A new dual approach for a class of phase transitions with memory

E. Rocca

Università degli Studi di Milano

Fluid-Structure Interactions and Related Topics
Nečas Center for Mathematical Modelling

Prague, October 30–November 2, 2007

joint work with E. Bonetti (Pavia) and M. Frémond (Roma)

Plan of the Talk

- Discuss a phase transition model based on a **dual formulation - in the sense of Convex Analysis - of the energy balance**

Plan of the Talk

- Discuss a phase transition model based on a **dual formulation - in the sense of Convex Analysis - of the energy balance**
- Introduce the state variable: in particular, the **entropy** in place of the temperature

Plan of the Talk

- Discuss a phase transition model based on a **dual formulation - in the sense of Convex Analysis - of the energy balance**
- Introduce the state variable: in particular, the **entropy** in place of the temperature
- Describe equilibria by the **internal energy functional** in place of the free energy

Plan of the Talk

- Discuss a phase transition model based on a **dual formulation - in the sense of Convex Analysis - of the energy balance**
- Introduce the state variable: in particular, the **entropy** in place of the temperature
- Describe equilibria by the **internal energy functional** in place of the free energy
- Present our mathematical results: **Existence and long-time behaviour of solutions** for the resulting **doubly nonlinear PDE system**
  [E. Bonetti, M. Frémond, E.R., *to appear on J. Math. Pure Appl.*]

Phase transitions and phase-field models

Phase transitions phenomena occur in several processes of physical and industrial interest (like **solid-liquid systems, solid-solid phase transitions in SMA, damage in elastic material**).

Phase transitions and phase-field models

Phase transitions phenomena occur in several processes of physical and industrial interest (like **solid-liquid systems, solid-solid phase transitions in SMA, damage in elastic material**).
Assume that the two **phases can coexist at every point**, then one needs a **parameter** χ characterizing the different phases (e.g. the concentration of one of the two phases in a point).

Phase transitions and phase-field models

Phase transitions phenomena occur in several processes of physical and industrial interest (like **solid-liquid systems, solid-solid phase transitions in SMA, damage in elastic material**).
Assume that the two **phases can coexist at every point**, then one needs a **parameter** χ characterizing the different phases (e.g. the concentration of one of the two phases in a point).
To describe the system we use the basic laws of continuum mechanics

- The generalized **principle of virtual power for microscopic forces** by [M. Frémond, Non-smooth Thermomechanics, 2002]
- A **dual formulation of the energy balance**

together with a proper choice of our **internal energy functional** (depending on the state variables) and of **the pseudo-potential of dissipation** (depending on the dissipative variables).

The internal energy: a dual formulation

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^{t}, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^{t}, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^{t}, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^{t})$.

We choose

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$.

We choose

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$.

We choose

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

where

- σ and λ are smooth functions accounting for the non-convex part and the latent heat in E_P, $\nu \geq 0$ an interfacial energy coefficient
- $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ is a general proper convex and lower-semicontinuous function
- $\widehat{\alpha} : \mathbb{R} \to [0, +\infty)$ is convex, increasing, and suitably regular

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$.

We choose

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

where

- σ and λ are smooth functions accounting for the non-convex part and the latent heat in E_P, $\nu \geq 0$ an interfacial energy coefficient
- $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ is a general proper convex and lower-semicontinuous function
- $\widehat{\alpha} : \mathbb{R} \to [0, +\infty)$ is convex, increasing, and suitably regular

It corresponds

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$.

We choose

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

where

- σ and λ are smooth functions accounting for the non-convex part and the latent heat in E_P, $\nu \geq 0$ an interfacial energy coefficient
- $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ is a general proper convex and lower-semicontinuous function
- $\widehat{\alpha} : \mathbb{R} \to [0, +\infty)$ is convex, increasing, and suitably regular

It corresponds - due to the standard thermodynamic relation linking Ψ_P $(= \Psi - \Psi_H)$ and E_P -

$$\Psi_P(\vartheta, \chi, \nabla\chi) = -(E_P^*(\vartheta, \chi, \nabla\chi))$$

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$.

We choose

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

where

- σ and λ are smooth functions accounting for the non-convex part and the latent heat in E_P, $\nu \geq 0$ an interfacial energy coefficient
- $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ is a general proper convex and lower-semicontinuous function
- $\widehat{\alpha} : \mathbb{R} \to [0, +\infty)$ is convex, increasing, and suitably regular

It corresponds - due to the standard thermodynamic relation linking Ψ_P $(= \Psi - \Psi_H)$ and E_P -

$$\Psi_P(\vartheta, \chi, \nabla\chi) = -\sup_s\{\langle\vartheta, s\rangle - E_P(s, \chi, \nabla\chi)\}, \ \vartheta = \frac{\partial E_P}{\partial s}$$

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$.

We choose

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

where

- σ and λ are smooth functions accounting for the non-convex part and the latent heat in E_P, $\nu \geq 0$ an interfacial energy coefficient
- $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ is a general proper convex and lower-semicontinuous function
- $\widehat{\alpha} : \mathbb{R} \to [0, +\infty)$ is convex, increasing, and suitably regular

It corresponds

to the following general free energy functional:

$$\Psi_P(\vartheta, \chi, \nabla\chi) = -\widehat{\alpha}^*(\vartheta) - \lambda(\chi)\vartheta + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

- $\widehat{\alpha}^* : \mathbb{R} \to \mathbb{R}$ is the convex conjugate of $\widehat{\alpha}$

The internal energy: a dual formulation

The state variables: $(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Longrightarrow (s, \widetilde{\nabla s}^t, \chi, \nabla\chi)$

The functional: $\Psi(\vartheta, \widetilde{\nabla\vartheta}^t, \chi, \nabla\chi) \Rightarrow E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$.

We choose

$$E_P(s,\chi,\nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

where

- σ and λ are smooth functions accounting for the non-convex part and the latent heat in E_P, $\nu \geq 0$ an interfacial energy coefficient
- $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ is a general proper convex and lower-semicontinuous function
- $\widehat{\alpha} : \mathbb{R} \to [0, +\infty)$ is convex, increasing, and suitably regular

It corresponds

to the standard one in case $\widehat{\alpha}^*(\vartheta) = c_v\vartheta(\log\vartheta - 1)$:

$$\Psi_P(\vartheta,\chi,\nabla\chi) = c_v\vartheta(1 - \log\vartheta) - \lambda(\chi)\vartheta + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

The History part of the internal energy

The History part of the internal energy

- We refer to the theory introduced in 1968 by [Gurtin and Pipkin] in order to model the fact that it is not reasonable to observe an immediate responce of the material to a disturbance at a distant point

The History part of the internal energy

- We refer to the theory introduced in 1968 by [Gurtin and Pipkin] in order to model the fact that it is not reasonable to observe an immediate responce of the material to a disturbance at a distant point
- In our dual formulation, we consider as state variable the summed past history of $\nabla\vartheta$ ($\vartheta = \partial E/\partial s$) up to time t:

$$\widetilde{\nabla s}^t(\tau) := \int_0^\tau \nabla[\partial E/\partial s](t-\iota)\,d\iota\,.$$

The History part of the internal energy

- We refer to the theory introduced in 1968 by [Gurtin and Pipkin] in order to model the fact that it is not reasonable to observe an immediate responce of the material to a disturbance at a distant point
- In our dual formulation, we consider as state variable the summed past history of $\nabla\vartheta$ ($\vartheta = \partial E/\partial s$) up to time t:

$$\widetilde{\nabla s}^t(\tau) := \int_0^\tau \nabla[\partial E/\partial s](t-\iota)\,d\iota\,.$$

- Following the idea of Gurtin and Pipkin we choose as History part of the internal energy:

$$\boxed{E_H(\widetilde{\nabla s}^t) := \frac{1}{2}\int_0^{+\infty} h(\tau)\widetilde{\nabla s}^t(\tau)\cdot\widetilde{\nabla s}^t(\tau)\,d\tau}$$

where

- $h:\ (0,+\infty)\to(0,+\infty)$ denotes a continuous, decreasing function such that $\displaystyle\int_0^{+\infty}\tau^2\,h(\tau)\,d\tau<\infty$.

Possible choices of $\widehat{\alpha}$'s

Possible choices of $\widehat{\alpha}$'s

Take the caloric part of the entropy $\boxed{u = s - \lambda(\chi)}$.

Possible choices of $\widehat{\alpha}$'s

Take the caloric part of the entropy $\boxed{u = s - \lambda(\chi)}$.

- If we consider the standard caloric part of the Free Energy

Possible choices of $\widehat{\alpha}$'s

Take the caloric part of the entropy $\boxed{u = s - \lambda(\chi)}$.

- If we consider the standard caloric part of the Free Energy $\widehat{\alpha}^*(\vartheta) = c_v\vartheta(\log\vartheta - 1),\ c_v$ constant [standard Ginzburg-Landau Free energy functional]

Possible choices of $\widehat{\alpha}$'s

Take the caloric part of the entropy $\boxed{u = s - \lambda(\chi)}$.

- If we consider the standard caloric part of the Free Energy $\widehat{\alpha}^*(\vartheta) = c_v \vartheta(\log\vartheta - 1),\ c_v$ constant [standard Ginzburg-Landau Free energy functional]
  $\Longrightarrow$

  $\widehat{\alpha}(u) = \exp(c\,u)$ for some $c \in \mathbb{R}$

Possible choices of $\widehat{\alpha}$'s

Take the caloric part of the entropy $\boxed{u = s - \lambda(\chi)}$.

- If we consider the standard caloric part of the Free Energy $\widehat{\alpha}^*(\vartheta) = c_v\vartheta(\log\vartheta - 1)$, c_v constant [standard Ginzburg-Landau Free energy functional]
  $\Longrightarrow$

  $\widehat{\alpha}(u) = \exp(c\, u)$ for some $c \in \mathbb{R}$

- Since, c_v in the applications may also not be constant, we can allow every form for $c_v = c_v(\vartheta)$ such that $\widehat{\alpha}^*(\vartheta)$ is convex - e.g., if $c_V(\vartheta) = \vartheta^\sigma$, for $\vartheta \in (0, \bar{\vartheta})$ with $\sigma \geq 0$ - since $c_v(\vartheta) = -\vartheta\left(\partial^2\Psi/\partial\vartheta^2\right)$, then we have $\widehat{\alpha}^*(\vartheta) = \vartheta^{\sigma+1}/\left[\sigma(\sigma+1)\right]$

Possible choices of $\widehat{\alpha}$'s

Take the caloric part of the entropy $\boxed{u = s - \lambda(\chi)}$.

- If we consider the standard caloric part of the Free Energy $\widehat{\alpha}^*(\vartheta) = c_v\vartheta(\log\vartheta - 1)$, c_v constant [standard Ginzburg-Landau Free energy functional]
  $\Longrightarrow$

  $\widehat{\alpha}(u) = \exp(c\,u)$ for some $c \in \mathbb{R}$

- Since, c_v in the applications may also not be constant, we can allow every form for $c_v = c_v(\vartheta)$ such that $\widehat{\alpha}^*(\vartheta)$ is convex - e.g., if $c_V(\vartheta) = \vartheta^\sigma$, for $\vartheta \in (0, \bar{\vartheta})$ with $\sigma \geq 0$ - since $c_v(\vartheta) = -\vartheta\left(\partial^2\Psi/\partial\vartheta^2\right)$, then we have $\widehat{\alpha}^*(\vartheta) = \vartheta^{\sigma+1}/\left[\sigma(\sigma+1)\right]$
  $\Longrightarrow$

  $\widehat{\alpha}(u) = u^{\frac{\sigma+1}{\sigma}}/(\sigma+1)$

The pseudo-potential of dissipation

The dissipative variables: $(\nabla\vartheta, \chi_t) \Longrightarrow (\nabla u, \chi_t)$ being $u = s - \lambda(\chi)$

The pseudo-potential of dissipation

The dissipative variables: $(\nabla\vartheta, \chi_t) \Longrightarrow (\nabla u, \chi_t)$ being $u = s - \lambda(\chi)$
The pseudo-potential: $\Phi(\nabla\vartheta, \chi_t) \Rightarrow p = p(\nabla u, \chi_t)$.

The pseudo-potential of dissipation

The dissipative variables: $(\nabla\vartheta, \chi_t) \Longrightarrow (\nabla u, \chi_t)$ being $u = s - \lambda(\chi)$
The pseudo-potential: $\Phi(\nabla\vartheta, \chi_t) \Rightarrow p = p(\nabla u, \chi_t)$.

We choose:

$$p(\nabla u, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{\alpha'(u)}{2}|\nabla u|^2$$

The pseudo-potential of dissipation

The dissipative variables: $(\nabla\vartheta, \chi_t) \Longrightarrow (\nabla u, \chi_t)$ being $u = s - \lambda(\chi)$
The pseudo-potential: $\Phi(\nabla\vartheta, \chi_t) \Rightarrow p = p(\nabla u, \chi_t)$.

We choose:

$$p(\nabla u, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{\alpha'(u)}{2}|\nabla u|^2$$

Justification of this choice.

- We use the relation linking Φ and p - i.e.

$$\Phi(\nabla\vartheta, \chi_t) = p^*(\nabla\vartheta, \chi_t) = \sup_{\nabla u}\{\langle \nabla u, \nabla\vartheta\rangle - p(\nabla u, \chi_t)\}$$

The pseudo-potential of dissipation

The dissipative variables: $(\nabla\vartheta, \chi_t) \Longrightarrow (\nabla u, \chi_t)$ being $u = s - \lambda(\chi)$

The pseudo-potential: $\Phi(\nabla\vartheta, \chi_t) \Rightarrow p = p(\nabla u, \chi_t)$.

We choose:

$$p(\nabla u, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{\alpha'(u)}{2}|\nabla u|^2$$

Justification of this choice.

- We use the relation linking Φ and p - i.e.

$$\Phi(\nabla\vartheta, \chi_t) = p^*(\nabla\vartheta, \chi_t) = \sup_{\nabla u}\{\langle\nabla u, \nabla\vartheta\rangle - p(\nabla u, \chi_t)\}$$

- Then we have $\nabla\vartheta = \dfrac{\partial p}{\partial(\nabla u)}$, indeed $\nabla\vartheta = \nabla\alpha(u) = \alpha'(u)\nabla u$

The pseudo-potential of dissipation

The dissipative variables: $(\nabla\vartheta, \chi_t) \Longrightarrow (\nabla u, \chi_t)$ being $u = s - \lambda(\chi)$
The pseudo-potential: $\Phi(\nabla\vartheta, \chi_t) \Rightarrow p = p(\nabla u, \chi_t)$.

We choose:

$$p(\nabla u, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{\alpha'(u)}{2}|\nabla u|^2$$

Justification of this choice.

- We use the relation linking Φ and p - i.e.

$$\Phi(\nabla\vartheta, \chi_t) = p^*(\nabla\vartheta, \chi_t) = \sup_{\nabla u}\{\langle\nabla u, \nabla\vartheta\rangle - p(\nabla u, \chi_t)\}$$

- Then we have $\nabla\vartheta = \dfrac{\partial p}{\partial(\nabla u)}$, indeed $\nabla\vartheta = \nabla\alpha(u) = \alpha'(u)\nabla u$
- Hence, the choice of p corresponds to the general choice of the pseudo-potential of dissipation

$$\Phi(\nabla\vartheta, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{1}{2\alpha'(\alpha^{-1}(\vartheta))}|\nabla\vartheta|^2$$

The pseudo-potential of dissipation

The dissipative variables: $(\nabla\vartheta, \chi_t) \Longrightarrow (\nabla u, \chi_t)$ being $u = s - \lambda(\chi)$
The pseudo-potential: $\Phi(\nabla\vartheta, \chi_t) \Rightarrow p = p(\nabla u, \chi_t)$.

We choose:

$$p(\nabla u, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{\alpha'(u)}{2}|\nabla u|^2$$

Justification of this choice.

- We use the relation linking Φ and p - i.e.

$$\Phi(\nabla\vartheta, \chi_t) = p^*(\nabla\vartheta, \chi_t) = \sup_{\nabla u}\{\langle\nabla u, \nabla\vartheta\rangle - p(\nabla u, \chi_t)\}$$

- Then we have $\nabla\vartheta = \frac{\partial p}{\partial(\nabla u)}$, indeed $\nabla\vartheta = \nabla\alpha(u) = \alpha'(u)\nabla u$
- Hence, the choice of p corresponds to the general choice of the pseudo-potential of dissipation

$$\Phi(\nabla\vartheta, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{1}{2\alpha'(\alpha^{-1}(\vartheta))}|\nabla\vartheta|^2$$

- In case $\alpha(u) = c\exp(u)$ leads to the standard case

$$\Phi(\nabla\vartheta, \chi_t) = \frac{1}{2}|\chi_t|^2 + \frac{1}{2\vartheta}|\nabla\vartheta|^2.$$

Principle of virtual power for microscopic motion

For any subdomain $D \subset \Omega$ and any virtual microscopic velocity v,

$$P_{\mathrm{int}}(D, v) + P_{\mathrm{ext}}(D, v) = \mathbf{0},$$

Principle of virtual power for microscopic motion

For any subdomain $D \subset \Omega$ and any virtual microscopic velocity v,

$$P_{\rm int}(D,v) + P_{\rm ext}(D,v) = \mathbf{0},$$

where (B and $\mathbf{H}$ new interior forces)

$$P_{\rm int}(D,v) := -\int_D (Bv + \mathbf{H}\cdot\nabla v),$$
$$P_{\rm ext}(D,v) := \int_D A\cdot v + \int_{\partial D} a\cdot v = 0.$$

Principle of virtual power for microscopic motion

For any subdomain $D \subset \Omega$ and any virtual microscopic velocity v,

$$P_{\text{int}}(D, v) + P_{\text{ext}}(D, v) = \mathbf{0},$$

where (B and $\mathbf{H}$ new interior forces)

$$P_{\text{int}}(D, v) := -\int_D (Bv + \mathbf{H} \cdot \nabla v),$$

$$P_{\text{ext}}(D, v) := \int_D A \cdot v + \int_{\partial D} a \cdot v = 0.$$

Here

- B is an energy density per units of concentration χ
- $\mathbf{H}$ is an energy flux per density

Principle of virtual power for microscopic motion

For any subdomain $D \subset \Omega$ and any virtual microscopic velocity v,

$$P_{\text{int}}(D, v) + P_{\text{ext}}(D, v) = \mathbf{0},$$

where (B and $\mathbf{H}$ new interior forces)

$$P_{\text{int}}(D, v) := -\int_D (Bv + \mathbf{H} \cdot \nabla v),$$

$$P_{\text{ext}}(D, v) := \int_D A \cdot v + \int_{\partial D} a \cdot v = 0.$$

Here

- B is an energy density per units of concentration χ
- $\mathbf{H}$ is an energy flux per density
- A and a are the volume and surface amounts of mechanical energy provided to the system by microscopic actions (e.g. electrical, chemical, or radiative external actions).

Principle of virtual power for microscopic motion

For any subdomain $D \subset \Omega$ and any virtual microscopic velocity v,

$$P_{\rm int}(D,v) + P_{\rm ext}(D,v) = \mathbf{0},$$

where (B and $\mathbf{H}$ new interior forces)

$$P_{\rm int}(D,v) := -\int_D (Bv + \mathbf{H}\cdot\nabla v),$$

$$P_{\rm ext}(D,v) := \int_D A\cdot v + \int_{\partial D} a\cdot v = 0.$$

Here

- B is an energy density per units of concentration χ
- $\mathbf{H}$ is an energy flux per density
- A and a are the volume and surface amounts of mechanical energy provided to the system by microscopic actions (e.g. electrical, chemical, or radiative external actions).

In absence of external actions, we derive an equilibrium equation in Ω

$$\boxed{B - \operatorname{div}\mathbf{H} = 0}$$

Principle of virtual power for microscopic motion

For any subdomain $D \subset \Omega$ and any virtual microscopic velocity v,

$$P_{\text{int}}(D, v) + P_{\text{ext}}(D, v) = \mathbf{0},$$

where (B and $\mathbf{H}$ new interior forces)

$$P_{\text{int}}(D, v) := -\int_D (Bv + \mathbf{H} \cdot \nabla v),$$

$$P_{\text{ext}}(D, v) := \int_D A \cdot v + \int_{\partial D} a \cdot v = 0.$$

Here

- B is an energy density per units of concentration χ
- $\mathbf{H}$ is an energy flux per density
- A and a are the volume and surface amounts of mechanical energy provided to the system by microscopic actions (e.g. electrical, chemical, or radiative external actions).

In absence of external actions, we derive an equilibrium equation in Ω

$$\boxed{B - \operatorname{div} \mathbf{H} = 0}$$

with the natural associated boundary condition on $\partial\Omega$

$$\mathbf{H} \cdot \boldsymbol{n} = 0.$$

The phase inclusion

Using the constitutive laws

$$B = \frac{\partial E_P}{\partial \chi} + \frac{\partial p}{\partial \chi_t}, \quad \mathbf{H} = \frac{\partial E_P}{\partial (\nabla \chi)}$$

The phase inclusion

Using the constitutive laws

$$B = \frac{\partial E_P}{\partial \chi} + \frac{\partial p}{\partial \chi_t}, \quad \mathbf{H} = \frac{\partial E_P}{\partial (\nabla \chi)}$$

and recalling

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

$$\Downarrow$$

The phase inclusion

Using the constitutive laws

$$B = \frac{\partial E_P}{\partial \chi} + \frac{\partial p}{\partial \chi_t}, \quad \mathbf{H} = \frac{\partial E_P}{\partial(\nabla \chi)}$$

and recalling

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

$$\Downarrow$$

$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \alpha(s - \lambda(\chi))\lambda'(\chi) \ni 0 \quad \text{in } \Omega$$

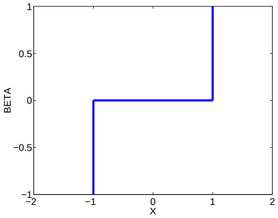
$$\text{and } \partial_{\mathbf{n}}\chi = 0 \text{ on } \partial\Omega$$

- $\alpha = \widehat{\alpha}'$ and $\beta = \partial\widehat{\beta}$.

Possible choices of the potentials $\widehat{\beta}$

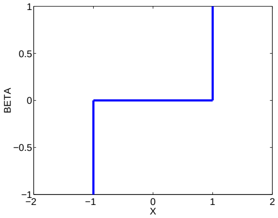
Possible choices of the potentials $\widehat{\beta}$

Subdifferential case: $\beta := \partial\widehat{\beta} = \partial I_{[-1,1]}$:

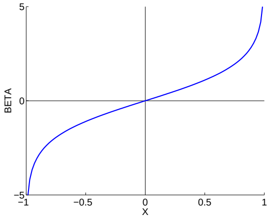


Possible choices of the potentials $\widehat{\beta}$

Subdifferential case: $\beta := \partial\widehat{\beta} = \partial I_{[-1,1]}$:



Logarithmic case: $\beta := \partial\widehat{\beta} = \log(1+\chi) - \log(1-\chi)$:



The first Principle of Thermodynamics

For any subdomain $D \subset \Omega$ and in absence of external actions, it reads

$$\frac{d}{dt}\int_{\mathcal{D}} E\, d\Omega = -\mathcal{P}_i(\mathcal{D}, \mathcal{X}_t).$$

The first Principle of Thermodynamics

For any subdomain $D \subset \Omega$ and in absence of external actions, it reads

$$\frac{d}{dt}\int_{\mathcal{D}} E \, d\Omega = -\mathcal{P}_i(\mathcal{D}, \chi_t).$$

If, in agreement with [Frémond, 2002], we take the following form for the power of internal actions

$$\mathcal{P}_i(\mathcal{D}, \chi_t) = -\int_{\mathcal{D}} (B\chi_t + \mathbf{H} \cdot \nabla\chi_t) \, d\Omega,$$

and remember that

$$B = \frac{\partial E}{\partial \chi} + \frac{\partial p}{\partial \chi_t}, \quad \mathbf{H} = \frac{\partial E}{\partial(\nabla\chi)},$$

The first Principle of Thermodynamics

For any subdomain $D \subset \Omega$ and in absence of external actions, it reads

$$\frac{d}{dt} \int_{\mathcal{D}} E \, d\Omega = -\mathcal{P}_i(\mathcal{D}, \chi_t).$$

If, in agreement with [Frémond, 2002], we take the following form for the power of internal actions

$$\mathcal{P}_i(\mathcal{D}, \chi_t) = -\int_{\mathcal{D}} (B\chi_t + \mathbf{H} \cdot \nabla\chi_t) \, d\Omega,$$

and remember that

$$B = \frac{\partial E}{\partial \chi} + \frac{\partial p}{\partial \chi_t}, \quad \mathbf{H} = \frac{\partial E}{\partial(\nabla\chi)},$$

we get exactly that there exists $\mathbf{q}$ such that

$$E_t + \operatorname{div} \mathbf{q} = \frac{\partial E}{\partial \chi}\chi_t + \frac{\partial p}{\partial \chi_t}\chi_t + \frac{\partial E}{\partial(\nabla\chi)}\nabla\chi_t \quad \text{in } \Omega.$$

The energy balance

The first principle of thermodynamics hence reads

$$E_t + \operatorname{div} \mathbf{q} = \frac{\partial E}{\partial \chi}\chi_t + \frac{\partial E}{\partial(\nabla \chi)}\nabla\chi_t + \chi_t^2 \quad \text{in } \Omega.$$

The energy balance

The first principle of thermodynamics hence reads

$$E_t + \operatorname{div} \mathbf{q} = \frac{\partial E}{\partial \chi}\chi_t + \frac{\partial E}{\partial(\nabla \chi)}\nabla \chi_t + \chi_t^2 \quad \text{in } \Omega.$$

With $E = E_P(s, \chi, \nabla\chi) + E_H(\widetilde{\nabla s}^t)$ and

$$E_P(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

$$E_H(\widetilde{\nabla s}^t) = \frac{1}{2}\int_0^{+\infty} h(\tau)\widetilde{\nabla s}^t(\tau) \cdot \widetilde{\nabla s}^t(\tau)\, d\tau,$$

and, denoting by $u = s - \lambda(\chi)$, it gives:

The energy balance

The first principle of thermodynamics hence reads

$$E_t + \operatorname{div} \mathbf{q} = \frac{\partial E}{\partial \chi}\chi_t + \frac{\partial E}{\partial(\nabla\chi)}\nabla\chi_t + \chi_t^2 \quad \text{in } \Omega.$$

With $E = E_P(s,\chi,\nabla\chi) + E_H(\widetilde{\nabla s}^t)$ and

$$E_P(s,\chi,\nabla\chi) = \widehat{\alpha}(s-\lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

$$E_H(\widetilde{\nabla s}^t) = \frac{1}{2}\int_0^{+\infty} h(\tau)\widetilde{\nabla s}^t(\tau)\cdot\widetilde{\nabla s}^t(\tau)\,d\tau,$$

and, denoting by $u = s - \lambda(\chi)$, it gives:

$$\alpha(u)\,(s_t + \operatorname{div} \mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2 \text{ in } \Omega$$

where we have chosen

- $\mathbf{Q} = \dfrac{\mathbf{q}}{\alpha(u)} = -\displaystyle\int_0^{+\infty} h(\tau)\widetilde{\nabla s}^t(\tau)\,d\tau - \nabla u$
- $\alpha(u) = \widehat{\alpha}'(u)\left(= \vartheta = \dfrac{\partial E}{\partial s}\right)$, $r^{int} = \dfrac{1}{2}\displaystyle\int_0^{+\infty} h(\tau)\frac{d}{d\tau}\left|\widetilde{\nabla s}^t(\tau)\right|^2 d\tau.$

Thermodynamical consistency

Thermodynamical consistency

Assume that in

$$\alpha(u)\,(s_t + \operatorname{div}\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

Thermodynamical consistency

Assume that in

$$\alpha(u)\,(s_t + \mathrm{div}\,\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

- $\alpha(u) = \alpha(s - \lambda(\chi)) = \widehat{\alpha}'(s - \lambda(\chi)) = \frac{\partial E}{\partial s}(= \vartheta) > 0$ (ϑ is the absolute temperature), $\alpha' > 0$ ($\widehat{\alpha}$ is convex), and

Thermodynamical consistency

Assume that in

$$\boxed{\alpha(u)\,(s_t + \operatorname{div}\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2}$$

- $\alpha(u) = \alpha(s - \lambda(\chi)) = \widehat{\alpha}'(s - \lambda(\chi)) = \frac{\partial E}{\partial s}(= \vartheta) > 0$ (ϑ is the absolute temperature), $\alpha' > 0$ ($\widehat{\alpha}$ is convex), and
- $r^{int}\left(= \frac{1}{2}\int_0^{+\infty} h(\tau)\frac{d}{d\tau}\left|\widetilde{\nabla s}^t(\tau)\right|^2 d\tau\right) \geq 0$

Thermodynamical consistency

Assume that in

$$\alpha(u)\,(s_t + \operatorname{div}\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

- $\alpha(u) = \alpha(s - \lambda(\chi)) = \widehat{\alpha}'(s - \lambda(\chi)) = \dfrac{\partial E}{\partial s} (= \vartheta) > 0$ (ϑ is the absolute temperature), $\alpha' > 0$ ($\widehat{\alpha}$ is convex), and
- $r^{int} \left(= \frac{1}{2}\int_0^{+\infty} h(\tau)\frac{d}{d\tau}\left|\widetilde{\nabla s}^t(\tau)\right|^2 d\tau\right) \geq 0$
- This can be done introducing the auxiliary kernel k such that $h = -k'$, with k, k', $k'' \in L^1(0, +\infty)$ and $\lim_{\tau\to+\infty} k(\tau) = 0$, then $r^{int} = \frac{1}{2}\int_0^{+\infty} k''(\tau)\left|\widetilde{\nabla s}^t(\tau)\right|^2 d\tau$ with $k'' \geq 0$ (being h decreasing), and hence $r^{int} \geq 0$

Thermodynamical consistency

Assume that in

$$\alpha(u)\,(s_t + \operatorname{div}\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

- $\alpha(u) = \alpha(s-\lambda(\chi)) = \widehat{\alpha}'(s-\lambda(\chi)) = \frac{\partial E}{\partial s} (= \vartheta) > 0$ (ϑ is the absolute temperature), $\alpha' > 0$ ($\widehat{\alpha}$ is convex), and
- $r^{int} \left(= \frac{1}{2} \int_0^{+\infty} h(\tau) \frac{d}{d\tau} \left| \widetilde{\nabla s}^t(\tau) \right|^2 d\tau \right) \geq 0$
- This can be done introducing the auxiliary kernel k such that $h = -k'$, with k, k', $k'' \in L^1(0,+\infty)$ and $\lim_{\tau\to+\infty} k(\tau) = 0$, then $r^{int} = \frac{1}{2}\int_0^{+\infty} k''(\tau) \left|\widetilde{\nabla s}^t(\tau)\right|^2 d\tau$ with $k'' \geq 0$ (being h decreasing), and hence $r^{int} \geq 0$

Dividing by $\alpha(u) > 0$ the internal energy balance, we get

$$s_t + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = s_t + \operatorname{div}\mathbf{Q} \geq 0,$$

that is just the pointwise Clausius-Duhem inequality

The PDE equation for u

The PDE equation for u

- Use the following energy conservation principle

$$\boxed{\alpha(u)\,(s_t + \operatorname{div}\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2}$$

where $u = s - \lambda(\chi)$

The PDE equation for u

- Use the following energy conservation principle

$$\boxed{\alpha(u)\,(s_t + \mathrm{div}\,\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2}$$

where $u = s - \lambda(\chi)$

- Divide by $\alpha(u)$

The PDE equation for u

- Use the following energy conservation principle

$$\alpha(u)\,(s_t + \operatorname{div}\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

where $u = s - \lambda(\chi)$

- Divide by $\alpha(u)$
- Use the small perturbations assumption (cf. [Germain]) - which allows to neglect the higher order dissipative contributions on the right hand side - smaller with respect to the other terms

The PDE equation for u

- Use the following energy conservation principle

$$\alpha(u)\,(s_t + \operatorname{div} \mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

  where $u = s - \lambda(\chi)$
- Divide by $\alpha(u)$
- Use the small perturbations assumption (cf. [Germain]) - which allows to neglect the higher order dissipative contributions on the right hand side - smaller with respect to the other terms
- Recall the form of the entropy flux

$$\mathbf{Q} = -\int_{-\infty}^{t} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau - \nabla u$$

The PDE equation for u

- Use the following energy conservation principle

$$\alpha(u)\,(s_t + \operatorname{div}\mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

where $u = s - \lambda(\chi)$

- Divide by $\alpha(u)$
- Use the small perturbations assumption (cf. [Germain]) - which allows to neglect the higher order dissipative contributions on the right hand side - smaller with respect to the other terms
- Recall the form of the entropy flux

$$\mathbf{Q} = -\int_{-\infty}^{t} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau - \nabla u$$

we obtain the following equation for u

The PDE equation for u

- Use the following energy conservation principle

$$\alpha(u)\,(s_t + \operatorname{div} \mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2$$

where $u = s - \lambda(\chi)$

- Divide by $\alpha(u)$
- Use the small perturbations assumption (cf. [Germain]) - which allows to neglect the higher order dissipative contributions on the right hand side - smaller with respect to the other terms
- Recall the form of the entropy flux

$$\mathbf{Q} = -\int_{-\infty}^{t} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau - \nabla u$$

we obtain the following equation for u

$$(u + \lambda(\chi))_t - \Delta u - \operatorname{div}\int_{-\infty}^{t} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau = 0.$$

The PDE system

The PDE system

We **generalize** now the system:

- let $\alpha = \partial\widehat{\alpha}$ be a general **maximal monotone graph** (maybe also multivalued)

The PDE system

We **generalize** now the system:

- let $\alpha = \partial\widehat{\alpha}$ be a general **maximal monotone graph** (maybe also multivalued)

Take the auxiliary variable $\boxed{u = s - \lambda(\chi)}$

The PDE system

We **generalize** now the system:

- let $\alpha = \partial\widehat{\alpha}$ be a general **maximal monotone graph** (maybe also multivalued)

Take the auxiliary variable $\boxed{u = s - \lambda(\chi)}$ and suppose the past history term $\operatorname{div} \int_{-\infty}^{0} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau$ to be known - put it on the right hand side.

The PDE system

We **generalize** now the system:

- let $\alpha = \partial\widehat{\alpha}$ be a general **maximal monotone graph** (maybe also multivalued)

Take the auxiliary variable $\boxed{u = s - \lambda(\chi)}$ and suppose the past history term $\operatorname{div} \int_{-\infty}^{0} k(t-\tau)\nabla\alpha(u(\tau))\, d\tau$ to be known - put it on the right hand side.

Aim: find suitably regular (u, χ) solving in a proper functional framework the following initial-boundary value problem:

The PDE system

We **generalize** now the system:

- let $\alpha = \partial\widehat{\alpha}$ be a general **maximal monotone graph** (maybe also multivalued)

Take the auxiliary variable $\boxed{u = s - \lambda(\chi)}$ and suppose the past history term $\operatorname{div} \int_{-\infty}^{0} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau$ to be known - put it on the right hand side.

Aim: find suitably regular (u, χ) solving in a proper functional framework the following initial-boundary value problem:

$$
\begin{aligned}
&(u + \lambda(\chi))_t - \Delta(u + k * \alpha(u)) \ni r \quad \text{in } \Omega \times (0,T)\\
&\partial_{\mathbf{n}}(u + k * \alpha(u)) \ni g \quad \text{on } \partial\Omega \times (0,T)\\
&\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\alpha(u) \ni 0 \quad \text{in } \Omega \times (0,T)\\
&\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega \times (0,T)\\
&u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega
\end{aligned}
$$

where $(k * \alpha(u))(t) := \int_0^t k(t-s)\alpha(u)(s)\,ds$.

The PDE system

We **generalize** now the system:

- let $\alpha = \partial\widehat{\alpha}$ be a general **maximal monotone graph** (maybe also multivalued)

Take the auxiliary variable $u = s - \lambda(\chi)$ and suppose the past history term $\operatorname{div} \int_{-\infty}^{0} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau$ to be known - put it on the right hand side.

Aim: find suitably regular (u, χ) solving in a proper functional framework the following initial-boundary value problem:

$$(u + \lambda(\chi))_t - \Delta(u + k * \alpha(u)) \ni r \quad \text{in } \Omega \times (0, T)$$

$$\partial_{\mathbf{n}}(u + k * \alpha(u)) \ni g \quad \text{on } \partial\Omega \times (0, T)$$

$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\alpha(u) \ni 0 \quad \text{in } \Omega \times (0, T)$$

$$\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega \times (0, T)$$

$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega$$

where $(k * \alpha(u))(t) := \int_0^t k(t-s)\alpha(u)(s)\,ds$. We must suppose from now on λ' constant ($= 1$ for simplicity).

Our main results

- **Existence of (weak) solutions**

Our main results

- **Existence of (weak) solutions**
  - under general assumptions on the nonlinearity α, for a graph β with domain the whole $\mathbb{R}$ and with at most a polynomial growth at ∞

Our main results

- **Existence of (weak) solutions**
  - under general assumptions on the nonlinearity α, for a graph β with domain the whole $\mathbb{R}$ and with at most a polynomial growth at ∞
  - in case $\alpha = \exp$ and for a general β

Our main results

- **Existence of (weak) solutions**
  - under general assumptions on the nonlinearity α, for a graph β with domain the whole $\mathbb{R}$ and with at most a polynomial growth at ∞
  - in case $\alpha = \exp$ and for a general β
- **Long-time behaviour of solutions: analysis of the ω-limit** in both cases

Our main results

- **Existence of (weak) solutions**
  - under general assumptions on the nonlinearity α, for a graph β with domain the whole $\mathbb{R}$ and with at most a polynomial growth at ∞
  - in case $\alpha = \exp$ and for a general β
- **Long-time behaviour of solutions: analysis of the ω-limit** in both cases
- **Uniqueness** of solutions in case α is Lipschitz-continuous and for a general β

Hypothesis 1

Hypothesis 1

- $\Omega \subset \mathbb{R}^3$ bdd connected domain with Lipschitz boundary $\Gamma := \partial\Omega$
- $t \in [0, \infty]$, $Q_t := \Omega \times (0, t)$, $\Sigma_t := \Gamma \times (0, t)$,
- $V := H^1(\Omega) \hookrightarrow H := L^2(\Omega) \equiv H' \hookrightarrow V'$ the Hilbert triplet.

Hypothesis 1

- $\Omega \subset \mathbb{R}^3$ bdd connected domain with Lipschitz boundary $\Gamma := \partial\Omega$
- $t \in [0, \infty]$, $Q_t := \Omega \times (0, t)$, $\Sigma_t := \Gamma \times (0, t)$,
- $V := H^1(\Omega) \hookrightarrow H := L^2(\Omega) \equiv H' \hookrightarrow V'$ the Hilbert triplet.

Suppose moreover that

$\widehat{\beta}, \widehat{\alpha} : \mathbb{R} \to [0, +\infty]$ are proper, convex, l.s.c. function, $\alpha := \partial\widehat{\alpha}$, $\beta := \partial\widehat{\beta}$

$|\xi| \leq c_\beta + c'_\beta \min\{|r|^{5-\eta}, |\widehat{\beta}(r)|\} \quad \forall r \in \mathbb{R}, \xi \in \beta(r)$, and for some $\eta > 0$

$\sigma \in C^2(\mathbb{R}), \quad \sigma'' \in L^\infty(\mathbb{R})$

$k \in W^{2,1}(\mathbb{R}), \quad k(0) > 0, \quad \nu > 0$

$$\langle F(t), v\rangle = \int_\Omega r(\cdot, t)v + \int_\Gamma g(\cdot, v)v_{|_\Gamma}, \quad v \in V, \quad F \in W^{1,1}(0, T; V')$$

$u_0 \in H, \quad \widehat{\alpha}(u_0) \in L^1(\Omega), \quad \chi_0 \in V$

being c_β and c'_β two positive constants depending only on β.

Maximal monotone operators in V'-V

Associate to $\widehat{\alpha}$ the functionals $\widehat{\alpha}_H$ and $\widehat{\alpha}_V$ (defined on H and V respectively)

$$\widehat{\alpha}_H(v) = \int_\Omega \widehat{\alpha}(v(x))dx \quad \text{if} \quad v \in H \quad \text{and} \quad \widehat{\alpha}(v) \in L^1(\Omega),$$
$$\widehat{\alpha}_H(v) = +\infty \quad \text{if} \quad v \in H \quad \text{and} \quad \widehat{\alpha}(v) \notin L^1(\Omega),$$
$$\widehat{\alpha}_V(v) = \widehat{\alpha}_H(v) \quad \text{if} \quad v \in V.$$

Maximal monotone operators in V'-V

Associate to $\widehat{\alpha}$ the functionals $\widehat{\alpha}_H$ and $\widehat{\alpha}_V$ (defined on H and V respectively)

$$\widehat{\alpha}_H(v) = \int_\Omega \widehat{\alpha}(v(x))dx \quad \text{if} \quad v \in H \quad \text{and} \quad \widehat{\alpha}(v) \in L^1(\Omega),$$
$$\widehat{\alpha}_H(v) = +\infty \quad \text{if} \quad v \in H \quad \text{and} \quad \widehat{\alpha}(v) \notin L^1(\Omega),$$
$$\widehat{\alpha}_V(v) = \widehat{\alpha}_H(v) \quad \text{if} \quad v \in V.$$

Their subdifferentials (cf. [Barbu, '76])

$$\mathcal{A} := \partial_{V,V'}\widehat{\alpha}_V : V \to 2^{V'}$$

and (cf. [Brezis, '73])

$$\partial_H\widehat{\alpha}_H : H \to 2^H$$

are maximal monotone operators.

Existence result for a general α

Existence result for a general α

PROBLEM 1. Find $(u, \chi, \vartheta, \xi)$ with the regularity properties

$$u \in H^1(0,T;V') \cap L^2(0,T;V), \; \vartheta \in L^2(0,T;V')$$
$$k * \vartheta \in L^2(0,T;V) \cap C^0([0,T];H),$$
$$\chi \in H^1(0,T;H) \cap L^\infty(0,T;V), \; \xi \in L^\infty(0,T;L^{6/(5-\eta)}(\Omega)),$$

and satisfying

$$\partial_t(u+\chi) + Au + A(k*\vartheta) = F \quad \text{in } V', \quad \text{a.e. in } (0,T), \tag{1}$$
$$\partial_t\chi + \nu A(\chi) + \xi + \sigma'(\chi) - \vartheta = 0 \quad \text{in } V', \quad \text{a.e. in } (0,T), \tag{2}$$
$$\vartheta \in \mathcal{A}(u) \text{ in } V' \text{ a.e. in } (0,T), \quad \xi \in \beta(\chi) \text{ a.e. in } Q_T,$$
$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega.$$

Existence result for a general α

PROBLEM 1. Find $(u, \chi, \vartheta, \xi)$ with the regularity properties

$$u \in H^1(0,T;V') \cap L^2(0,T;V), \ \vartheta \in L^2(0,T;V')$$
$$k * \vartheta \in L^2(0,T;V) \cap C^0([0,T];H),$$
$$\chi \in H^1(0,T;H) \cap L^\infty(0,T;V), \ \xi \in L^\infty(0,T;L^{6/(5-\eta)}(\Omega)),$$

and satisfying

$$\partial_t(u+\chi) + Au + A(k*\vartheta) = F \quad \text{in } V', \quad \text{a.e. in } (0,T), \qquad (1)$$
$$\partial_t\chi + \nu A(\chi) + \xi + \sigma'(\chi) - \vartheta = 0 \quad \text{in } V', \quad \text{a.e. in } (0,T), \qquad (2)$$
$$\vartheta \in \mathcal{A}(u) \text{ in } V' \text{ a.e. in } (0,T), \quad \xi \in \beta(\chi) \text{ a.e. in } Q_T,$$
$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega.$$

THEOREM 1. Let T be a positive final time and let HYPOTHESIS 1 be satisfied, then PROBLEM 1 has at least a solution $(u, \chi, \vartheta, \xi)$ on the time interval $(0,T)$.

Choices of α

Choices of α

- $\boxed{\alpha(u) = \exp(u) (= \vartheta)}$: we recover the model proposed by [Bonetti, Colli, Frémond, '03]

$$(u + \chi)_t - \Delta(u + k * \exp(u)) = r$$
$$\chi_t - \nu \Delta \chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0.$$

Choices of α

- $\boxed{\alpha(u) = \exp(u)(= \vartheta)}$: we recover the model proposed by [Bonetti, Colli, Frémond, '03]

$$(u + \chi)_t - \Delta(u + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0.$$

With the choice $\mathbf{q} = -\kappa\nabla(\alpha^2(u))$ we recover the model proposed by [Bonetti, Colli, Fabrizio, Gilardi, '06]

$$(u + \chi)_t - \Delta(\exp(u) + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0$$

Choices of α

- $\boxed{\alpha(u) = \exp(u)(= \vartheta)}$: we recover the model proposed by [Bonetti, Colli, Frémond, '03]

$$(u + \chi)_t - \Delta(u + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0.$$

With the choice $\mathbf{q} = -\kappa\nabla(\alpha^2(u))$ we recover the model proposed by [Bonetti, Colli, Fabrizio, Gilardi, '06]

$$(u + \chi)_t - \Delta(\exp(u) + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0$$

- $\boxed{\alpha(u) = -1/u}$: we recover, e.g., the Penrose-Fife system

Choices of α

- $\boxed{\alpha(u) = \exp(u)(= \vartheta)}$: we recover the model proposed by [Bonetti, Colli, Frémond, '03]

$$(u + \chi)_t - \Delta(u + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0.$$

With the choice $\mathbf{q} = -\kappa\nabla(\alpha^2(u))$ we recover the model proposed by [Bonetti, Colli, Fabrizio, Gilardi, '06]

$$(u + \chi)_t - \Delta(\exp(u) + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0$$

- $\boxed{\alpha(u) = -1/u}$: we recover, e.g., the Penrose-Fife system

Choices of β

Choices of α

- $\boxed{\alpha(u) = \exp(u) (= \vartheta)}$: we recover the model proposed by [Bonetti, Colli, Frémond, '03]

$$(u + \chi)_t - \Delta(u + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0.$$

With the choice $\mathbf{q} = -\kappa\nabla(\alpha^2(u))$ we recover the model proposed by [Bonetti, Colli, Fabrizio, Gilardi, '06]

$$(u + \chi)_t - \Delta(\exp(u) + k * \exp(u)) = r$$
$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \exp(u) \ni 0$$

- $\boxed{\alpha(u) = -1/u}$: we recover, e.g., the Penrose-Fife system

Choices of β

- The growth condition of $\widehat{\beta}(r)$ at most like a power $(6 - \eta)$ $(\eta > 0)$ as $|r| \nearrow \infty$ is needed only in the 3D (in space) case
- This condition exclude the choice $\widehat{\beta} = I_{[0,1]}$, but it includes the smooth double-well potential $\beta(\chi) + \sigma'(\chi) \sim \chi^3 - \chi$, e.g.

The case α Lipschitz continuous

PROPSITION 1. Suppose, beside HYPOTHESIS 1 that $D(\widehat{\alpha}) \equiv \mathbb{R}$.

The case α Lipschitz continuous

PROPSITION 1. Suppose, beside HYPOTHESIS 1 that $D(\widehat{\alpha}) \equiv \mathbb{R}$.Then $\vartheta(t) \in L^1(\Omega)$ for all $t \in [0, T]$ and the relation $\vartheta \in \mathcal{A}(u)$ can be rewritten as

$$\vartheta \in \alpha(u) \quad \text{a.e. in } Q_T.$$

The case α Lipschitz continuous

PROPSITION 1. Suppose, beside HYPOTHESIS 1 that $D(\widehat{\alpha}) \equiv \mathbb{R}$.Then $\vartheta(t) \in L^1(\Omega)$ for all $t \in [0, T]$ and the relation $\vartheta \in \mathcal{A}(u)$ can be rewritten as

$$\vartheta \in \alpha(u) \quad \text{a.e. in } Q_T .$$

PROPOSITION 2. Let us prescribe, in addition to HYPOTHESIS 1, the following hypothesis

$$\alpha \text{ is a Lipschitz continuous function on } \mathbb{R} .$$

The case α Lipschitz continuous

PROPSITION 1. Suppose, beside HYPOTHESIS 1 that $D(\widehat{\alpha}) \equiv \mathbb{R}$.Then $\vartheta(t) \in L^1(\Omega)$ for all $t \in [0, T]$ and the relation $\vartheta \in \mathcal{A}(u)$ can be rewritten as

$$\vartheta \in \alpha(u) \quad \text{a.e. in } Q_T.$$

PROPOSITION 2. Let us prescribe, in addition to HYPOTHESIS 1, the following hypothesis

$$\alpha \text{ is a Lipschitz continuous function on } \mathbb{R}\ .$$

Then, PROBLEM 1 (where relation $\vartheta \in \mathcal{A}(u)$ in V' can be rewritten as $\vartheta \in \alpha(u)$ a.e. in Q_T, due to PROPOSITION 1) admits a unique solution with the further regularity:

$$u,\, \vartheta \in H^1(0, T; V') \cap L^2(0, T; V) \cap C^0([0, T]; H).$$

The case α Lipschitz continuous

PROPSITION 1. Suppose, beside HYPOTHESIS 1 that $D(\widehat{\alpha}) \equiv \mathbb{R}$.Then $\vartheta(t) \in L^1(\Omega)$ for all $t \in [0, T]$ and the relation $\vartheta \in \mathcal{A}(u)$ can be rewritten as

$$\vartheta \in \alpha(u) \quad \text{a.e. in } Q_T .$$

PROPOSITION 2. Let us prescribe, in addition to HYPOTHESIS 1, the following hypothesis

$$\alpha \text{ is a Lipschitz continuous function on } \mathbb{R} .$$

Then, PROBLEM 1 (where relation $\vartheta \in \mathcal{A}(u)$ in V' can be rewritten as $\vartheta \in \alpha(u)$ a.e. in Q_T, due to PROPOSITION 1) admits a unique solution with the further regularity:

$$u, \vartheta \in H^1(0, T; V') \cap L^2(0, T; V) \cap C^0([0, T]; H).$$

Note: PROPOSITION 2 still holds true under the following weaker assumptions on the data

$$k \in W^{1,1}(0, T),\ k(0) \geq 0,\ k \equiv 0 \text{ if } k(0) = 0,$$

$$\beta \text{ proper, convex, l.s.c., without growth conditions},$$

$$F \in L^2(0, T; V'), \quad \nu \geq 0 .$$

Remark in case $\alpha = \exp$

Remark in case $\alpha = \exp$

REMARK 1. In case $\alpha = \exp$ (cf. [Bonetti, Colli, Frémond, 2003]) one can prove the existence of solutions on $[0, T]$ under the following more general assumptions on β

$$\nu \geq 0 \text{ if } D(\beta) \text{ is bounded and } \nu > 0 \text{ if } D(\beta) \text{ is unbounded.}$$

Remark in case $\alpha = \exp$

REMARK 1. In case $\alpha = \exp$ (cf. [Bonetti, Colli, Frémond, 2003]) one can prove the existence of solutions on $[0, T]$ under the following more general assumptions on β

$$\nu \geq 0 \text{ if } D(\beta) \text{ is bounded and } \nu > 0 \text{ if } D(\beta) \text{ is unbounded.}$$

Moreover, in this case, due to PROPOSITION 1, the relation $\vartheta \in \alpha(u)$ holds true a.e. and the solution has the following regularity

$$u \in H^1(0,T;V') \cap L^2(0,T;V), \quad \vartheta \in L^{5/3}(Q_T),$$
$$\chi \in H^1(0,T;H), \quad \nu\chi \in L^\infty(0,T;V) \cap L^{5/3}(0,T;W^{2,5/3}(\Omega)),$$
$$\xi \in L^{5/3}(Q_T), \quad k(0)(1 * \vartheta) \in L^\infty(0,T;V).$$

Remark in case $\alpha = \exp$

REMARK 1. In case $\alpha = \exp$ (cf. [Bonetti, Colli, Frémond, 2003]) one can prove the existence of solutions on $[0, T]$ under the following more general assumptions on β

$\nu \geq 0$ if $D(\beta)$ is bounded and $\nu > 0$ if $D(\beta)$ is unbounded.

Moreover, in this case, due to PROPOSITION 1, the relation $\vartheta \in \alpha(u)$ holds true a.e. and the solution has the following regularity

$$u \in H^1(0,T;V') \cap L^2(0,T;V), \quad \vartheta \in L^{5/3}(Q_T),$$
$$\chi \in H^1(0,T;H), \quad \nu\chi \in L^\infty(0,T;V) \cap L^{5/3}(0,T;W^{2,5/3}(\Omega)),$$
$$\xi \in L^{5/3}(Q_T), \quad k(0)(1 * \vartheta) \in L^\infty(0,T;V).$$

REMARK 2. In [Bonetti, E.R., '07] the long-time behaviour of solutions has been studied for this system.

THE CASE ν, $k = 0$ has been studied in [Bonetti, Frémond, 2003] and in [Bonetti, in "Dissipative phase transitions" (ed. P. Colli, N. Kenmochi, J. Sprekels), 2006]

Main difficulty: lack of regularity in the ϑ-component

The case α "general".

Main difficulty: lack of regularity in the ϑ-component

The case α “general”. Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

Main difficulty: lack of regularity in the ϑ-component

The case α "general". Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

(i) Test "formally" (1) $\times\, \vartheta$ + (2) $\times\, \chi_t$ ("$\vartheta \in \alpha(u)$") :

Main difficulty: lack of regularity in the ϑ-component

The case α "general". Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

(i) Test "formally" $(1) \times \vartheta + (2) \times \chi_t$ ("$\vartheta \in \alpha(u)$"):

- handle $\int_0^t \langle F, \vartheta - (\vartheta)_\Omega \rangle$ by means of $k(0)\, |\nabla(1 * \vartheta)(t)|_H^2$ on the l.h.s. and by Poincaré inequality

Main difficulty: lack of regularity in the ϑ-component

The case α "general". Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

(i) Test "formally" $(1) \times \vartheta + (2) \times \chi_t$ ("$\vartheta \in \alpha(u)$"):

- handle $\int_0^t \langle F, \vartheta - (\vartheta)_\Omega \rangle$ by means of $k(0)\,|\nabla(1 * \vartheta)(t)|_H^2$ on the l.h.s. and by Poincaré inequality
- estimate $\int_0^t \langle F, (\vartheta)_\Omega \rangle$ testing $(2) \times 1$ and using $|\beta(\chi)| \leq c_\beta + c_{\beta'}|\widehat{\beta}(\chi)|$ (cf. Hypothesis 1)

Main difficulty: lack of regularity in the ϑ-component

The case α "general". Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

(i) Test "formally" (1) $\times\, \vartheta$ + (2) $\times\, \chi_t$ ("$\vartheta \in \alpha(u)$"):

- handle $\int_0^t \langle F, \vartheta - (\vartheta)_\Omega \rangle$ by means of $k(0)\, |\nabla(1 * \vartheta)(t)|_H^2$ on the l.h.s. and by Poincaré inequality
- estimate $\int_0^t \langle F, (\vartheta)_\Omega \rangle$ testing (2) $\times$ 1 and using $|\beta(\chi)| \leq c_\beta + c_{\beta'} |\widehat{\beta}(\chi)|$ (cf. Hypothesis 1)

This gives $|\nabla(1 * \vartheta)|_{L^\infty(0,T;H)}$, $|\chi|_{H^1(0,T;H)\cap L^\infty(0,T;V)}$, $|\widehat{\beta}(\chi)|_{L^\infty(0,T;L^1(\Omega))} \leq c$.

Main difficulty: lack of regularity in the ϑ-component

The case α "general". Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

(i) Test "formally" (1) $\times\ \vartheta$ + (2) $\times\ \chi_t$ ("$\vartheta \in \alpha(u)$"):

- handle $\int_0^t \langle F, \vartheta - (\vartheta)_\Omega \rangle$ by means of $k(0)\,|\nabla(1 * \vartheta)(t)|_H^2$ on the l.h.s. and by Poincaré inequality
- estimate $\int_0^t \langle F, (\vartheta)_\Omega \rangle$ testing (2) $\times$ 1 and using $|\beta(\chi)| \leq c_\beta + c_{\beta'}|\widehat{\beta}(\chi)|$ (cf. Hypothesis 1)

This gives $|\nabla(1 * \vartheta)|_{L^\infty(0,T;H)}$, $|\chi|_{H^1(0,T;H)\cap L^\infty(0,T;V)}$, $|\widehat{\beta}(\chi)|_{L^\infty(0,T;L^1(\Omega))} \leq c$.

(ii) Estimate $\beta(\chi)$ in $L^\infty(0,T;L^{4/3}(\Omega))$ by using $|\beta(s)| \leq c_\beta + c'_\beta|s|^p$, $p < 5$ (cf. Hypothesis 1), the Sobolev embedding in 3D domains, and the boundedness of χ in $L^\infty(0,T;V)$

Main difficulty: lack of regularity in the ϑ-component

The case α "general". Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

(i) Test "formally" (1) $\times\, \vartheta$ + (2) $\times\, \chi_t$ ("$\vartheta \in \alpha(u)''$) :

- handle $\int_0^t \langle F, \vartheta - (\vartheta)_\Omega \rangle$ by means of $k(0)\,|\nabla(1 * \vartheta)(t)|_H^2$ on the l.h.s. and by Poincaré inequality
- estimate $\int_0^t \langle F, (\vartheta)_\Omega \rangle$ testing (2) $\times$ 1 and using $|\beta(\chi)| \le c_\beta + c_{\beta'} |\widehat{\beta}(\chi)|$ (cf. Hypothesis 1)

This gives $|\nabla(1 * \vartheta)|_{L^\infty(0,T;H)}$, $|\chi|_{H^1(0,T;H)\cap L^\infty(0,T;V)}$, $|\widehat{\beta}(\chi)|_{L^\infty(0,T;L^1(\Omega))} \le c$.

(ii) Estimate $\beta(\chi)$ in $L^\infty(0,T;L^{4/3}(\Omega))$ by using $|\beta(s)| \le c_\beta + c'_\beta |s|^p$, $p < 5$ (cf. Hypothesis 1), the Sobolev embedding in $3D$ domains, and the boundedness of χ in $L^\infty(0,T;V)$

(iii) Then (1) $\times\, u$ gives $|u|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c$ and, by comparison, $|\partial_t u|_{L^2(0,T;V')} \le c$

Main difficulty: lack of regularity in the ϑ-component

The case α "general". Approximate the problem:

regularizing α and $\beta \Longrightarrow \alpha_\varepsilon$ and β_ε (their Yosida approximation).

(i) Test "formally" (1) $\times\, \vartheta$ + (2) $\times\, \chi_t$ ("$\vartheta \in \alpha(u)''$):

- handle $\int_0^t \langle F, \vartheta - (\vartheta)_\Omega \rangle$ by means of $k(0)\,|\nabla(1*\vartheta)(t)|_H^2$ on the l.h.s. and by Poincaré inequality
- estimate $\int_0^t \langle F, (\vartheta)_\Omega \rangle$ testing (2) $\times$ 1 and using $|\beta(\chi)| \le c_\beta + c_{\beta'}|\widehat{\beta}(\chi)|$ (cf. Hypothesis 1)

This gives $|\nabla(1*\vartheta)|_{L^\infty(0,T;H)}$, $|\chi|_{H^1(0,T;H)\cap L^\infty(0,T;V)}$, $|\widehat{\beta}(\chi)|_{L^\infty(0,T;L^1(\Omega))} \le c$.

(ii) Estimate $\beta(\chi)$ in $L^\infty(0,T;L^{4/3}(\Omega))$ by using $|\beta(s)| \le c_\beta + c'_\beta|s|^p$, $p<5$ (cf. Hypothesis 1), the Sobolev embedding in $3D$ domains, and the boundedness of χ in $L^\infty(0,T;V)$

(iii) Then (1) $\times\, u$ gives $|u|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c$ and, by comparison, $|\partial_t u|_{L^2(0,T;V')} \le c$

(iv) Pass to the limit proving a strong convergence of $\nabla\chi_\varepsilon$ in H by a Cauchy argument and identify β and α by means of monotonicity arguments

Main difficulty: lack of regularity in the ϑ-component

The case $\alpha(u) = \exp(u)$.

Main difficulty: lack of regularity in the ϑ-component

The case $\alpha(u) = \exp(u)$. The system can be rewritten as

$$\partial_t(\log\vartheta + \chi) + A(\log\vartheta) + A(k * \vartheta) = F \quad \text{a.e. in } Q_T, \tag{1L}$$

$$\partial_t\chi + \nu A(\chi) + \beta(\chi) + \sigma'(\chi) - \vartheta \ni 0 \quad \text{a.e. in } Q_T. \tag{2L}$$

Main difficulty: lack of regularity in the ϑ-component

The case $\alpha(u) = \exp(u)$. The system can be rewritten as

$$\partial_t(\log\vartheta + \chi) + A(\log\vartheta) + A(k * \vartheta) = F \quad \text{a.e. in } Q_T, \tag{1L}$$

$$\partial_t\chi + \nu A(\chi) + \beta(\chi) + \sigma'(\chi) - \vartheta \ni 0 \quad \text{a.e. in } Q_T. \tag{2L}$$

The main issue here is to prove sufficient regularity in ϑ to pass to the limit in the "approximation scheme".

Main difficulty: lack of regularity in the ϑ-component

<u>**The case $\alpha(u) = \exp(u)$.**</u> The system can be rewritten as

$$\partial_t(\log\vartheta + \chi) + A(\log\vartheta) + A(k * \vartheta) = F \quad \text{a.e. in } Q_T, \qquad (1L)$$

$$\partial_t\chi + \nu A(\chi) + \beta(\chi) + \sigma'(\chi) - \vartheta \ni 0 \quad \text{a.e. in } Q_T. \qquad (2L)$$

The main issue here is to prove sufficient regularity in ϑ to pass to the limit in the "approximation scheme".

(i) Testing "formally" $(1L) \times \vartheta + (2L) \times \chi_t$ we get

$$\vartheta^{1/2} \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

Main difficulty: lack of regularity in the ϑ-component

The case $\alpha(u) = \exp(u)$. The system can be rewritten as

$$\partial_t(\log\vartheta + \chi) + A(\log\vartheta) + A(k * \vartheta) = F \quad \text{a.e. in } Q_T\,, \qquad (1L)$$

$$\partial_t\chi + \nu A(\chi) + \beta(\chi) + \sigma'(\chi) - \vartheta \ni 0 \quad \text{a.e. in } Q_T\,. \qquad (2L)$$

The main issue here is to prove sufficient regularity in ϑ to pass to the limit in the "approximation scheme".

(i) Testing "formally" $(1L) \times \vartheta + (2L) \times \chi_t$ we get

$$\vartheta^{1/2} \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

(ii) Using the 3D Gagliardo Nieremberg inequality

$$\|\vartheta^{1/2}\|_{L^{10/3}(\Omega)} \le C\|\vartheta^{1/2}\|_V^{3/5}\|\vartheta^{1/2}\|_H^{2/5} \quad \forall v \in V\,.$$

we get $\vartheta \in L^{5/3}(Q_T)$ (more than L^1) which is sufficient to pass to the limit

Main difficulty: lack of regularity in the ϑ-component

The case $\alpha(u) = \exp(u)$. The system can be rewritten as

$$\partial_t(\log\vartheta + \chi) + A(\log\vartheta) + A(k * \vartheta) = F \quad \text{a.e. in } Q_T, \qquad (1L)$$

$$\partial_t\chi + \nu A(\chi) + \beta(\chi) + \sigma'(\chi) - \vartheta \ni 0 \quad \text{a.e. in } Q_T. \qquad (2L)$$

The main issue here is to prove sufficient regularity in ϑ to pass to the limit in the "approximation scheme".

(i) Testing "formally" $(1L) \times \vartheta + (2L) \times \chi_t$ we get

$$\vartheta^{1/2} \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

(ii) Using the 3D Gagliardo Nieremberg inequality

$$\|\vartheta^{1/2}\|_{L^{10/3}(\Omega)} \le C\|\vartheta^{1/2}\|_V^{3/5}\|\vartheta^{1/2}\|_H^{2/5} \quad \forall v \in V.$$

we get $\vartheta \in L^{5/3}(Q_T)$ (more than L^1) which is sufficient to pass to the limit but not to get uniqueness of solutions, which follows instead authomatically in case α is Lipschitz continuous on $\mathbb{R}$

Main difficulty: lack of regularity in the ϑ-component

The case $\alpha(u) = \exp(u)$. The system can be rewritten as

$$\partial_t(\log\vartheta + \chi) + A(\log\vartheta) + A(k * \vartheta) = F \quad \text{a.e. in } Q_T, \qquad (1L)$$
$$\partial_t\chi + \nu A(\chi) + \beta(\chi) + \sigma'(\chi) - \vartheta \ni 0 \quad \text{a.e. in } Q_T. \qquad (2L)$$

The main issue here is to prove sufficient regularity in ϑ to pass to the limit in the "approximation scheme".

(i) Testing "formally" $(1L) \times \vartheta + (2L) \times \chi_t$ we get

$$\vartheta^{1/2} \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

(ii) Using the 3D Gagliardo Nieremberg inequality

$$\|\vartheta^{1/2}\|_{L^{10/3}(\Omega)} \le C\|\vartheta^{1/2}\|_V^{3/5}\|\vartheta^{1/2}\|_H^{2/5} \quad \forall v \in V.$$

we get $\vartheta \in L^{5/3}(Q_T)$ (more than L^1) which is sufficient to pass to the limit but not to get uniqueness of solutions, which follows instead authomatically in case α is Lipschitz continuous on $\mathbb{R}$

(iii) Note that in this case we do not need to estimate ϑ from the second equation, hence we do not have restrictions on the growth of β

The long-time behaviour of solutions for general α

THEOREM 2. Let HYPOTHESIS 1 hold and suppose that

(i) $k \in W^{1,1}(0,\infty)$ is of strongly positive type, i.e. $\exists\,\eta > 0$ such that

$$\widetilde{k}(t) := k(t) - \eta \exp(-t) \quad \text{is of positive type}$$

(ii) r, h sufficiently regular, $\lim_{|r|\to+\infty} |r|^{-2}\widehat{\beta}(r) = +\infty$.

The long-time behaviour of solutions for general α

THEOREM 2. Let HYPOTHESIS 1 hold and suppose that

(i) $k \in W^{1,1}(0,\infty)$ is of strongly positive type, i.e. $\exists\, \eta > 0$ such that

$$\widetilde{k}(t) := k(t) - \eta \exp(-t) \quad \text{is of positive type}$$

(ii) r, h sufficiently regular, $\lim_{|r|\to+\infty} |r|^{-2}\widehat{\beta}(r) = +\infty$.

Then, the ω-limit of a single trajectory (u,χ):

$$\omega(u,\chi) := \{(u_\infty,\chi_\infty) \in H \times V : \text{ there exists } t_n \nearrow \infty, \\ (u(t_n),\chi(t_n)) \to (u_\infty,\chi_\infty) \text{ in } V' \times H\}$$

is a nonempty, compact, and connected subset of $V' \times H$.

The long-time behaviour of solutions for general α

THEOREM 2. Let HYPOTHESIS 1 hold and suppose that

(i) $k \in W^{1,1}(0,\infty)$ is of strongly positive type, i.e. $\exists\, \eta > 0$ such that

$$\widetilde{k}(t) := k(t) - \eta \exp(-t) \quad \text{is of positive type}$$

(ii) r, h sufficiently regular, $\lim_{|r|\to+\infty} |r|^{-2}\widehat{\beta}(r) = +\infty$.

Then, the ω-limit of a single trajectory (u, χ):

$$\omega(u,\chi) := \{(u_\infty, \chi_\infty) \in H \times V : \text{ there exists } t_n \nearrow \infty,\ (u(t_n), \chi(t_n)) \to (u_\infty, \chi_\infty) \text{ in } V' \times H\}$$

is a nonempty, compact, and connected subset of $V' \times H$.
$\forall\, (u_\infty, \chi_\infty) \in \omega(u,\chi)\ \exists\, \vartheta_\infty \in V'$, $\xi_\infty \in L^{6/(5-\eta)}(\Omega)$
s.t. $(u_\infty, \chi_\infty, \xi_\infty, \vartheta_\infty)$ solves

The long-time behaviour of solutions for general α

THEOREM 2. Let HYPOTHESIS 1 hold and suppose that

(i) $k \in W^{1,1}(0,\infty)$ is of strongly positive type, i.e. $\exists\,\eta > 0$ such that

$$\widetilde{k}(t) := k(t) - \eta \exp(-t) \quad \text{is of positive type}$$

(ii) r, h sufficiently regular, $\lim_{|r|\to+\infty} |r|^{-2}\widehat{\beta}(r) = +\infty$.

Then, the ω-limit of a single trajectory (u,χ):

$$\omega(u,\chi) := \{(u_\infty,\chi_\infty) \in H \times V : \text{ there exists } t_n \nearrow \infty, \\ (u(t_n),\chi(t_n)) \to (u_\infty,\chi_\infty) \text{ in } V' \times H\}$$

is a nonempty, compact, and connected subset of $V' \times H$.
$\forall\,(u_\infty,\chi_\infty) \in \omega(u,\chi)\ \exists\,\vartheta_\infty \in V'$, $\xi_\infty \in L^{6/(5-\eta)}(\Omega)$
s.t. $(u_\infty,\chi_\infty,\xi_\infty,\vartheta_\infty)$ solves

$$u_\infty = \frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right) \quad \text{a.e. in } \Omega,$$

$$\nu A\chi_\infty + \xi_\infty + \sigma'(\chi_\infty) - \vartheta_\infty = 0 \quad \text{in } V',$$

$$\xi_\infty \in \beta(\chi_\infty) \quad \text{a.e. in } \Omega, \qquad \vartheta_\infty \in \mathcal{A}\left(\frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right)\right) \quad \text{in } V',$$

where $c_0 := \int_\Omega u_0 + \int_\Omega \chi_0$, $m := \int_0^\infty \left(\int_\Omega r(s) + \int_\Gamma h(s)\right) ds$.

Convergence of the whole trajectories for special nonlinearities

PROPSITION 3. Under the assumptions of THEOREM 2, letting $\sigma'(\chi) \equiv \vartheta_c$ and supposing that

α is not multivalued,

we can conclude in addition to THEOREM 2 that χ_∞ is constant a.e. in Ω.

Convergence of the whole trajectories for special nonlinearities

PROPSITION 3. Under the assumptions of THEOREM 2, letting $\sigma'(\chi) \equiv \vartheta_c$ and supposing that

$$\alpha \text{ is not multivalued,}$$

we can conclude in addition to THEOREM 2 that χ_∞ is constant a.e. in Ω. Moreover, if we assume in addition that

$$\beta + \widetilde{\alpha} \text{ is injective, where } \widetilde{\alpha}(\cdot) := -\alpha\left(-(\cdot) + \frac{1}{|\Omega|}(c_0 + m)\right),$$

then the couple $(u_\infty, \chi_\infty) \in \omega(u, \chi)$ is uniquely determined as the solution of the following system

$$u_\infty = -\chi_\infty + \frac{1}{|\Omega|}(c_0 + m),$$

$$\beta(\chi_\infty) - \alpha\left(-\chi_\infty + \frac{1}{|\Omega|}(c_0 + m)\right) \ni -\vartheta_c \quad \text{a.e. in } \Omega,$$

being c_0 and m defined as before.

Convergence of the whole trajectories for special nonlinearities

PROPSITION 3. Under the assumptions of THEOREM 2, letting $\sigma'(\chi) \equiv \vartheta_c$ and supposing that

$$\alpha \text{ is not multivalued,}$$

we can conclude in addition to THEOREM 2 that χ_∞ is constant a.e. in Ω. Moreover, if we assume in addition that

$$\beta + \widetilde{\alpha} \text{ is injective, where } \widetilde{\alpha}(\cdot) := -\alpha\left(-(\cdot) + \frac{1}{|\Omega|}(c_0 + m)\right),$$

then the couple $(u_\infty, \chi_\infty) \in \omega(u, \chi)$ is uniquely determined as the solution of the following system

$$u_\infty = -\chi_\infty + \frac{1}{|\Omega|}(c_0 + m),$$

$$\beta(\chi_\infty) - \alpha\left(-\chi_\infty + \frac{1}{|\Omega|}(c_0 + m)\right) \ni -\vartheta_c \quad \text{a.e. in } \Omega,$$

being c_0 and m defined as before. In particular, the whole trajectory $(u(t), \chi(t))$ tends to (u_∞, χ_∞) weakly in $H \times V$ and strongly in $V' \times H$ as $t \nearrow \infty$.

Regarding well-posedness in finite time intervals

Regarding well-posedness in finite time intervals

- To prove **uniqueness** in case of a general α (not Lipschitz-continuous). **Problem:** the doubly-nonlinear character of the system.

Regarding well-posedness in finite time intervals

- To prove **uniqueness** in case of a general α (not Lipschitz-continuous). **Problem:** the doubly-nonlinear character of the system.
- To study the case of **two general multivalued operators** α (as in our case) and β in the phase equation (e.g. $\beta = \partial I_{[0,1]}$).

Regarding well-posedness in finite time intervals

- To prove **uniqueness** in case of a general α (not Lipschitz-continuous). **Problem:** the doubly-nonlinear character of the system.
- To study the case of **two general multivalued operators** α (as in our case) and β in the phase equation (e.g. $\beta = \partial I_{[0,1]}$).
- To study the general inclusion

$$\alpha(u)\,(u_t + \ell\chi_t) + \operatorname{div}\mathbf{q} \ni \chi_t^2$$

without the small perturbations assumption for a **suitable nonlinear function** α and **suitable choices of the heat flux q** and of the phase dynamics (cf. [Feireisl, 2005] for an entropy inequality appoach and a different notion of solution)

Regarding the long-time behaviour

Regarding the long-time behaviour

– To study the **convergence of the whole trajectories** in case the anti-monotone part σ' is present in the phase equation:

Regarding the long-time behaviour

– To study the **convergence of the whole trajectories** in case the anti-monotone part σ' is present in the phase equation: no uniqueness of the stationary states is expected

$$-\nu\Delta\chi_\infty + \beta(\chi_\infty) + \sigma'(\chi_\infty) \ni \exp(u_\infty)$$

by employing the **Lojasiewicz technique** in case of **analytical potentials** (cf., e.g., [Aizicovici, Feireisl, 2001] $\hookrightarrow$ Caginalp model with memory; [Feireisl, Schimperna, 2005] $\hookrightarrow$ Penrose-Fife systems; [Grasselli, Petzeltova, Schimperna, 2007], etc.).

Regarding the long-time behaviour

– To study the **convergence of the whole trajectories** in case the anti-monotone part σ' is present in the phase equation: no uniqueness of the stationary states is expected

$$-\nu\Delta\chi_\infty + \beta(\chi_\infty) + \sigma'(\chi_\infty) \ni \exp(u_\infty)$$

by employing the **Lojasiewicz technique** in case of **analytical potentials** (cf., e.g., [Aizicovici, Feireisl, 2001] $\hookrightarrow$ Caginalp model with memory; [Feireisl, Schimperna, 2005] $\hookrightarrow$ Penrose-Fife systems; [Grasselli, Petzeltova, Schimperna, 2007], etc.). Or use **other techniques**, cf. [Krejčí, Zheng, 2005] $\hookrightarrow$ phase-relaxation systems with **non-smooth potentials**

Regarding the long-time behaviour

- To study the **convergence of the whole trajectories** in case the anti-monotone part σ' is present in the phase equation: no uniqueness of the stationary states is expected

$$-\nu\Delta\chi_\infty + \beta(\chi_\infty) + \sigma'(\chi_\infty) \ni \exp(u_\infty)$$

  by employing the **Lojasiewicz technique** in case of **analytical potentials** (cf., e.g., [Aizicovici, Feireisl, 2001] $\hookrightarrow$ Caginalp model with memory; [Feireisl, Schimperna, 2005] $\hookrightarrow$ Penrose-Fife systems; [Grasselli, Petzeltova, Schimperna, 2007], etc.). Or use **other techniques**, cf. [Krejčí, Zheng, 2005] $\hookrightarrow$ phase-relaxation systems with **non-smooth potentials**
- To study the existence of the **attractors**: in case α Lipschitz continuous $\hookrightarrow$ uniqueness of solutions and in case $\alpha = \exp$ or more general α's $\hookrightarrow$ no uniqueness, cf. **the theories of J. Ball, Vishik, etc.**

Regarding the long-time behaviour

- To study the **convergence of the whole trajectories** in case the anti-monotone part σ' is present in the phase equation: no uniqueness of the stationary states is expected

  $$-\nu\Delta\chi_\infty + \beta(\chi_\infty) + \sigma'(\chi_\infty) \ni \exp(u_\infty)$$

  by employing the **Lojasiewicz technique** in case of **analytical potentials** (cf., e.g., [Aizicovici, Feireisl, 2001] $\hookrightarrow$ Caginalp model with memory; [Feireisl, Schimperna, 2005] $\hookrightarrow$ Penrose-Fife systems; [Grasselli, Petzeltova, Schimperna, 2007], etc.). Or use **other techniques**, cf. [Krejčí, Zheng, 2005] $\hookrightarrow$ phase-relaxation systems with **non-smooth potentials**
- To study the existence of the **attractors**: in case α Lipschitz continuous $\hookrightarrow$ uniqueness of solutions and in case $\alpha = \exp$ or more general α's $\hookrightarrow$ no uniqueness, cf. **the theories of J. Ball, Vishik, etc.**
- The problem both for recovering uniqueness of solutions and existence of the attractor is the **lack of regularity of the ϑ-component**