QUASI-OPTIMALITY OF THE SUPG METHOD FOR THE ONE-DIMENSIONAL ADVECTION-DIFFUSION PROBLEM

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Abstract. In this paper we propose quasi-optimal error estimates, in various norms, for the *Streamline-Upwind Petrov-Galerkin* (SUPG) method applied to the linear one-dimensional advection-diffusion problem. We follow the classical argument due to Babuška and Brezzi, therefore the goal of this work is the proof of the inf-sup and of the continuity conditions for the bilinear stabilized variational form, with respect to suitable norms. These norms are suggested by our previous work [13], in which we analyze the continuous multi-dimensional advection-diffusion operator. We obtain these results by means of function space interpolation.

Key words. convection-diffusion, finite element method, inf-sup condition.

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1. Introduction. In this work we shall consider the one-dimensional advection-diffusion operator:

$$\mathcal{L}_\varepsilon w := -\varepsilon w'' + w', \tag{1.1}$$

and, given a source term f, the related Dirichlet homogeneous boundary value problem:

$$\begin{cases} \mathcal{L}_\varepsilon u = f & \text{in } (0,1) \\ u(0) = 0 \\ u(1) = 0. \end{cases} \tag{1.2}$$

We assume the diffusion parameter ε to be positive; when it is small, i.e., in the advection-dominated regime, (1.2) represents one of the simplest examples of singularly perturbed boundary value differential problems. We think of it as a prototype of more general problems, where a skew-symmetric operator (represented by the first-order derivative) is perturbed by a symmetric operator of higher order (the second-order derivative in the example).

The associated variational problem reads

$$\begin{cases} \text{find } u \in H^1_0 \equiv H^1_0(0,1) \text{ such that} \\ a_\varepsilon(u,v) = {}_{H^{-1}}\langle f, v\rangle_{H^1_0} \qquad \forall v \in H^1_0, \end{cases} \tag{1.3}$$

where $a_\varepsilon(w,v) := \varepsilon \int_0^1 w'(x)v'(x)\,dx + \int_0^1 w'(x)v(x)\,dx$ and f is assumed to be in H^{-1}. This problem fits into the Lax-Milgram framework, but its solution, when ε is small, depends on the source term f in a very sensitive way with respect to the usual norms on H^1_0 and H^{-1}; actually

$$\|\mathcal{L}_\varepsilon^{-1}\|_{L(H^{-1},H^1_0)} := \sup_{w\in H^1_0} \frac{\|w\|_{H^1_0}}{\|\mathcal{L}_\varepsilon w\|_{H^{-1}}} \approx \varepsilon^{-1}.$$

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Nevertheless problem (1.2) is well posed for any $\varepsilon > 0$, and indeed in [14] and [13] we defined suitable norms $\|\cdot\|_W$ and $\|\cdot\|_V$ such that both the continuity and the inf-sup conditions

$$a_\varepsilon(w,v) \le \kappa \|w\|_W \|v\|_V, \quad \forall w \in H_0^1, \forall v \in H_0^1, \tag{1.4}$$

$$\inf_{w\in H_0^1} \sup_{v\in H_0^1} \frac{a_\varepsilon(w,v)}{\|w\|_W \|v\|_V} \ge \gamma > 0, \tag{1.5}$$

hold true with κ and γ independent of ε. The results in this paper are based on the approach proposed in [13]: actually the analysis of [13] is more general, since it deals with the multi-dimensional operator, where the advection term has an anisotropic structure. In §2 we shall specialize the results of [13] to the simpler one-dimensional problem (1.2).

It is well known that the standard Galerkin numerical method, when applied to (1.2), is unstable (e.g., see [11]). The most popular methods for (1.2) are actually the *Streamline-Upwind Petrov-Galerkin* (SUPG) method and its variants—e.g., the *Galerkin Least Squares* (GaLS) method—introduced by Hughes and coworkers (see [7] and [9]) in the eighties. In this work we shall consider the one-dimensional version of SUPG, whose detailed presentation is postponed to §2. We only recall now that even though these methods are quite satisfactory for practical situations, their error analysis does not fit into the classical theory due to Babuška and Brezzi (see [1] and [3] and (2.16)–(2.18) in the sequel); as a result it is usually very hard to prove that these methods are *quasi-optimal*, namely to show that the their numerical solution u_h is close to the exact solution u as the best fit of u in the trial space W_h (up to a multiplicative constant C independent of ε and with respect to a suitable norm $\|\cdot\|$):

$$\|u - u_h\| \le C \inf_{w_h \in W_h} \|u - w_h\|.$$

More recent numerical methods for the advection diffusion problem are, among others, the *Residual-free Bubbles FEM* (proposed in [6] and analyzed in [4], [5], [8] and [12]) and the method with *negative-order stabilization* (see [2]). Those methods are closely related to SUPG—in some cases they lead to the same numerical algorithm—but they actually improve the theoretical understanding of this subject; for our purposes here, we only note here that those recent analyses are, roughly speaking, *close* to the ideal Babuška-Brezzi framework, and the methods are proved to be *close* to exhibiting the quasi-optimal behavior.

In this work we actually prove a family of quasi-optimal error estimates for the SUPG method for solving (1.2). We apply the general theory stated in §2, by showing that the method verifies the continuity and inf-sup conditions (2.16)–(2.17) with respect to suitable norms, by means of function space interpolation tools. In particular, we show that the method is quasi-optimal (see (3.37)) with respect to a norm whose part independent of ε is of differentiability-order 1/2, which is in accordance with [2], [5] and [13].

We are restricting this analysis to the one-dimensional problem because we are not able, at the present time, to deal with the anisotropic structure of the convection term from the numerical point of view; on the other hand, we will not exploit other special properties of the one-dimensional operator. We refer to §4 for a discussion on further extensions of our approach.

The outline of the paper is as follows. In §2 we present the notation and assumptions, we recall the results of [13] and specialize and extend them to the one-dimensional case. In §3 we develop the error analysis for the SUPG method in the present setting. Finally, in §4, we give some comments about the results proved and outline possible extensions.

2. Preliminaries. We denote by $L^2 \equiv L^2(0,1)$ the usual Lebesgue space endowed with the norm $\|\cdot\|_{L^2}$, by $L^2_0 \equiv L^2_0(0,1)$ its subset containing zero mean-value functions and by $\Pi_0 : L^2 \to L^2_0$ the L^2-projection onto L^2_0; we also denote by $\overline{w}$ the mean value of a generic function $w \in L^2$, so that $w = \Pi_0 w + \overline{w}$. Moreover $H^1 \equiv H^1(0,1)$ is the usual Sobolev space endowed with the norm $\|\cdot\|_{H^1}$ and seminorm $|\cdot|_{H^1}$; $H^1_{\#}$ denotes its subspace of functions w such that $w(0) = w(1)$, while H^1_0 denotes the subspace of functions vanishing at 0 and 1, endowed with the norm $|\cdot|_{H^1}$; finally $H^{-1} \equiv H^{-1}(0,1) := (H^1_0)^*$ denotes the dual space of H^1_0 endowed with the dual norm $\|\cdot\|_{H^{-1}}$ and the usual pairing $\langle\cdot,\cdot\rangle \equiv {}_{H^{-1}}\langle\cdot,\cdot\rangle_{H^1_0}$; the dual (norm, space, ...) is always denoted by the superscripted star. We shall make use of the interpolation theory of function spaces; more specifically, we shall use the *K-method* and we refer to [15] for its definition, notation and properties.

In the sequel C denotes a generic constant whose value, possibly different at various occurrences, does not depend on any other mathematical quantity appearing in the analysis (e.g., ε, θ, p, h, u, w, f, ϕ). We also adopt the notational convention

$$\begin{aligned} \alpha \preceq \beta \quad &\Longleftrightarrow \quad \alpha \le C\beta, \\ \alpha \simeq \beta \quad &\Longleftrightarrow \quad \alpha \preceq \beta \text{ and } \beta \preceq \alpha. \end{aligned}$$

We now revise the analysis proposed in [13] and specialize it to the one-dimensional case. Following [13] we define

$$\begin{aligned} \|w\|_{A_0} &:= \varepsilon |w|_{H^1} + \|\Pi_0 w\|_{L^2} \qquad && \forall w \in A_0 := H^1_0, \\ \|w\|_{A_1} &:= |w|_{H^1} && \forall w \in A_1 := H^1_0; \end{aligned} \tag{2.1}$$

where we have used $\|\Pi_0 w\|_{L^2}$ instead of the equivalent norm $\|w'\|_{H^{-1}}$. Therefore one has the equivalence between $\|w\|_{A_0}$ and $\|\mathcal{L}_\varepsilon w\|_{A_1^*}$, i.e.,

$$\varepsilon |w|_{H^1} + \|\Pi_0 w\|_{L^2} \simeq \sup_{v \in H^1_0} \frac{a_\varepsilon(w,v)}{|v|_{H^1}} \quad \forall w \in H^1_0; \tag{2.2}$$

one half of (2.2)—the continuity of $\mathcal{L}_\varepsilon$—is obvious, while the other half actually follows from the coercivity $\varepsilon |w|^2_{H^1} \preceq a_\varepsilon(w,w)$. By means of a duality argument we obtain from (2.2) the other estimate

$$|w|_{H^1} \simeq \sup_{v \in H^1_0} \frac{a_\varepsilon(w,v)}{\varepsilon |v|_{H^1} + \|\Pi_0 v\|_{L^2}} \quad \forall w \in H^1_0, \tag{2.3}$$

i.e., $\|w\|_{A_1}$ and $\|\mathcal{L}_\varepsilon w\|_{A_0^*}$ are equivalent.

Both (2.2) and (2.3) state that $\mathcal{L}_\varepsilon$ is an isomorphism uniformly with respect to ε; the dependence on ε of the operator has been included into the norms themselves.

We briefly comment on (2.3): it allows control of the H^1_0 norm of the solution u of (1.2) in terms of the source term f, despite the presence of the boundary layer near $x = 1$. It is due to the structure of $\|\cdot\|_{A_0^*}$: when $f \in L^2_0$, then $\|f\|_{A_0^*} \le \|f\|_{L^2}$ and actually there is no boundary layer; otherwise, if $f \in L^2$ has a non-zero mean value, then $\|f\|_{A_0^*}$ behaves asymptotically as $\varepsilon^{-1/2}$ for $\varepsilon \to 0$, and accordingly, the presence of a thin layer on the related solution u makes $|u|_{H^1}$ behave in the same way.

We can infer from (2.2)–(2.3) a family of intermediate estimates: given θ and p with $0 < \theta < 1$, $1 \le p \le +\infty$ and denoting by p' the conjugate of p, i.e., $1/p + 1/p' = 1$, by means of the interpolation theory we have (see [13])

$$\varepsilon^{1-\theta}|w|_{H^1} + \|w'\|_{(H^{-1},L^2_0)_{\theta,p}} \simeq \sup_{v\in H^1_0} \frac{a_\varepsilon(w,v)}{\varepsilon^{\theta}|v|_{H^1} + \|v'\|_{(H^{-1},L^2_0)_{1-\theta,p'}}} \quad \forall w \in H^1_0, \tag{2.4}$$

where we have also made use of the equivalence

$$\|w\|_{(A_0,A_1)_{\theta,p}} \simeq \varepsilon^{1-\theta}|w|_{H^1} + \|w'\|_{(H^{-1},L^2_0)_{\theta,p}}, \tag{2.5}$$

that is the one-dimensional counterpart of [13, Proposition 2]. Condition (2.4) brings our model problem into the framework (1.4)–(1.5).

We also proved the Poincaré-like estimate

$$\|w\|_{L^2} \preceq \|w'\|_{(H^{-1},L^2_0)_{1/2,1}} \qquad \forall w \in H^1_0. \tag{2.6}$$

and, as a consequence,

$$\|w\|_{L^2} \preceq \|w\|_{(A_0,A_1)_{\theta,p}}\ \forall w \in H^1_0 \quad \Leftrightarrow \quad \theta > 1/2 \text{ or } (\theta,p) = (1/2,1), \tag{2.7}$$

$$\|\phi\|_{(A_0,A_1)^*_{\theta,p}} \preceq \|\phi\|_{L^2}\ \forall \phi \in H^{-1} \quad \Leftrightarrow \quad \theta > 1/2 \text{ or } (\theta,p) = (1/2,1). \tag{2.8}$$

Finally, we show the relation between the fractional order norm appearing in the equivalence (2.4) and a more usual Besov norm. We focus our attention on the case $\theta = 1/2$, which will be of special interest in the sequel; the case of a generic θ is similar but more technical (in order to obtain the optimal dependence on θ).

PROPOSITION 2.1. *For $1 \le p \le +\infty$, we have*

$$(L^2, H^1_\#)_{1/2,p} = \{w \in L^2 | w' \in (H^{-1}, L^2_0)_{1/2,p}\},$$

and also

$$\|w\|_{(L^2,H^1_\#)_{1/2,p}} \simeq \|w\|_{L^2} + \|w'\|_{(H^{-1},L^2_0)_{1/2,p}}, \tag{2.9}$$

for any $w \in (L^2, H^1_\#)_{1/2,p}$.

Proof. Let w be a generic function in L^2 and $p \neq +\infty$; we have, by definition and by the triangle inequality,

$$\begin{aligned}
\|w\|_{(L^2,H^1_\#)_{1/2,p}} &\le \left[\int_0^{+\infty} \left(t^{-1/2}\|w_0(t)\|_{L^2} + t^{1/2}\|w_1(t)\|_{H^1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&\le \left[\int_0^{1} \left(t^{-1/2}\|w_0(t)\|_{L^2} + t^{1/2}\|w_1(t)\|_{H^1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&\quad + \left[\int_1^{+\infty} \left(t^{-1/2}\|w_0(t)\|_{L^2} + t^{1/2}\|w_1(t)\|_{H^1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&= I + II
\end{aligned}$$

for any $w_0(t)$ and $w_1(t)$ with $w = w_0(t) + w_1(t)$, $w_0(t) \in L^2$, $w_1(t) \in H^1_\#$ and $0 < t < +\infty$; I and II actually depend on $w_0(t)$ and $w_1(t)$. Moreover we assume, for $0 < t < 1$, that $w_0(t) \in L^2_0$, yielding $\|w_0(t)\|_{L^2} = \|w_0'(t)\|_{H^{-1}}$. Recalling the notation

$w_1(t) = \overline{w_1(t)} + \Pi_0 w_1(t)$, where $\overline{w_1(t)} = \overline{w}$, we use the continuity of the mean value in L^2 for $\overline{w}$ and the Bramble-Hilbert lemma for $\Pi_0 w_1(t)$ to get

$$\begin{aligned}\|w_1(t)\|_{H^1} &\leq \|\overline{w_1(t)}\|_{H^1} + \|\Pi_0 w_1(t)\|_{H^1}\\ &\preceq \|\overline{w_1(t)}\|_{L^2} + |\Pi_0 w_1(t)|_{H^1}\\ &\preceq \|w\|_{L^2} + \|w_1'(t)\|_{L^2}.\end{aligned}$$

Therefore we have

$$\begin{aligned} I &\leq \left[\int_0^1 \left(t^{-1/2}\|w_0'(t)\|_{H^{-1}} + t^{1/2}\|w_1'(t)\|_{L^2}\right)^p \frac{dt}{t}\right]^{1/p}\\ &\quad + \left[\int_0^1 \left(t^{1/2}\|w\|_{L^2}\right)^p \frac{dt}{t}\right]^{1/p} \\ &\preceq \left[\int_0^1 \left(t^{-1/2}\|w_0'(t)\|_{H^{-1}} + t^{1/2}\|w_1'(t)\|_{L^2}\right)^p \frac{dt}{t}\right]^{1/p} + \|w\|_{L^2}.\end{aligned} \tag{2.10}$$

Now the key point is that, for any decomposition $w' = \phi_0(t) + \phi_1(t)$, $\phi_0(t) \in H^{-1}$, $\phi_1(t) \in L_0^2$ on $0 < t < 1$, we can define $w_0(t)$ and $w_1(t)$ that satisfy $w_0'(t) = \phi_0(t)$, $w_1'(t) = \phi_1(t)$ and all the conditions given above: simply take, for any $0 < t < 1$, $w_1(t)$ as the primitive of $\phi_1(t)$ with $\overline{w_1(t)} = \overline{w}$, and $w_0(t) = w - w_1(t)$. In particular $\phi_1(t) \in L_0^2$ yields $w_1(t) \in H^1_\#$. Therefore we can rewrite (2.10) in terms of $\phi_0(t)$ and $\phi_1(t)$,

$$I \preceq \left[\int_0^1 \left(t^{-1/2}\|\phi_0(t)\|_{H^{-1}} + t^{1/2}\|\phi_1(t)\|_{L^2}\right)^p \frac{dt}{t}\right]^{1/p} + \|w\|_{L^2}, \tag{2.11}$$

and take the infimum with respect to $\phi_0(t)$ and $\phi_1(t)$, obtaining

$$I \preceq \|w'\|_{(H^{-1},L_0^2)_{1/2,p}} + \|w\|_{L^2}. \tag{2.12}$$

Otherwise, taking $w_0(t) = w$ and $w_1(t) = 0$ for $1 < t < +\infty$ we have

$$\begin{aligned} II &\leq \left[\int_1^{+\infty} \left(t^{-1/2}\|w\|_{L^2}\right)^p \frac{dt}{t}\right]^{1/p}\\ &\leq \frac{2}{p}\|w\|_{L^2}.\end{aligned} \tag{2.13}$$

Therefore (2.12) and (2.13) yield $\|w\|_{(L^2,H^1_\#)_{1/2,p}} \preceq \|w\|_{L^2} + \|w'\|_{(H^{-1},L_0^2)_{1/2,p}}$, and the inclusion $\{w \in L^2 | w' \in (H^{-1}, L_0^2)_{1/2,p}\} \subset (L^2, H^1_\#)_{1/2,p}$. With obvious modification one could deal with the case $p = +\infty$.

The remaining part, i.e., the inclusion

$$(L^2, H^1_\#)_{1/2,p} \subset \{w \in L^2 | w' \in (H^{-1}, L_0^2)_{1/2,p}\}$$

and the related estimate, is given by the interpolation theorem [15, §1.3.3 (a)], since the derivative operator is both continuous from $H^1_\#$ into L_0^2, and from L^2 into H^{-1}. ∎

We recall that $(L^2, H^1_\#)_{1/2,p}$ is the space of functions whose periodic extension, say, on $(-1, 2)$, belongs to $B^{1/2}_{2,p}(-1, 2)$.

It is useful for the sequel to notice that (2.6) and (2.9) give a variant of (2.5), as stated in the following corollary.

COROLLARY 2.2. *We have*

$$\|w\|_{(A_0,A_1)_{1/2,1}} \simeq \varepsilon^{1/2}|w|_{H^1} + \|w\|_{(L^2,H^1_\#)_{1/2,1}} \quad \forall w \in H^1_0. \tag{2.14}$$

Now we turn our attention to the numerical solution of (1.2). We briefly recall the general error estimation theory, due to Babuška and Brezzi (see [10, Proposition 5.5.1]). Let u be the solution of (1.2); the finite element formulation reads

$$\begin{cases} \text{find } u_h \in W_h \text{ such that} \\ a_{\varepsilon,h}(u_h, v_h) = \langle f_h, v_h\rangle \qquad \forall v_h \in V_h, \end{cases} \tag{2.15}$$

where the spaces W_h and V_h are finite-dimensional subsets of H^1_0, while $a_{\varepsilon,h}$ and f_h give a consistent discretization of the continuous problem, namely $a_{\varepsilon,h}(u, v_h) = \langle f_h, v_h\rangle, \forall v_h \in V_h$. If there exist constants $\widetilde{\kappa} < +\infty$ and $\widetilde{\gamma} > 0$, independent of ε and h, such that

$$a_{\varepsilon,h}(u - w_h, v_h) \leq \widetilde{\kappa}\|u - w_h\|_{W,h}\|v_h\|_{V_h} \quad \forall w_h \in W_h, \forall v_h \in V_h, \tag{2.16}$$

and

$$\inf_{w_h\in W_h}\sup_{v_h\in V_h} \frac{a_{\varepsilon,h}(w_h, v_h)}{\|w_h\|_{W,h}\|v_h\|_{V_h}} \geq \widetilde{\gamma}, \tag{2.17}$$

then the method is *quasi-optimal*:

$$\|u - u_h\|_{W,h} \leq (\widetilde{\kappa}\widetilde{\gamma}^{-1} + 1)\inf_{w_h\in W_h}\|u - w_h\|_{W,h}. \tag{2.18}$$

The notation $\|\cdot\|_{W,h}$ refers to a norm on the space W (we postpone the definition of W to (3.20)), which can depend on the discretization. Conditions (2.16)–(2.17) are the analogs of (1.4)–(1.5), and this motivates our interest in (2.4). In order to verify them, we shall look for norms $\|\cdot\|_{W,h}$ and $\|\cdot\|_{V_h}$ that are the discrete counterparts of the ones in (2.4).

We shall consider the very simple case of a uniform subdivision of $(0, 1)$ into N open elements T_i of size $h = N^{-1}$:

$$T_i \equiv T_{i,h} := \{x : (i-1)h < x < ih\} \qquad \forall i = 1, 2, \ldots, N, \tag{2.19}$$

and the corresponding space of continuous piecewise linear elements:

$$W_h \equiv V_h := \left\{ \begin{matrix} v \in H^1_0 : v|_{T_i} \text{ is affine} \\ \forall i = 1, \ldots, N \end{matrix} \right\}; \tag{2.20}$$

the SUPG method, proposed by Hughes and coworkers in [7], adds a *weighted residual stabilization* to the continuous variational problem:

$$\begin{aligned} a_{\varepsilon,h}(w, v_h) &:= a_\varepsilon(w, v_h) + \sum_{i=1}^N \tau \int_{T_i} (\mathcal{L}_\varepsilon w)(x)\, v_h'(x)\, dx \\ \langle f_h, v_h\rangle &:= \langle f, v_h\rangle + \sum_{i=1}^N \tau \int_{T_i} f(x)\, v_h'(x)\, dx; \end{aligned} \tag{2.21}$$

this definition requires both f and w to be regular in the interior of the elements, which is not restrictive for applications. The amount of *streamline*[1] *diffusion* τ, is a parameter of the method and its value is a relevant point. It could depend on ε and h; here we assume that the problem is advection-dominated, namely $\varepsilon \leq h$, and therefore a usual assumption is that

$$\tau \simeq h. \tag{2.22}$$

We shall return to a more detailed discussion of the optimal value of τ in §4.

3. Main results. This section is devoted to the error analysis of the SUPG method in the framework (2.16)– (2.18).

First, we analyze the inf-sup condition (2.17).

LEMMA 3.1. *The SUPG method* (2.19)–(2.22) *satisfies the estimates:*

$$\varepsilon |w_h|_{H^1} + \|\Pi_0 w_h\|_{L^2} \preceq \sup_{v_h \in W_h} \frac{a_{\varepsilon,h}(w_h, v_h)}{|v_h|_{H^1}} \quad \forall w_h \in W_h, \tag{3.1}$$

$$|w_h|_{H^1} \preceq \sup_{v_h \in W_h} \frac{a_{\varepsilon,h}(w_h, v_h)}{\varepsilon |v_h|_{H^1} + \|\Pi_0 v_h\|_{L^2}} \quad \forall w_h \in W_h. \tag{3.2}$$

Proof. First, recall that any $w_h \in W_h$ is piecewise linear, therefore the higher order term in (2.21) vanishes:

$$a_{\varepsilon,h}(w_h, v_h) = (\varepsilon + \tau) \int_0^1 w_h'(x) v_h'(x)\, dx + \int_0^1 w_h'(x) v_h(x)\, dx \quad \forall v_h \in V_h.$$

Thanks to the coercivity of $a_{\varepsilon,h}$:

$$\varepsilon |w_h|_{H^1}^2 \leq a_{\varepsilon,h}(w_h, w_h), \tag{3.3}$$

we have immediately

$$\varepsilon |w_h|_{H^1} \preceq \sup_{v_h \in W_h} \frac{a_{\varepsilon,h}(w_h, v_h)}{|v_h|_{H^1}}. \tag{3.4}$$

We have therefore to prove that

$$\|\Pi_0 w_h\|_{L^2} \preceq \sup_{v_h \in W_h} \frac{a_{\varepsilon,h}(w_h, v_h)}{|v_h|_{H^1}}. \tag{3.5}$$

In order to do this, we shall define $\widetilde{w}_h \in W_h$, depending on w_h, such that

$$a_{\varepsilon,h}(w_h, \widetilde{w}_h) = \|\Pi_0 w_h\|_{L^2}^2, \tag{3.6}$$
$$|\widetilde{w}_h|_{H^1} \preceq \|\Pi_0 w_h\|_{L^2}; \tag{3.7}$$

such a $\widetilde{w}_h$ is the solution of the discrete variational problem

$$a_{\varepsilon,h}(v_h, \widetilde{w}_h) = \int_0^1 (\Pi_0 w_h)(x) v_h(x)\, dx \quad \forall v_h \in W_h. \tag{3.8}$$

[1] In the one-dimensional case there are no *streamline directions*; we are just following the general terminology.

To verify (3.7), we define $\widetilde{w} \in H_0^1$ as the solution of

$$-(\varepsilon+\tau)\widetilde{w}'' - \widetilde{w}' = \Pi_0 w_h, \tag{3.9}$$

such that $a_{\varepsilon,h}(v_h, \widetilde{w}_h - \widetilde{w}) = 0$ for all $v_h \in W_h$; moreover (3.9) has the same structure as (1.2), with $\varepsilon + \tau \simeq h$ instead of ε, so that we can make use of the analogue of (2.3):

$$\begin{aligned} |\widetilde{w}|_{H^1} &\preceq \sup_{v \in H_0^1} \frac{(\varepsilon+\tau)\int_0^1 \widetilde{w}'(x)v'(x)\,dx - \int_0^1 \widetilde{w}'(x)v(x)\,dx}{(\varepsilon+\tau)|v|_{H^1} + \|\Pi_0 v\|_{L^2}} \\ &= \sup_{v \in H_0^1} \frac{\int_0^1 \Pi_0 w_h(x)v(x)\,dx}{(\varepsilon+\tau)|v|_{H^1} + \|\Pi_0 v\|_{L^2}} \\ &\le \sup_{v \in H_0^1} \frac{\int_0^1 \Pi_0 w_h(x)v(x)\,dx}{\|\Pi_0 v\|_{L^2}} \\ &= \sup_{v \in H_0^1} \frac{\int_0^1 \Pi_0 w_h(x)\Pi_0 v(x)\,dx}{\|\Pi_0 v\|_{L^2}} \\ &= \|\Pi_0 w_h\|_{L^2}. \end{aligned} \tag{3.10}$$

We also need to introduce the nodal interpolant $\widetilde{w}_I \in W_h$ of $\widetilde{w}$, which satisfies

$$|\widetilde{w} - \widetilde{w}_I|_{H^1} + h^{-1}\|\widetilde{w} - \widetilde{w}_I\|_{L^2} \preceq |\widetilde{w}|_{H^1}. \tag{3.11}$$

Therefore we have, by using the coercivity of $a_{\varepsilon,h}$, (3.8), (3.9) and (3.11),

$$\begin{aligned} |\widetilde{w} - \widetilde{w}_h|_{H^1}^2 &= (\varepsilon+\tau)^{-1} a_{\varepsilon,h}(\widetilde{w} - \widetilde{w}_h, \widetilde{w} - \widetilde{w}_h) \\ &= (\varepsilon+\tau)^{-1} a_{\varepsilon,h}(\widetilde{w} - \widetilde{w}_I, \widetilde{w} - \widetilde{w}_h) \\ &\le |\widetilde{w} - \widetilde{w}_h|_{H^1}\left[|\widetilde{w} - \widetilde{w}_I|_{H^1} + (\varepsilon+\tau)^{-1}\|\widetilde{w} - \widetilde{w}_I\|_{L^2}\right] \\ &\preceq |\widetilde{w} - \widetilde{w}_h|_{H^1}|\widetilde{w}|_{H^1}, \end{aligned}$$

that gives $|\widetilde{w} - \widetilde{w}_h|_{H^1} \preceq |\widetilde{w}|_{H^1}$, and then, by using (3.10),

$$\begin{aligned} |\widetilde{w}_h|_{H^1} &\le |\widetilde{w}|_{H^1} + |\widetilde{w} - \widetilde{w}_h|_{H^1} \\ &\preceq |\widetilde{w}|_{H^1} \\ &\preceq \|\Pi_0 w_h\|_{L^2}, \end{aligned}$$

which gives (3.7); (3.6) follows immediately from (3.8) and (3.1) is satisfied.

We obtain (3.2) from (3.1) by a duality argument: we now associate to a generic $w_h \in W_h$ the function $\widetilde{w}_h \in W_h$ which satisfies

$$a_{\varepsilon,h}(v_h, \widetilde{w}_h) = \int_0^1 w_h'(x)v_h'(x)\,dx \quad \forall v_h \in W_h. \tag{3.12}$$

The left-hand side of (3.12) is the dual of $a_{\varepsilon,h}(\widetilde{w}_h, v_h)$: we can proceed as before to obtain the analog of (3.1); in particular,

$$\begin{aligned} \varepsilon|\widetilde{w}_h|_{H^1} + \|\Pi_0\widetilde{w}_h\|_{L^2} &\preceq \sup_{v_h \in W_h} \frac{a_{\varepsilon,h}(v_h, \widetilde{w}_h)}{|v_h|_{H^1}} \\ &= \sup_{v_h \in W_h} \frac{\int_0^1 w_h'(x)v_h'(x)\,dx}{|v_h|_{H^1}} \\ &= |w_h|_{H^1}; \end{aligned}$$

then

$$|w_h|_{H^1} = \frac{\int_0^1 w_h'(x)w_h'(x)\,dx}{|w_h|_{H^1}} = \frac{a_{\varepsilon,h}(w_h,\widetilde{w}_h)}{|w_h|_{H^1}} \preceq \frac{a_{\varepsilon,h}(w_h,\widetilde{w}_h)}{\varepsilon|\widetilde{w}_h|_{H^1} + \|\Pi_0\widetilde{w}_h\|_{L^2}}$$

which gives (3.2). □

The estimates (3.1)–(3.2) state inf-sup conditions with respect to the same norms as in (2.2)–(2.3). Focusing on (3.1), for example, we see that one could replace $\varepsilon|w_h|_{H^1}$, on the left-hand side, with $(\varepsilon+\tau)|w_h|_{H^1}$, as it seems natural from (3.3)–(3.4). Actually this leads to an equivalent estimate at the discrete level, because of the inverse inequality $(\varepsilon+\tau)|w_h|_{H^1} \preceq \|\Pi_0 w_h\|_{L^2}$. On the other hand, in order to obtain in the sequel a meaningful error estimate (2.18), our aim here is to make use of the "natural" norms (for the continuous problem).

Let us now define the discrete counterpart of (2.1):

$$\begin{aligned} \|w_h\|_{A_{0,h}} &:= \varepsilon|w_h|_{H^1} + \|\Pi_0 w_h\|_{L^2} \qquad && \forall w_h \in A_{0,h} := W_h, \\ \|w_h\|_{A_{1,h}} &:= |w_h|_{H^1} && \forall w_h \in A_{1,h} := W_h. \end{aligned} \tag{3.13}$$

We construct, from (3.1)–(3.2), a family of intermediate inf-sup conditions by means of function space interpolation.

PROPOSITION 3.2. *Let $0<\theta<1$, $1\le p\le +\infty$ The 1-D SUPG method* (2.19)–(2.22) *satisfies the estimates:*

$$\|w_h\|_{(A_{0,h},A_{1,h})_{\theta,p}} \preceq \sup_{v_h\in W_h} \frac{a_{\varepsilon,h}(w_h,v_h)}{\|v_h\|_{(A_{0,h},A_{1,h})_{1-\theta,p'}}} \quad \forall w_h \in W_h, \tag{3.14}$$

where $1/p + 1/p' = 1$.

Proof. The bilinear form $a_{\varepsilon,h} : W_h \times W_h \to \mathbb{R}$ induces the linear operator $\mathcal{L}_{\varepsilon,h} : W_h \to W_h^*$ in the usual way:

$$_{W_h^*}\langle \mathcal{L}_{\varepsilon,h} w_h, v_h\rangle_{W_h} := a_{\varepsilon,h}(w_h,v_h) \quad \forall w_h, v_h \in W_h$$

which turns out to be invertible, thanks to (3.1)–(3.2); in particular

$$\|\mathcal{L}_{\varepsilon,h}^{-1}\phi_h\|_{A_{0,h}} \preceq \|\phi_h\|_{A_{1,h}^*}, \tag{3.15}$$

$$\|\mathcal{L}_{\varepsilon,h}^{-1}\phi_h\|_{A_{1,h}} \preceq \|\phi_h\|_{A_{0,h}^*}, \tag{3.16}$$

for any $\phi_h \in W_h^*$. Therefore, by means of interpolation (see [15, §1.3.3 (a)]), we obtain

$$\|\mathcal{L}_{\varepsilon,h}^{-1}\phi_h\|_{(A_{0,h},A_{1,h})_{\theta,p}} \preceq \|\phi_h\|_{(A_{1,h}^*,A_{0,h}^*)_{\theta,p}}. \tag{3.17}$$

By means of [15, §1.11.2], we also have that the norm on $(A_{1,h}^*, A_{0,h}^*)_{\theta,p}$ is actually equivalent to the dual norm on $(A_{1,h}, A_{0,h})_{\theta,p'} \equiv (A_{0,h}, A_{1,h})_{1-\theta,p'}$; notice in particular that, for the case $p=1$ and $p'=+\infty$, the mentioned result follows because $(A_{1,h}, A_{0,h})_{\theta,+\infty} \equiv (A_{1,h}, A_{0,h})^0_{\theta,+\infty}$ in the algebraic sense: actually from the algebraic point of view $A_{0,h} \equiv A_{1,h} \equiv V_h$, we are just defining norms that have a different dependence on the parameter ε. Finally

$$\|w_h\|_{(A_{0,h},A_{1,h})_{\theta,p}} \preceq \|\mathcal{L}_{\varepsilon,h} w_h\|_{(A_{0,h},A_{1,h})^*_{1-\theta,p}}. \tag{3.18}$$

that is just (3.14). □

Now we turn our attention to the continuity of $a_{\varepsilon,h}$ (i.e., estimate (2.16)), that is, contrary to expectation, the most difficult point. For the sake of clarity, we write $a_{\varepsilon,h}(\cdot,\cdot) = a_\varepsilon(\cdot,\cdot) + s(\cdot,\cdot) + c(\cdot,\cdot)$, where $s : H_0^1 \times H_0^1 \to \mathbb{R}$ denotes the stabilizing term

$$s(w,v) := \tau \int_0^1 w'(x)\, v'(x)\, dx \tag{3.19}$$

while the term $c(\cdot,\cdot)$ makes the numerical formulation consistent; it requires the trial functions w to be regular inside any element: therefore we set

$$W := \left\{ w \in H_0^1 | w''_{|T_i} \in L^2(T_i),\ i = 1, \ldots, N \right\}, \tag{3.20}$$

equipped with the graph norm, and we define $c : W \times H_0^1 \to \mathbb{R}$ as

$$c(w,v) = -\tau\varepsilon \sum_{i=1}^{N} \int_{T_i} w''(x)\, v'(x)\, dx. \tag{3.21}$$

First, we consider $a_\varepsilon(\cdot,\cdot)$.

LEMMA 3.3. *Assume* $0 < \theta < 1$, $1 \leq p \leq +\infty$, *and* $1/p + 1/p' = 1$; *we have*

$$a_\varepsilon(w, v_h) \preceq \|w\|_{(A_0,A_1)_{\theta,p}} \|v_h\|_{(A_{0,h},A_{1,h})_{1-\theta,p'}} \quad \forall w \in H_0^1, \forall v_h \in W_h. \tag{3.22}$$

Proof. Let $\widetilde{\mathcal{L}}_\varepsilon : H_0^1 \to W_h^*$ be the linear operator given by

$${}_{W_h^*}\langle \widetilde{\mathcal{L}}_\varepsilon w, v_h \rangle_{W_h} := a_\varepsilon(w, v_h) \quad \forall w \in H_0^1, \forall v_h \in W_h,$$

(notice that it differs from $\mathcal{L}_\varepsilon$ because we are now considering discrete test functions). The Cauchy-Schwarz inequality gives, for any $w \in H_0^1$,

$$\|\widetilde{\mathcal{L}}_\varepsilon w\|_{A_{1,h}^*} \preceq \|w\|_{A_0},$$

and

$$\|\widetilde{\mathcal{L}}_\varepsilon w\|_{A_{0,h}^*} \preceq \|w\|_{A_1};$$

therefore, proceeding as in the proof of Lemma 3.2, we obtain

$$\|\widetilde{\mathcal{L}}_\varepsilon w\|_{(A_{1,h},A_{0,h})_{\theta,p}^*} \preceq \|w\|_{(A_0,A_1)_{\theta,p}}.$$

which is (3.22). □

We need the following inverse inequalities for the forthcoming analysis.

LEMMA 3.4. *We have*

$$h\|v_h'\|_{L^2} \preceq \|v_h'\|_{H^{-1}} \quad \forall v_h \in W_h. \tag{3.23}$$

Proof. Actually (3.23) follows from $\|v_h'\|_{L^2} = \|(\Pi_0 v_h)'\|_{L^2}$, $\|v_h'\|_{H^{-1}} = \|\Pi_0 v_h\|_{L^2}$, and from the more usual inverse inequality $h\|(\Pi_0 v_h)'\|_{L^2} \preceq \|\Pi_0 v_h\|_{L^2}$. □

LEMMA 3.5. *We have*

$$h\|v_h'\|_{(L^2,H_0^1)_{1/2,+\infty}} \preceq \|v_h'\|_{(H^{-1},L^2)_{1/2,+\infty}} \quad \forall v_h \in W_h. \tag{3.24}$$

Proof. We shall show that

$$h\|v_h'\|_{(L^2,H_0^1)_{1/2,+\infty}} \preceq h^{1/2}\|v_h'\|_{L^2} \quad \forall v_h \in W_h, \tag{3.25}$$

$$h\|v_h'\|_{(L^2,H_0^1)_{1/2,+\infty}} \preceq h^{-1/2}\|v_h'\|_{H^{-1}} \quad \forall v_h \in W_h, \tag{3.26}$$

which give (3.24) by means of the interpolation theorem [15, §1.3.3 (a)]. We only focus on (3.25), as (3.26) follows from (3.25) and (3.23).

We have, by definition,

$$\|v_h'\|_{(L^2,H_0^1)_{1/2,+\infty}} \leq \sup_{0<t<+\infty} \left(t^{-1/2}\|\phi_0(t)\|_{L^2} + t^{1/2}\|\phi_1(t)\|_{H^1}\right) \tag{3.27}$$

for any $\phi_0(t)$ and $\phi_1(t)$ with $v_h' = \phi_0(t) + \phi_1(t)$, $\phi_0(t) \in L^2$, $\phi_1(t) \in H_0^1$ and $0 < t < +\infty$. We choose now suitable $\phi_0(t)$ and $\phi_1(t)$. If $t > h$ it suffices to take $\phi_0(t) = v_h'$, and accordingly $\phi_1(t) = 0$, to obtain

$$t^{-1/2}\|\phi_0(t)\|_{L^2} + t^{1/2}\|\phi_1(t)\|_{H^1} \leq h^{-1/2}\|v_h'\|_{L^2} \quad \forall v_h \in W_h. \qquad (t > h \text{ case})$$

Otherwise, when $t \leq h$, set $\delta \equiv \delta(t,x) := \min\{x, 1-x, t/2\}$ and

$$\phi_1(t)(x) := t^{-1}\int_{x-\delta}^{x+\delta} v_h'(\xi)\, d\xi;$$

the effect of this definition is shown in Figure 3.1. As a result, one has

$$t^{-1/2}\|\phi_0(t)\|_{L^2} + t^{1/2}\|\phi_1(t)\|_{H^1} \preceq \left[\sum_{i=0}^{N}\left(v_h'|_{T_{i+1}} - v_h'|_{T_i}\right)^2\right]^{1/2},$$

where $v_h'|_{T_0} = v_h'|_{T_{N+1}} := 0$ by convention, and

$$\sum_{i=0}^{N}\left(v_h'|_{T_{i+1}} - v_h'|_{T_i}\right)^2 \preceq \sum_{i=0}^{N}\left(v_h'|_{T_i}\right)^2 = h^{-1}\|v_h'\|_{L^2}^2,$$

therefore

$$t^{-1/2}\|\phi_0(t)\|_{L^2} + t^{1/2}\|\phi_1(t)\|_{H^1} \preceq h^{-1/2}\|v_h'\|_{L^2} \quad \forall v_h \in W_h. \qquad (t \leq h \text{ case})$$

Collecting the ($t > h$ case) and ($t \leq h$ case) with (3.27), we obtain (3.25). □

LEMMA 3.6. *Assume* $1/2 < \theta < 1$ *and* $1 \leq p \leq +\infty$*, or* $\theta = 1/2$ *and* $p = 1$*; we have*

$$s(w, v_h) \preceq \|w\|_{(A_0,A_1)_{\theta,p}}\|v_h\|_{(A_{0,h},A_{1,h})_{1-\theta,p'}} \quad \forall w \in H_0^1, \forall v_h \in W_h, \tag{3.28}$$

where $1/p + 1/p' = 1$.

Proof. Let $w \in H_0^1$ and $v_h \in W_h$. Assume for a moment that the estimates

$$s(w, v_h) \preceq \|w'\|_{(H^{-1},L_0^2)_{1/2,1}}\|v_h'\|_{(H^{-1},L_0^2)_{1/2,+\infty}}, \tag{3.29}$$

$$s(w, v_h) \preceq \|w'\|_{L^2}\|v_h'\|_{H^{-1}}, \tag{3.30}$$

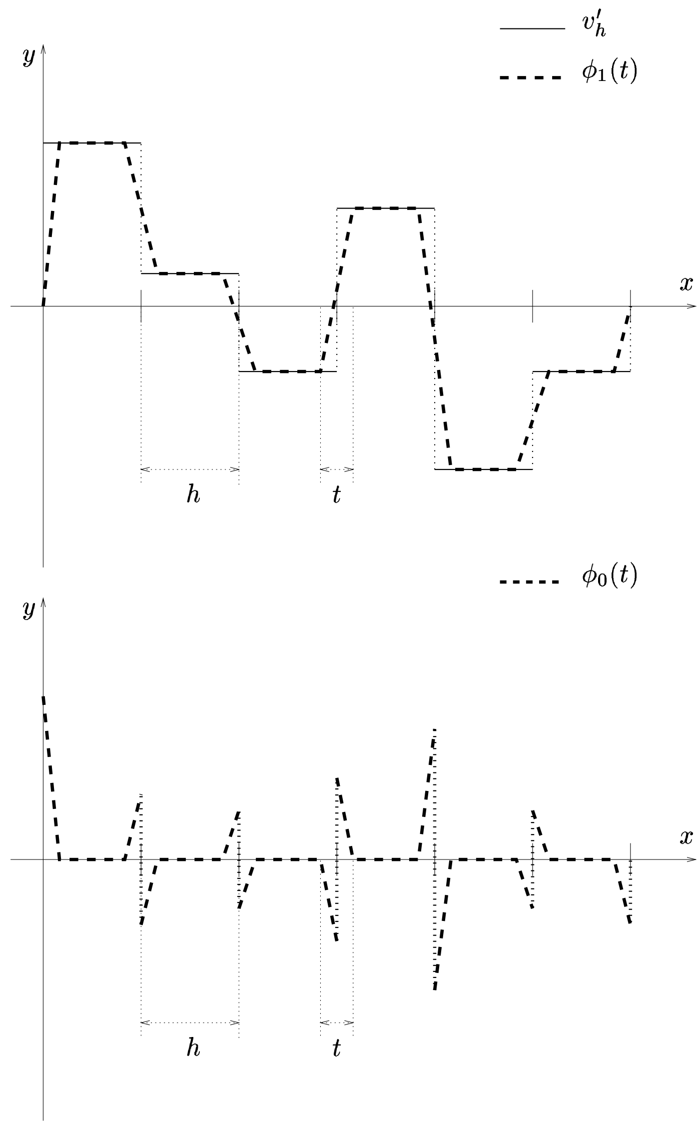

FIG. 3.1. *Construction of $\phi_0(t)$ and $\phi_1(t)$ inside the proof of Lemma 3.5*

hold true. Therefore, recalling (2.5) and as $\|v_h\|_{(A_0,A_1)_{1/2,+\infty}} \le \|v_h\|_{(A_{0,h},A_{1,h})_{1/2,+\infty}}$, we also have

$$s(w,v_h) \preceq \|w\|_{(A_0,A_1)_{1/2,1}} \|v_h\|_{(A_{0,h},A_{1,h})_{1/2,+\infty}},$$
$$s(w,v_h) \preceq \|w\|_{A_1} \|v_h\|_{A_{0,h}},$$

and we can apply a new interpolation, with parameters $0 < \eta < 1$ and $1 \le q \le +\infty$, reasoning as in the proof of Lemma 3.3, in order to obtain

$$s(w,v_h) \preceq \|w\|_{((A_0,A_1)_{1/2,1},A_1)_{\eta,q}} \|v_h\|_{((A_{0,h},A_{1,h})_{1/2,+\infty},A_{0,h})_{\eta,q'}},$$

where $1/q + 1/q' = 1$. This gives (3.28), thanks to the reiteration theorem (see [15, §1.10.2]), with $\theta = 1/2 + \eta/2$ and $p = q$.

First, let us focus on (3.30): it follows from the Cauchy-Schwarz inequality, from Lemma 3.4 and from (2.22).

Now we focus on (3.29). The Cauchy-Schwarz inequality yields

$$s(w, v_h) \preceq \|w'\|_{(H^{-1},L^2)_{1/2,1}} \|\tau v_h'\|_{(H^{-1},L^2)^*_{1/2,1}}, \tag{3.31}$$

and, of course, we have

$$\|w'\|_{(H^{-1},L^2)_{1/2,1}} \leq \|w'\|_{(H^{-1},L^2_0)_{1/2,1}}; \tag{3.32}$$

on the other hand, thanks to [15, §1.11.2], (2.22) and Lemma 3.5, we also have

$$\begin{aligned} \|\tau v_h'\|_{(H^{-1},L^2)^*_{1/2,1}} &\preceq \|\tau v_h'\|_{(L^2,H^1_0)_{1/2,+\infty}} \\ &\preceq \|v_h'\|_{(H^{-1},L^2)_{1/2,+\infty}} \\ &\preceq \|v_h'\|_{(H^{-1},L^2_0)_{1/2,+\infty}}. \end{aligned} \tag{3.33}$$

Finally, (3.31)–(3.33) give (3.29). □

It is worth noting that the stabilizing term $s(\cdot,\cdot)$ is continuous, with respect to the norms $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ and $\|\cdot\|_{(A_{0,h},A_{1,h})_{1-\theta,p'}}$, for any value of θ and p; nevertheless the uniformity with respect to ε requires the restrictions stated in Lemma 3.6. In this sense, these restrictions are optimal: notice that the norms in the left-hand side of (3.24) can not be replaced by stronger norms, since v_h' is discontinuous.

In order to deal with $c(\cdot,\cdot)$, we define an *ad hoc* semi-norm:

$$\|\phi\|_{\theta-1,p} = \sup_{v_h \in W_h} \frac{\sum_{i=1}^N \tau \int_{T_i} \phi(x)\, v_h'(x)\, dx}{\|v_h\|_{(A_{0,h},A_{1,h})_{1-\theta,p'}}}; \tag{3.34}$$

the continuity of $c(\cdot,\cdot)$ follows immediately.

LEMMA 3.7. *Assume* $0 < \theta < 1$, $1 \leq p \leq +\infty$, *and* $1/p + 1/p' = 1$*; we have*

$$c(w, v_h) \preceq \varepsilon \|w''\|_{\theta-1,p} \|v_h\|_{(A_{0,h},A_{1,h})_{1-\theta,p'}}. \tag{3.35}$$

The framework is complete: we can now state our main result.

THEOREM 3.8. *Assume that* $f \in L^2$, $1/2 < \theta < 1$ *and* $1 \leq p \leq +\infty$, *or* $\theta = 1/2$ *and* $p = 1$. *The 1-D SUPG method* (2.19)–(2.22) *satisfies the classical continuity and inf-sup conditions* (2.16)–(2.17) *with respect to the norms*

$$\begin{aligned} \|w\|_{W,h} &:= \|w\|_{(A_0,A_1)_{\theta,p}} + \varepsilon \|w''\|_{\theta-1,p}, \\ \|v_h\|_{V_h} &:= \|v_h\|_{(A_{0,h},A_{1,h})_{1-\theta,p'}}. \end{aligned}$$

Proof. The continuity (2.16) follows from Lemmas 3.3, 3.6 and 3.7. Moreover, since $\|w_h\|_{(A_0,A_1)_{\theta,p}} \leq \|w_h\|_{(A_{0,h},A_{1,h})_{\theta,p}}$ and $\varepsilon \|w_h''\|_{\theta-1,p} = 0$, $\forall v_h \in W_h$, we get

$$\|w\|_{W,h} \leq \|w\|_{(A_{0,h},A_{1,h})_{\theta,p}}, \tag{3.36}$$

whence the inf-sup condition (2.17) follows from Proposition 3.2 and (3.36). □

As a result, we can state the *quasi-optimality* (2.18) of the 1-D SUPG method with respect to the norm $\|\cdot\|_{W,h}$ (for $1/2 < \theta < 1$ or $\theta = 1/2$ and $p = 1$); the most interesting case is for $\theta = 1/2$: thanks to (2.6) and (2.9) we can state the following result.

COROLLARY 3.9. *Assume that $f \in L^2$; let u be the solution of* (1.2) *and u_h be the numerical solution given by the 1-D SUPG method* (2.19)–(2.22). *Then,*

$$\begin{aligned} &\varepsilon^{1/2}|u-u_h|_{H^1} + \|u-u_h\|_{(L^2,H^1_\#)_{1/2,1}} + \varepsilon|\!|\!|(u-u_h)''|\!|\!|_{-1/2,1} \\ &\quad\preceq \inf_{w_h\in V_h}\left[\varepsilon^{1/2}|u-w_h|_{H^1} + \|u-w_h\|_{(L^2,H^1_\#)_{1/2,1}} + \varepsilon|\!|\!|(u-w_h)''|\!|\!|_{-1/2,1}\right]. \end{aligned} \tag{3.37}$$

Estimate (3.37) is interesting because the norm appearing there is "natural" for the problem, in the sense that $\varepsilon^{1/2}|u|_{H^1} + \|u\|_{(L^2,H^1_\#)_{1/2,1}} + \varepsilon|\!|\!|u''|\!|\!|_{-1/2,1}$ does not grow as a negative power of ε when $\varepsilon \to 0$. Actually it is not uniformly bounded with respect to ε, but it behaves as $\log(\varepsilon)$, in the worst case. Indeed we have

$$\begin{aligned} \sum_{i=1}^N \tau \int_{T_i} \varepsilon u''(x)\, v_h'(x)\,dx &= +\sum_{i=1}^N \tau \int_{T_i} u'(x)\, v_h'(x)\,dx \\ &\quad - \sum_{i=1}^N \tau \int_{T_i} f\, v_h'(x)\,dx \\ &\preceq s(u,v_h) + \|f\|_{L^2}\|\tau v_h'\|_{L^2}. \end{aligned} \tag{3.38}$$

This, together with (3.28), (2.22) and (3.23) yield $\varepsilon|\!|\!|u''|\!|\!|_{-1/2,1} \preceq \|u\|_{(A_0,A_1)_{1/2,1}} + \|f\|_{L^2}$; then, thanks to (2.4), (2.5), (2.8), (2.14) and [13, equation (22)], we finally get

$$\varepsilon^{1/2}|u|_{H^1} + \|u\|_{(L^2,H^1_\#)_{1/2,1}} + \varepsilon|\!|\!|u''|\!|\!|_{-1/2,1} \preceq (1+|\log\varepsilon|)\,\|f\|_{L^2}. \tag{3.39}$$

Usually, one can infer the convergence of the numerical method from an estimate like (2.18) (in particular, (3.37). This is not the case here. We recall that we are interested in uniform convergence with respect to ε, in the advection-dominated regime $\varepsilon < h$. In fact, since we are using piecewise linear elements, we easily see that $|\!|\!|(u-u_h)''|\!|\!|_{-1/2,1} = |\!|\!|u''|\!|\!|_{-1/2,1}$. Furthermore, $\varepsilon^{1/2}|u-u_h|_{H^1}$, as well as $\|u-u_h\|_{(L^2,H^1_\#)_{1/2,1}}$, cannot vanish uniformly with respect to ε when $h \to 0$. This because the boundary layer can not be captured within the discrete space W_h, when $\varepsilon \ll h$. On the other hand, this is not surprising: convergence results are indeed obtained from estimates like (2.18) by assuming *extra* regularity on the solution u. In our case, this is not possible since, for example, $\|u\|_{(A_0,A_1)_{\theta,p}}$ is strongly dependent on ε for $\theta > 1/2$.

4. Conclusion and further extension. In this paper we proved the *quasi-optimality* of the SUPG method for the one-dimensional advection-diffusion problem on a uniform grid. Actually it is a very simple case; most of our analysis is based on our previous work [13] and therefore it is suitable for an extension to the multi-dimensional case; in other parts it depends on some special properties of the one-dimensional problem. We do not know at the moment whether the SUPG method preserves its quasi-optimality in the two-dimensional case, or whether a modification of the method is required for the purpose; we shall focus on possible extensions of the theory in further works.

We have assumed the amount of *streamline diffusion* (or, better, *artificial diffusion*) τ to be proportional to the *mesh size* h. It is well known that, from a practical point of view, the particular choice of τ is relevant for the accuracy of the method. Our analysis does not give any suggestion for this, because, for the sake of simplicity, our final estimate implicitly contains generic constants whose dependence on τ is not investigated. This investigation is indeed a very technical task, but one could perform this kind of analysis by a computational procedure: this has been done in a previous work [14] (see in particular §3 therein) where we perform a fine-tuning of τ based on a similar idea.

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