\Omega)$, $\chi_0(x) \in \mathcal{D}_C(\varphi)$ a. e. in Ω ,

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• $\alpha:\mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous in \mathbb{R} and bdd from below,

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- ▶ $\alpha: \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous in \mathbb{R} and bdd from below,
- ▶ $f = f[X, \vartheta] : L^1(Q_T) \times L^1(Q_T) \to L^\infty(Q_T)$ be a bdd (by the constant *C*) and Lipschitz continuous w.r.t. the first variable.

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There exists a unique solution $\chi \in L^{\infty}(Q_T)$ to

$$\alpha(\vartheta) \chi_t + \partial \varphi(\chi) \ni f[\chi, \vartheta], \chi(x, 0) = \chi_0(x) \text{ a. e. in } \Omega$$

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There exists a unique solution $\chi \in L^{\infty}(Q_T)$ to

$$\alpha(\vartheta) \chi_t + \partial \varphi(\chi) \ \ni \ f[\chi, \vartheta], \ \chi(x, 0) = \chi_0(x) \ \text{a. e. in } \Omega$$

such that $\chi(\mathbf{x},t) \in \mathcal{D}_{\mathbf{C}}(\varphi)$ a.e. in $\mathbf{Q}_{\mathcal{T}}, \chi_t \in L^{\infty}(\mathbf{Q}_{\mathcal{T}})$, and

$$|f[\chi, \vartheta](x, t) - \alpha(\vartheta(x, t)) \chi_t(x, t)| \leq C$$
 a.e. in Q_T .

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• We have the L^1 -Lipschitz continuity in case f is also Lipschitz continuous w.r.t. ϑ :

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• We have the L^1 -Lipschitz continuity in case f is also Lipschitz continuous w.r.t. ϑ : there exists M>0 such that the solutions $\chi_1,\chi_2\in L^\infty(Q_T)$ associated with $\chi_{01},\chi_{02}\in\mathcal{D}_C(\varphi),\,\vartheta_1,\vartheta_2\in L^1(Q_T)$ satisfy a. e. in Q_∞ :

$$|(\chi_1 - \chi_2)_t(\mathbf{x}, t)| + \frac{\partial}{\partial t} |(\chi_1 - \chi_2)(\mathbf{x}, t)|$$

$$\leq M \Big(|(\vartheta_1 - \vartheta_2)(\mathbf{x}, t)| + |(f[\chi_1, \vartheta_1] - f[\chi_2, \vartheta_2])(\mathbf{x}, t)| \Big).$$

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In case f is also Lipschitz continuous w.r.t. ϑ , integrating over $\Omega \times (0,t)$ and using Gronwall lemma, we obtain the L^1 -Lipschitz continuity.

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In case f is also Lipschitz continuous w.r.t. ϑ , integrating over $\Omega \times (0,t)$ and using Gronwall lemma, we obtain the L^1 -Lipschitz continuity.

 Only if φ is C¹ with locally Lipschitz continuous derivative it can be extended to L^p(Q_T) for p > 1.

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$$|(\chi_1 - \chi_2)_t(\mathbf{x}, t)| + \frac{\partial}{\partial t} |(\chi_1 - \chi_2)(\mathbf{x}, t)|$$

$$\leq M \Big(|(\vartheta_1 - \vartheta_2)(\mathbf{x}, t)| + |(f[\chi_1, \vartheta_1] - f[\chi_2, \vartheta_2])(\mathbf{x}, t)| \Big).$$

In case f is also Lipschitz continuous w.r.t. ϑ , integrating over $\Omega \times (0,t)$ and using Gronwall lemma, we obtain the L^1 -Lipschitz continuity.

- Only if φ is C¹ with locally Lipschitz continuous derivative it can be extended to L^p(Q_T) for p > 1.
- Strong continuity L¹(Q_T) → L^p(Q_T) of the solution mapping for p < ∞ follows however from the uniform L[∞]-bound.

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Fix a final time T>0 and sufficiently regular and bdd r, $h_1,\ h_2,\ \gamma,\ u_\Gamma,\ u_0:\ \gamma,r,u_\Gamma\geq 0,\ a(x,t)\geq a_*>0,$ and $u_*\psi_*< u_*\psi_1(x)\leq u_0(x)\leq u^*$ a. e.

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$$\left\{ \begin{array}{l} \displaystyle \int_{\Omega} a u_t w \, dx + \int_{\Omega} \left\langle \nabla u, \nabla w \right\rangle \, dx + \int_{\partial \Omega} \gamma (u - u_\Gamma) w \, ds \\ = \displaystyle \int_{\Omega} (r(x,t) + h_1(x,t) \, u + h_2(x,t) \, \frac{u}{u} |\log |u||) \, w \, dx \, , \\ u(x,0) = u_0(x) \quad \text{a. e.} \end{array} \right.$$

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Fix a final time T>0 and sufficiently regular and bdd r, $h_1, h_2, \gamma, u_{\Gamma}, u_0: \gamma, r, u_{\Gamma} \geq 0, a(x,t) \geq a_* > 0$, and $u_*\psi_* < u_*\psi_1(x) \leq u_0(x) \leq u^*$ a. e. Then the following PROBLEM in Q_T (for all $w \in H^1$)

$$\begin{cases} \int_{\Omega} a u_t w \, dx + \int_{\Omega} \langle \nabla u, \nabla w \rangle \, dx + \int_{\partial \Omega} \gamma (u - u_{\Gamma}) w \, ds \\ = \int_{\Omega} (r(x,t) + h_1(x,t) \, u + h_2(x,t) \, \frac{u}{u} |\log |u||) \, w \, dx \,, \\ u(x,0) = u_0(x) \quad \text{a. e.} \end{cases}$$

has a unique solution

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Fix a final time T>0 and sufficiently regular and bdd r, $h_1, h_2, \gamma, u_{\Gamma}, u_0: \gamma, r, u_{\Gamma} \geq 0, a(x,t) \geq a_* > 0$, and $u_*\psi_* < u_*\psi_1(x) \leq u_0(x) \leq u^*$ a. e. Then the following PROBLEM in Q_T (for all $w \in H^1$)

$$\begin{cases} \int_{\Omega} au_t w \, dx + \int_{\Omega} \langle \nabla u, \nabla w \rangle \, dx + \int_{\partial \Omega} \gamma (u - u_{\Gamma}) w \, ds \\ = \int_{\Omega} (r(x,t) + h_1(x,t) \, u + h_2(x,t) \, \frac{u}{|\log |u|} |) \, w \, dx \,, \\ u(x,0) = u_0(x) \quad \text{a. e.} \end{cases}$$

has a unique solution and it holds

$$u_*\psi_1(x)e^{-H(t)} \le u(x,t) \le K e^{H(t)}$$
 a. e.,

where $H(t) = \frac{A}{a_*} \int_0^t (1 + e^{B\tau}) d\tau$, and K, A, and B depend only on the data (not on T).

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Let the following assumptions hold true:

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Let the following assumptions hold true:

$$\begin{array}{l} \text{(i)} \;\; \mathcal{H} \; : \; L^{\infty}_{\text{loc}}(Q_{\infty}) \to L^{\infty}_{\text{loc}}(Q_{\infty}) \; \text{satisfy} \; \forall u \in L^{\infty}_{\text{loc}}(Q_{\infty}) \\ \\ u(x,t) \, \mathcal{H}[u](x,t) \leq H_{1} |u(x,t)| + H_{0} |u(x,t)|^{2} \quad \text{a. e. in } Q_{\infty}, \end{array}$$

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Let the following assumptions hold true:

$$\begin{array}{l} \text{(i)} \ \, \mathcal{H} \, : \, L^{\infty}_{\mathrm{loc}}(\mathsf{Q}_{\infty}) \to L^{\infty}_{\mathrm{loc}}(\mathsf{Q}_{\infty}) \text{ satisfy } \forall u \in L^{\infty}_{\mathrm{loc}}(\mathsf{Q}_{\infty}) \\ \\ u(x,t) \, \mathcal{H}[u](x,t) \leq H_{1}|u(x,t)| + H_{0}|u(x,t)|^{2} \quad \text{a. e. in } \mathsf{Q}_{\infty}, \end{array}$$

(ii) h, γ are suff. regular, $\gamma \geq 0$, a, a_t, u^0, u_Γ are sufficiently regular and bdd functions : $0 < a_0 \leq a(x,t), |a_t(x,t)| \leq a_1$ a.e. in Q_∞ ,

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Let the following assumptions hold true:

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- (ii) h, γ are suff. regular, $\gamma \geq 0$, a, a_t, u^0, u_Γ are sufficiently regular and bdd functions: $0 < a_0 \leq a(x,t), |a_t(x,t)| \leq a_1$ a.e. in Q_∞ ,
- (iii) there exists $u \in L^{\infty}_{loc}(\mathbb{Q}_{\infty}) \cap L^{2}_{loc}(0,\infty;H^{1})$ solution of

$$egin{aligned} \left\{ egin{aligned} & a(x,t)u_t - \Delta u = \mathcal{H}[u] \ ext{a.e. on } Q_\infty \ & \partial_{\mathbf{n}} u(x,t) + \gamma(x) \left(h(x,t,u(x,t)) - u_\Gamma(x,t)
ight) = 0 \ ext{a.e. on } \Sigma_\infty \ & u(x,0) = u^0 \ ext{a.e. in } \Omega \,, \, \int_\Omega |u(x,t)| \ dx \leq rac{E_0}{2} \ ext{a.e. in } (0,\infty) \,. \end{aligned}$$

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Let the following assumptions hold true:

$$\begin{array}{l} \text{(i)} \ \, \mathcal{H} \, : \, L^{\infty}_{\mathrm{loc}}(Q_{\infty}) \to L^{\infty}_{\mathrm{loc}}(Q_{\infty}) \text{ satisfy } \forall u \in L^{\infty}_{\mathrm{loc}}(Q_{\infty}) \\ \\ u(x,t) \, \mathcal{H}[u](x,t) \leq H_{1}|u(x,t)| + H_{0}|u(x,t)|^{2} \quad \text{a. e. in } Q_{\infty}, \end{array}$$

- (ii) h, γ are suff. regular, $\gamma > 0$, a, a_t, u^0, u_Γ are sufficiently regular and bdd functions: $0 < a_0 \le a(x,t), |a_t(x,t)| \le a_1$ a.e. in Q_{∞} ,
- (iii) there exists $u \in L^{\infty}_{loc}(\mathbb{Q}_{\infty}) \cap L^{2}_{loc}(0,\infty; H^{1})$ solution of

$$\begin{cases} a(x,t)u_t - \Delta u = \mathcal{H}[u] \text{ a. e. on } Q_{\infty} \\ \partial_{\mathbf{n}} u(x,t) + \gamma(x) \left(h(x,t,u(x,t)) - u_{\Gamma}(x,t) \right) = 0 \text{ a. e. on } \Sigma_{\infty} \\ u(x,0) = u^0 \text{ a. e. in } \Omega, \ \int_{\Omega} |u(x,t)| \ dx \leq \underline{E}_0 \text{ a. e. in } (0,\infty). \end{cases}$$

Then there exists $C^* > 0$, depending on the data (but not on a_1 , H_0 , H_1 , E_0) such that, for a. e. t > 0, it holds:

$$|u(t)|_{L^{\infty}(\Omega)} \leq C^* (1 + H_0 + a_1)^{1+N/2} (1 + H_1 + E_0).$$

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TECHNIQUE: approximations such that $\mu \mapsto \mu_{\varrho}$, $1 - \log \vartheta \mapsto f_{\varrho}(\vartheta)$, $\vartheta \mapsto |\vartheta|$,

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$$\begin{split} \mu_{\varrho}(\vartheta) \; &= \; \left\{ \begin{array}{l} \mu(|\vartheta|) & \text{for } |\vartheta| \leq \varrho \,, \\ \mu(\varrho) + \mu_*(|\vartheta| - \varrho) & \text{for } |\vartheta| > \varrho \,; \end{array} \right. \\ f_{\varrho}(\vartheta) \; &= \; \left\{ \begin{array}{l} 1 & \text{for } \vartheta \leq 0 \,, \\ 1 - \log \vartheta & \text{for } 0 < \vartheta < \varrho \,, \\ 1 - \log \varrho & \text{for } \vartheta \geq \varrho \,, \end{array} \right. \end{split}$$

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$$egin{aligned} \mu_{arrho}(artheta) &= \left\{ egin{array}{ll} \mu(|artheta|) & & ext{for } |artheta| \leq arrho\,, \ \mu(arrho) + \mu_*(|artheta| - arrho) & ext{for } |artheta| > arrho\,; \ \end{array}
ight. \ for \, artheta \leq 0\,, \ 1 - \log artheta & ext{for } 0 < artheta < arrho\,, \ 1 - \log arrho & ext{for } artheta \geq arrho\,, \end{aligned}$$

Then, via Faedo-Galerkin method, for T > 0, we find a pair (ϑ, χ) solving a. e. in Q_T the system coupling the same equations as before for ϑ , the same I.B.C. with

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$$\mu_{\varrho}(\vartheta)\chi_{t} + c'_{V}(\chi)\vartheta f_{\varrho}(\vartheta) + \lambda'(\chi) + \vartheta\sigma'(\chi) + (\beta + |\vartheta|)\partial\varphi(\chi) + \int_{\Omega} K(x,y)G'(\chi(x) - \chi(y)) dy \ni 0.$$

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$$\mu_{\varrho}(\vartheta)\chi_{t} + (\beta + |\vartheta|)\partial\varphi(\chi)$$

$$\ni - (c'_{V}(\chi)\vartheta f_{\varrho}(\vartheta) + \lambda'(\chi) + \vartheta\sigma'(\chi) + b[\chi]).$$

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$$\frac{\mu_{\varrho}(\vartheta)}{\beta + |\vartheta|} \chi_{t} + \partial \varphi(\chi) \ni -\frac{1}{\beta + |\vartheta|} \left(c'_{V}(\chi) \vartheta f_{\varrho}(\vartheta) + \lambda'(\chi) + \vartheta \sigma'(\chi) + b[\chi] \right).$$
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The equation (ChiR) is of the form of the differential inclusion introduced before with

$$\alpha(\vartheta) = \frac{\mu_{\varrho}(\vartheta)}{\beta + |\vartheta|} \ge \frac{\mu_{*}(1 + |\vartheta|)}{\beta + |\vartheta|} \ge \mu_{*} \min\left\{1, \frac{1}{\beta}\right\},$$

$$f[\chi, \vartheta] = -\frac{1}{\beta + |\vartheta|} \left(c'_{V}(\chi)\vartheta f_{\varrho}(\vartheta) + \lambda'(\chi) + \vartheta\sigma'(\chi) + b[\chi]\right).$$

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The equation (ChiR) is of the form of the differential inclusion introduced before with

$$\begin{split} \alpha(\vartheta) &= \frac{\mu_{\varrho}(\vartheta)}{\beta + |\vartheta|} \, \geq \, \frac{\mu_{*}(\mathbf{1} + |\vartheta|)}{\beta + |\vartheta|} \, \geq \, \mu_{*} \, \min\left\{\mathbf{1}, \frac{1}{\beta}\right\} \,, \\ f[\chi, \vartheta] &= -\frac{1}{\beta + |\vartheta|} \left(\mathcal{C}'_{V}(\chi) \vartheta f_{\varrho}(\vartheta) + \lambda'(\chi) + \vartheta \sigma'(\chi) + \boldsymbol{b}[\chi] \right). \end{split}$$

By Hypothesis $f[\vartheta, \chi]$ is bdd for all $\vartheta, \chi \in L^1(Q_T)$, $\chi \in \mathcal{D}(\varphi)$.

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$$\begin{split} \alpha(\vartheta) &= \frac{\mu_{\varrho}(\vartheta)}{\beta + |\vartheta|} \, \geq \, \frac{\mu_*(\mathbf{1} + |\vartheta|)}{\beta + |\vartheta|} \, \geq \, \mu_* \, \min\left\{\mathbf{1}, \frac{\mathbf{1}}{\beta}\right\} \,, \\ f[\chi, \vartheta] &= -\frac{\mathbf{1}}{\beta + |\vartheta|} \left(\mathcal{C}'_V(\chi) \vartheta f_{\varrho}(\vartheta) + \lambda'(\chi) + \vartheta \sigma'(\chi) + \boldsymbol{b}[\chi] \right). \end{split}$$

By Hypothesis $f[\vartheta,\chi]$ is bdd for all $\vartheta, \chi \in L^1(Q_T)$, $\chi \in \mathcal{D}(\varphi)$. Hence, the assumptions of PROPOSITION (DI) are satisfied. We thus may define the solution mapping

$$A_{\varrho}: L^{1}(Q_{T}) \to L^{\infty}(Q_{T}): \vartheta \mapsto \chi,$$

which with each $\vartheta \in L^1(Q_T)$ associates the solution χ to (??) with fixed initial condition χ_0 .

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We first notice that PROPOSITION (DI) yield

$$|\chi|_{L^{\infty}(Q_{\mathcal{T}})} + |\chi_t|_{L^{\infty}(Q_{\mathcal{T}})} + |\varphi(\chi)_t|_{L^{\infty}(Q_{\mathcal{T}})} \leq c(1 + \log \varrho).$$

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We first notice that Proposition (DI) yield

$$|\chi|_{L^{\infty}(Q_{T})} + |\chi_{t}|_{L^{\infty}(Q_{T})} + |\varphi(\chi)_{t}|_{L^{\infty}(Q_{T})} \leq c(1 + \log \varrho).$$

We may test with ϑ_t the equation for ϑ and integrate over (0, t) with $t \in (0, T]$, obtaining

$$|\vartheta_t|_{L^2(0,T;L^2(\Omega))} + |\vartheta|_{L^\infty(0,T;H^1(\Omega))} \leq c(1 + \log \varrho),$$

with c independent of ϱ .

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We may test with ϑ_t the equation for ϑ and integrate over (0, t) with $t \in (0, T]$, obtaining

$$|\vartheta_t|_{L^2(0,T;L^2(\Omega))} + |\vartheta|_{L^\infty(0,T;H^1(\Omega))} \leq c(1 + \log \varrho),$$

with c independent of ϱ . It remains to prove the positivity and a uniform upper bound for the ϑ -component of the solution to the approximating problem.

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 $(ChiR) \times \chi_t$ inserted in the equation for ϑ gives:

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Use the second auxiliary results

(ChiR) $\times \chi_t$ inserted in the equation for ϑ gives:

$$\begin{split} \boldsymbol{c}_{\boldsymbol{V}}(\boldsymbol{\chi})\boldsymbol{\vartheta}_t - \Delta\boldsymbol{\vartheta} &= \mu_{\varrho}(\boldsymbol{\vartheta})\,\boldsymbol{\chi}_t^2 + \boldsymbol{\vartheta}\boldsymbol{\sigma}(\boldsymbol{\chi})_t + |\boldsymbol{\vartheta}|\boldsymbol{\varphi}(\boldsymbol{\chi})_t \\ &\quad + \boldsymbol{c}_{\boldsymbol{V}}'(\boldsymbol{\chi})\boldsymbol{\vartheta}(\boldsymbol{f}_{\varrho}(\boldsymbol{\vartheta}) - \boldsymbol{1})\boldsymbol{\chi}_t \,. \end{split}$$

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Use the second auxiliary results

 $(ChiR) \times \chi_t$ inserted in the equation for ϑ gives:

$$\begin{split} \overline{c_{V}(\chi)\vartheta_{t} - \Delta\vartheta} &= \frac{\mu_{\varrho}(\vartheta)}{1 + |\vartheta|}\chi_{t}^{2} \\ &+ \vartheta \left(\sigma(\chi)_{t} + \operatorname{sign}(\vartheta) \left(\varphi(\chi)_{t} + \frac{\mu_{\varrho}(\vartheta)}{1 + |\vartheta|}\chi_{t}^{2}\right)\right) \\ &+ \vartheta |\log|\vartheta|| \left(c_{V}'(\chi) \frac{f_{\varrho}(\vartheta) - 1}{|\log|\vartheta||}\chi_{t}\right). \end{split}$$

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(ChiR)× χ_t inserted in the equation for ϑ gives:

$$\begin{split} c_V(\chi)\vartheta_t - \Delta\vartheta &= \frac{\mu_\varrho(\vartheta)}{1+|\vartheta|}\chi_t^2 \\ &+ \vartheta\left(\sigma(\chi)_t + \mathrm{sign}(\vartheta)\Big(\varphi(\chi)_t + \frac{\mu_\varrho(\vartheta)}{1+|\vartheta|}\chi_t^2\Big)\right) \\ &+ \vartheta|\log|\vartheta||\left(c_V'(\chi)\frac{f_\varrho(\vartheta) - 1}{|\log|\vartheta||}\chi_t\right). \end{split}$$

Choosing in the Maximum principle result $u = \vartheta$, and

$$\begin{aligned} & a(\mathbf{x},t) = \mathbf{c}_{V}(\mathbf{X}), \quad r(\mathbf{x},t) = \frac{\mu_{\varrho}(\vartheta)}{1 + |\vartheta|} \chi_{t}^{2}, \\ & h_{1}(\mathbf{x},t) = \sigma(\mathbf{X})_{t} + \operatorname{sign}(\vartheta) \Big(\varphi(\mathbf{X})_{t} + \frac{\mu_{\varrho}(\vartheta)}{1 + |\vartheta|} \chi_{t}^{2} \Big), \\ & h_{2}(\mathbf{x},t) = \mathbf{c}_{V}'(\mathbf{X}) \frac{f_{\varrho}(\vartheta) - 1}{|\log|\vartheta||} \chi_{t}, \end{aligned}$$

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(ChiR) $\times \chi_t$ inserted in the equation for ϑ gives:

$$\begin{split} \overline{\mathbf{c}_{V}(\chi)}\vartheta_{t} - \Delta\vartheta &= \frac{\mu_{\varrho}(\vartheta)}{1 + |\vartheta|}\chi_{t}^{2} \\ &+ \vartheta\left(\sigma(\chi)_{t} + \mathrm{sign}(\vartheta)\left(\varphi(\chi)_{t} + \frac{\mu_{\varrho}(\vartheta)}{1 + |\vartheta|}\chi_{t}^{2}\right)\right) \\ &+ \vartheta|\log|\vartheta||\left(\mathbf{c}_{V}'(\chi)\frac{f_{\varrho}(\vartheta) - 1}{|\log|\vartheta||}\chi_{t}\right). \end{split}$$

Choosing in the Maximum principle result $u = \vartheta$, and

$$\begin{aligned} a(x,t) &= c_V(\chi), \quad r(x,t) = \frac{\mu_{\varrho}(\vartheta)}{1+|\vartheta|} \chi_t^2, \\ h_1(x,t) &= \sigma(\chi)_t + \mathrm{sign}(\vartheta) \Big(\varphi(\chi)_t + \frac{\mu_{\varrho}(\vartheta)}{1+|\vartheta|} \chi_t^2 \Big), \\ h_2(x,t) &= c_V'(\chi) \frac{f_{\varrho}(\vartheta) - 1}{|\log |\vartheta||} \chi_t, \end{aligned}$$

we get $\theta > 0$ a. e. in Q_T , hence we can remove the absolute value from the truncated problem.

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Apply the Moser Iteration scheme to

$$c_V(\chi)\vartheta_t - \Delta\vartheta = -c_V'(\chi)\chi_t\vartheta - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t,$$
 with the choices $u = \vartheta$, $a(x, t) = c_V(\chi)$, and
$$\mathcal{H}[u] = -c_V'(\chi)\chi_t u - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t.$$

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Apply the Moser Iteration scheme to

$$\begin{split} c_V(\chi)\vartheta_t - \Delta\vartheta &= -c_V'(\chi)\chi_t\vartheta - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t\,,\\ \text{with the choices } u = \vartheta, \ a(x,t) = c_V(\chi), \ \text{and} \\ \mathcal{H}[u] &= -c_V'(\chi)\chi_t \, u - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t\,\,.\\ \text{Since } \boxed{u\mathcal{H}[u] \leq c\,(1 + \log\varrho)\,(|u| + |u|^2)}, \ \text{hence} \\ H_0, H_1 &= c(1 + \log\varrho), \ a_1 = c_0, \ \text{and so} \\ a_1 + H_0 + H_1 + E_0 &\leq c\,(1 + \log\varrho)\,\,, \end{split}$$

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$$c_V(\chi)\vartheta_t - \Delta\vartheta = -c'_V(\chi)\chi_t\vartheta - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t$$
, with the choices $u = \vartheta$, $a(x, t) = c_V(\chi)$, and

$$\mathcal{H}[u] = -c'_V(\chi)\chi_t \, u - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t.$$

Since
$$u\mathcal{H}[u] \le c (1 + \log \varrho) (|u| + |u|^2)$$
, hence $H_0, H_1 = c(1 + \log \varrho), a_1 = c_0$, and so $a_1 + H_0 + H_1 + E_0 \le c (1 + \log \varrho)$,

where, in order to determine the dependence on ϱ of E_0 we test the ϑ equation by ψ_1 .

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Apply the Moser Iteration scheme to

$$c_V(\chi)\vartheta_t - \Delta\vartheta = -c_V'(\chi)\chi_t\vartheta - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t$$
, with the choices $u = \vartheta$, $a(x, t) = c_V(\chi)$, and

$$\mathcal{H}[u] = -c'_V(\chi)\chi_t \, u - b[\chi]\chi_t - (\lambda(\chi) + \beta\varphi(\chi))_t.$$

Since
$$u\mathcal{H}[u] \le c(1 + \log \varrho)(|u| + |u|^2)$$
, hence $H_0, H_1 = c(1 + \log \varrho), a_1 = c_0$, and so

$$a_1 + H_0 + H_1 + E_0 \le c(1 + \log \varrho)$$
,

where, in order to determine the dependence on ϱ of E_0 we test the ϑ equation by ψ_1 . Hence, we have

$$|\vartheta(x,t)| \leq \bar{c}(1+\log\varrho)^{2+N/2},$$

with \bar{c} independent of ϱ .

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Apply the Moser Iteration scheme to

$$c_V(X)\vartheta_t - \Delta\vartheta = -c_V'(X)X_t\vartheta - b[X]X_t - (\lambda(X) + \beta\varphi(X))_t,$$
with the choices $u = \vartheta_t c(X, t) = a_t(X)$ and

with the choices $u = \vartheta$, $a(x, t) = c_V(\chi)$, and

$$\mathcal{H}[u] = -c'_{V}(X)X_{t} u - b[X]X_{t} - (\lambda(X) + \beta\varphi(X))_{t}.$$

Since
$$u\mathcal{H}[u] \le c(1 + \log \varrho)(|u| + |u|^2)$$
, hence $H_0, H_1 = c(1 + \log \varrho), a_1 = c_0$, and so

$$a_1 + H_0 + H_1 + E_0 \le c(1 + \log \varrho)$$
,

where, in order to determine the dependence on ϱ of E_0 we test the ϑ equation by ψ_1 . Hence, we have

$$|\vartheta(\mathbf{x},t)| \leq \bar{\mathbf{c}}(1+\log\varrho)^{2+N/2},$$

with $\bar{\mathbf{c}}$ independent of $\varrho.$ Choosing ϱ such that

$$\varrho > \bar{c}(1 + \log \varrho)^{2+N/2}$$
,

it follows that ϑ, χ are also solutions of the starting equations.

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• To study more general nonlocal systems including the case of a vectorial phase variable χ

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 To study more general nonlocal systems including the case of a vectorial phase variable X and the dependence on θ and X in the heat conductivity

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• To study more general nonlocal systems including the case of a vectorial phase variable χ and the dependence on ϑ and χ in the heat conductivity $(c_V\vartheta + \lambda(\chi) + \beta\varphi(\chi))_t + 2\chi_t \int_\Omega K(\cdot,y)G'(\chi(\cdot) - \chi(y))dy - \operatorname{div} (\kappa(\vartheta,\chi)\nabla\vartheta) = 0$

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To study more general nonlocal systems including the case of a vectorial phase variable *χ* and the dependence on *θ* and *χ* in the heat conductivity (c_V θ + λ(χ) + βφ(χ))_t + 2χ_t ∫_Ω K(·, y)G'(χ(·) - χ(y))dy - div (κ(ϑ, χ)∇ϑ) = 0 (cf. [P. Colli, P. Krejčí, E. R., J. Sprekels, work in

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• To study more general nonlocal systems including the case of a vectorial phase variable χ and the dependence on ϑ and χ in the heat conductivity $(c_V\vartheta + \lambda(\chi) + \beta\varphi(\chi))_t + 2\chi_t \int_\Omega K(\cdot,y)G'(\chi(\cdot) - \chi(y))dy \\ -\operatorname{div}\left(\kappa(\vartheta,\chi)\nabla\vartheta\right) = 0$

(cf. [P. Colli, P. Krejčí, E. R., J. Sprekels, work in progress]).

• To study the long-time behaviour of solutions:

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- To study more general nonlocal systems including the case of a vectorial phase variable χ and the dependence on ϑ and χ in the heat conductivity $(c_V\vartheta + \lambda(\chi) + \beta\varphi(\chi))_t + 2\chi_t \int_{\Omega} K(\cdot,y)G'(\chi(\cdot) \chi(y))dy$
 - $-\operatorname{div}(\kappa(\vartheta,\chi)\nabla\vartheta)=0$ (cf. [P. Colli, P. Krejčí, E. R., J. Sprekels, work in progress]).
- To study the long-time behaviour of solutions:
- in case of analytical potentials there are basically two results [Feireisl, Issard-Roch, Petzeltová, 2004] and [Grasselli, Petzeltová, Schimperna, to appear] employing the Lojasiewicz-Simon technique;

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 - $-\operatorname{div}(\kappa(\vartheta,\chi)\nabla\vartheta)=0$ (cf. [P. Colli, P. Krejčí, E. R., J. Sprekels, work in progress]).
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- \diamondsuit it does not apply to the case of $\varphi = I_K$, e.g.

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- To study more general nonlocal systems including the case of a vectorial phase variable χ and the dependence on ϑ and χ in the heat conductivity $(c_V\vartheta + \lambda(\chi) + \beta\varphi(\chi))_t + 2\chi_t \int_\Omega K(\cdot,y)G'(\chi(\cdot) \chi(y))dy$
 - $-\operatorname{div}(\kappa(\vartheta,\chi)\nabla\vartheta)=0$ (cf. [P. Colli, P. Krejčí, E. R., J. Sprekels, work in progress]).
- To study the long-time behaviour of solutions:
- in case of analytical potentials there are basically two results [Feireisl, Issard-Roch, Petzeltová, 2004] and [Grasselli, Petzeltová, Schimperna, to appear] employing the Lojasiewicz-Simon technique;
- \diamond it does not apply to the case of $\varphi = I_K$, e.g. One recent contribution in this direction is given by the paper [Krejčí, Zheng, 2005] in case of no nonlocal terms are present in the equations.

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