Mathematical Models and Methods in Applied Sciences

The Residual-Free Bubble numerical method with quadratic elements.

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In this paper we analyze the *Residual-Free Bubble* (RFB) method applied to the linear diffusion-advection-reaction problem. We propose a new *a priori* error analysis for the method and for its practical implementation in a quite general context, which allows, e.g., linear or quadratic elements on the resolvable scales. We also perform some numerical tests, showing in both cases the advantages of the method.

1. Introduction

This work is devoted to the numerical approximation of the linear diffusion-advection-reaction operator

$$\mathcal{L}_\kappa(\cdot) := -\kappa\Delta(\cdot) + \mathbf{a}\cdot\nabla(\cdot) + \sigma(\cdot), \tag{1.1}$$

and the related p.d.e. problem

$$\begin{cases} \mathcal{L}_\kappa u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the unknown u is a real function on a bounded polygonal domain $\Omega \in I\!R^2$ and f is the source term. The diffusion coefficient κ is assumed to be constant, while the vector-valued advection field $\mathbf{a}$ and the scalar reaction coefficient σ are functions on Ω.

This is a model problem in fluid dynamics and presents some of the difficulties that are encountered in the numerical simulations of fluid flows[12]. In some cases, solving (1.2) is part of solving a more complex real problem, possibly time-dependent.

Standard numerical methods—like central finite difference or standard Galerkin finite element methods—are inadequate when $\kappa/\|\mathbf{a}\|$ or $(\kappa/\sigma)^{1/2}$ are small compared to the discretization step size, since the numerical solutions exhibit unphysical oscillatory behavior. *Stabilized* methods have been introduced in order to overcome this undesirable feature; we shall deal here with the *Residual-Free Bubble* (RFB) method, originally proposed by Brezzi and Russo[8] for the diffusion-advection problem (i.e., where $\sigma = 0$), based on previous ideas[3,2], and extended to diffusion-advection-reaction by Asensio, Franca and Russo[1]; in the latter work the RFB method is shown to be effective for a wide range of coefficients and advantages of the method over other ones are discussed.

Roughly speaking, in the RFB formulation the *small scales*, represented by *bubble* functions, are not resolved, but their effect on the *resolvable scales*, which are represented by continuous and piecewise polynomial functions, is taken into account. In particular, previous practical implementations of the RFB method have been based on piecewise linear or bilinear functions as resolvable scales[8,4].

In the present work we investigate the effect of using quadratic shape-functions for the resolvable scale. We propose a new *a priori* error analysis which extends, in some sense, the results of a previous work[6], where the case of piecewise linear elements is considered. Other papers[7,14] have been devoted to the theoretical analysis of the general (i.e., high order elements) case, but the analysis therein assume that the *bubble* part is actually resolved and included into the numerical approximation. As we shall see in the sequel, from the practical point of view the final numerical scheme derived from the RFB approach is based on an approximated elimination of the *bubble* degrees of freedom. In the present error analysis we explicitly take into account the *bubble* elimination and the approximations introduced during the elimination itself. We obtain, at the end, a justification of the final algorithm based on RFB.

We present numerical tests using both linear and quadratic elements, showing that the RFB technique is very effective and confirming the validity of this approach as a general methodology.

The outline of the paper is as follows: in §2 we describe the notation and the RFB method in details; §3 is devoted to the *a priori* error analysis; in §4 we explain how to implement the method and show numerical examples.

2. Preliminaries

As usual in the Finite Element framework, we assume to have a family of partitions $\mathcal{T}_h$ of the domain Ω into triangles K; we assume $\mathcal{T}_h$ to be *admissible* (i.e., non-overlapping triangles, their union reproduces the domain, etc.) and *shape regular* (i.e., the triangles verify a minimum angle condition). We also denote by $\mathcal{E}_h$ the set of the edges of the elements of the triangulation $\mathcal{T}_h$, and by $\Gamma_h := \cup_{K\in\mathcal{T}_h}\partial K$ the skeleton of $\mathcal{T}_h$.

Given an open subset ω of Ω, we denote by $L^2(\omega) = H^0(\omega)$, $H^1(\omega)$,..., $H^m(\omega)$, $m \in \mathbb{N}$ the usual Sobolev spaces equipped with the standard norms $\|\cdot\|_{L^2(\omega)}$,..., $\|\cdot\|_{H^m(\omega)}$ and seminorms $|\cdot|_{H^m(\omega)}$; $H^1_0(\omega)$ is the space of functions contained in $H^1(\omega)$ with zero trace on $\partial\omega$, equipped with the norm $|\cdot|_{H^1(\omega)}$; finally $H^{-1}(\omega)$ denotes the dual space of $H^1_0(\omega)$ equipped with the dual norm $\|\cdot\|_{H^{-1}(\omega)}$. The standard notation $\langle\cdot,\cdot\rangle$ is used for the pairing between spaces in duality. We follow similar notations when the domain is a 1-dimensional piecewise regular manifold (e.g, the boundary of an element K): for example $L^2(\partial\omega)$ is the space of square integrable functions defined on the boundary of ω, equipped with the scalar product $(\cdot,\cdot)_{\partial\omega}$ and the related norm $\|\cdot\|_{L^2(\partial\omega)}$. The outward normal direction defined on the boundary of the element K is denoted by $\mathbf{n}_K$. We introduce the notation $[\![\cdot]\!]$ and $(\!|\cdot|\!)$ for the jump and, respectively, the maximum modulus of a piecewise continuous vector field across its discontinuities; more precisely $[\![\cdot]\!]$ and $(\!|\cdot|\!)$ take a vector field defined on Ω and give a scalar field defined on Γ_h in the following way: for any couple of adjacent element K_1 and K_2, which share one edge $e = \partial K_1 \cap \partial K_2$, then $[\![\mathbf{a}(\mathbf{x})]\!] := \mathbf{a}_{|K_1}(\mathbf{x})\cdot\mathbf{n}_{K_1}(\mathbf{x})+\mathbf{a}_{|K_2}(\mathbf{x})\cdot\mathbf{n}_{K_2}(\mathbf{x})$, when $\mathbf{x} \in e$, is the jump of the exterior normal component of $\mathbf{a}$, and $(\!|\mathbf{a}(\mathbf{x})|\!)$ is the maximum between $|\mathbf{a}_{|K_1}(\mathbf{x})\cdot\mathbf{n}_{K_1}(\mathbf{x})|$ and $|\mathbf{a}_{|K_2}(\mathbf{x})\cdot\mathbf{n}_{K_2}(\mathbf{x})|$; when $e \subset \partial\Omega$ we define $[\![\mathbf{a}]\!]_{|e}$ and $(\!|\mathbf{a}|\!)_{|e}$ similarly, after extending $\mathbf{a}$ to 0 outside Ω.

In the sequel C denotes a generic constant whose value, possibly different at any occurrence, can only depend on the explicitly indicated quantities. We also adopt the notational convention

$$\alpha \preceq \beta \quad \Longleftrightarrow \quad \alpha \leq C\beta.$$

We introduce the bilinear form $a : H^1_0(\Omega) \times H^1_0(\Omega) \to I\!R$ defined as

$$a(w,v) := \langle \mathcal{L}_\kappa w, v\rangle;$$

therefore the variational formulation of (1.2) is

$$\begin{cases} \text{Find } u \in H^1_0(\Omega), \text{ such that} \\ a(u,v) = \langle f, v\rangle, \qquad \forall v \in H^1_0(\Omega). \end{cases} \tag{2.3}$$

We assume, in the sense of *measures*,

$$\sigma - 1/2\,\mathrm{div}(\mathbf{a}) \geq 0, \tag{2.4}$$

which guarantees the well posedness of (1.2) for any $\kappa > 0$: indeed (2.4) makes a coercive on $H^1_0(\Omega)$:

$$a(w,w) \geq \kappa |w|^2_{H^1(\Omega)}. \tag{2.5}$$

We also assume that the coefficients $\mathbf{a}$ and σ are constant in each element $K \in \mathcal{T}_h$. Then hypothesis (2.4) becomes

$$\sigma \geq 0, \tag{2.6}$$

$$\mathrm{div}(\mathbf{a}) \leq 0, \tag{2.7}$$

where (2.7) has to be intended in the sense of *measures*, and actually means that $[\![\mathbf{a}]\!] \geq 0$ on any internal edge. We define $\mathcal{L}_0$ as the formal limit of $\mathcal{L}_\kappa$ for vanishing κ, i.e.,

$$\mathcal{L}_0 w := \mathbf{a} \cdot \nabla w + \sigma w;$$

$\mathcal{L}_\kappa^*$ and $\mathcal{L}_0^*$ denote their formal transpose:

$$\begin{aligned} \mathcal{L}_\kappa^* w &:= -\kappa \Delta w - \operatorname{div}(w\mathbf{a}) + \sigma w, \\ \mathcal{L}_0^* w &:= -\operatorname{div}(w\mathbf{a}) + \sigma w; \end{aligned}$$

the restrictions of $\mathcal{L}_\kappa^*$ and $\mathcal{L}_0^*$ inside any element K are simpler, thanks to $\operatorname{div}(w\mathbf{a})_{|K} = \mathbf{a}_{|K} \cdot \nabla w_{|K}$.

We define V_R to be a finite dimensional space (the space of *resolvable scales*) where the numerical approximation will be computed; we shall consider a usual space of continuous and piecewise polynomial finite elements, i.e.,

$$V_R \equiv V_R\left(\mathcal{T}_h, k\right) := \left\{ v \in H_0^1(\Omega) \text{ such that } v_{|K} \in I\!P_k, \forall K \in \mathcal{T}_h \right\},$$

where $I\!P_k$ denotes the space of polynomials of degree k, with the assumption $k \leq 2$.

We define the space $V_h \subset H_0^1(\Omega)$ as follows:

$$V_h \equiv V_h\left(\mathcal{T}_h, k\right) := \left\{ v_h \in H_0^1(\Omega) \text{ such that } v_{h|\partial K} \in I\!P_k, \forall K \in \mathcal{T}_h \right\};$$

The space V_h is actually an enrichment of V_R by means of the *bubble* space V_U, which represent the *unresolvable scales*:

$$V_U \equiv V_U(\mathcal{T}_h) := \left\{ v_U \in H_0^1(\Omega) \text{ such that } v_{U|\partial K} = 0, \forall K \in \mathcal{T}_h \right\}.$$

The Galerkin method on the space V_h reads

$$\begin{cases} \text{Find } u_h \in V_h, \text{ such that} \\ a(u_h, v_h) = \langle f, v_h \rangle, \qquad \forall v_h \in V_h. \end{cases} \tag{2.8}$$

We shall refer to (2.8) as the *complete* RFB method. This method is robust for any value of κ (the analysis proposed in other works[7,14] for the case $\sigma = 0$ can be extended to (1.2) with minor modifications) but it is not suitable for an actual numerical approximation of (1.2) because V_h is *infinite dimensional*. The usual way[9] of dealing with (2.8) is based on the numerical computation of the degrees of freedom of V_R, after the effect of the degrees of freedom in V_U is suitably taken into account. Any $v_h \in V_h$ admits a unique decomposition as

$$v_h = v_R \oplus v_U, \text{ with } v_R \in V_R, v_U \in V_U, \tag{2.9}$$

because we are restricting the order of polynomials to $1 \leq k \leq 2$ (see Remark 4 in the sequel for the case of $k > 2$). Therefore we split $u_h = u_R \oplus u_U$ where $u_R \in V_R$ and $u_U \in V_U$, and we test (2.8) using $v_R \in V_R$ first, and then $v_U \in V_U$, yielding

$$a\left(u_R, v_R\right) + a\left(u_U, v_R\right) = \langle f, v_R \rangle \qquad \forall v_R \in V_R, \tag{2.10}$$

$$a\left(u_R, v_U\right) + a\left(u_U, v_U\right) = \langle f, v_U \rangle \qquad \forall v_U \in V_U. \tag{2.11}$$

The equation (2.11) gives u_U from u_R and f: actually u_U solves in each element K the boundary value problem

$$\begin{cases} \mathcal{L}_\kappa u_U = f - \mathcal{L}_\kappa u_R & \text{in } K \\ u_U = 0 & \text{on } \partial K. \end{cases}$$

Therefore, by substituting u_U into (2.10), we are led to a closed form for u_R:

$$\begin{cases} \text{Find } u_R \in V_R, \text{ such that} \\ B(u_R, v_R) = L(v_R), \qquad \forall v_R \in V_R, \end{cases} \tag{2.12}$$

where

$$B(w,v) := a(w,v) + a(M(w),v), \qquad \forall w, v \in H_0^1(\Omega) \tag{2.13}$$

$$L(v) := \langle f, v\rangle - a(F(f),v), \qquad \forall v \in H_0^1(\Omega), \tag{2.14}$$

and $M(w)$, $F(f)$ are solutions of

$$\begin{cases} \text{Find } M(w) \in V_U, \text{ such that} \\ a(M(w), v_U) = -a(w, v_U), \qquad \forall v_U \in V_U, \end{cases}$$

and

$$\begin{cases} \text{Find } F(f) \in V_U, \text{ such that} \\ a(F(f), v_U) = \langle f, v_U\rangle, \qquad \forall v_U \in V_U. \end{cases}$$

In each element K, $M(w)_{|K}$ and $F(f)_{|K}$ verify

$$\begin{cases} \mathcal{L}_\kappa M(w)_{|K} = -\mathcal{L}_\kappa w & \text{in } K \\ M(w)_{|K} = 0 & \text{on } \partial K, \end{cases}$$

and

$$\begin{cases} \mathcal{L}_\kappa F(f)_{|K} = f & \text{in } K \\ F(f)_{|K} = 0 & \text{on } \partial K. \end{cases}$$

Setting $H(w) := w + M(w)$, (2.13) reads

$$B(w,v) := a(H(w), v); \tag{2.15}$$

integrating by parts and using $\mathcal{L}_\kappa H(w)_{|K} = 0, \forall K \in \mathcal{T}_h$ we also have

$$\begin{aligned} B(w,v) &= \kappa \sum_{K\in\mathcal{T}_h} \int_{\partial K} v\, \nabla(H(w)_{|K}) \cdot \mathbf{n}_K, \\ &= \kappa \sum_{e\in\mathcal{E}_h} \int_e [\![\nabla(H(w))]\!]\, v, \qquad \forall w \in H_0^1(\Omega) \cap H^2(\Omega); \end{aligned} \tag{2.16}$$

note that in (2.16) we assumed $\nabla(H(w)_{|K}) \cdot \mathbf{n}_K \in L^2(\partial K)$, which requires w more regular than $H_0^1(\Omega)$, but is not restrictive in this context.

We refer to (2.12) as the *exact* RFB method, because the *bubble* degrees of freedom have been taken into account exactly; even if it is a *finite dimensional* problem, it contains now the difficulty of computing the terms $a(M(w_R), v_R)$ and $a(F(f), v_R)$.

Assume for a moment that we can approximate, in each element, $M(w_R)$ and $F(f)$ by means of $\widetilde{M}(w_R)$ and $\widetilde{F}(f)$, given by

$$\begin{cases} \mathcal{L}_0\widetilde{M}(w_R) = -\mathcal{L}_0 w_R & \text{in } K \\ \widetilde{M}(w_R) = 0 & \text{on } \partial K^-, \end{cases} \tag{2.17}$$

and

$$\begin{cases} \mathcal{L}_0\widetilde{F}(f) = f & \text{in } K \\ \widetilde{F}(f) = 0 & \text{on } \partial K^-; \end{cases} \tag{2.18}$$

where ∂K^- is the inflow boundary of the element K, i.e.,

$$\partial K^- = \{\mathbf{x} \in \partial K : \mathbf{a}\cdot\mathbf{n}(\mathbf{x}) < 0\};$$

also assume that we can change $\mathcal{L}_\kappa^* v_R$ in $\mathcal{L}_0^* v_R$, in such a way that the approximations

$$a(M(w_R), v_R) \equiv \sum_{K\in\mathcal{T}_h}\int_K M(w_R)\mathcal{L}_\kappa^* v_R \approx \sum_{K\in\mathcal{T}_h}\int_K \widetilde{M}(w_R)\mathcal{L}_0^* v_R \tag{2.19}$$

$$a(F(f), v_R) \equiv \sum_{K\in\mathcal{T}_h}\int_K F(f)\mathcal{L}_\kappa^* v_R \approx \sum_{K\in\mathcal{T}_h}\int_K \widetilde{F}(f)\mathcal{L}_0^* v_R \tag{2.20}$$

make sense.

This motivates the definition of the following *approximated* RFB method[4] which is equivalent, when using linear elements, to the original formulation[8]:

$$\begin{cases} \text{Find } \widetilde{u}_R \in V_R, \text{ such that} \\ \widetilde{B}(\widetilde{u}_R, v_R) = \widetilde{L}(v_R), \qquad \forall v_R \in V_R, \end{cases} \tag{2.21}$$

where

$$\widetilde{B}(w,v) := a(w,v) + \sum_{K\in\mathcal{T}_h}\int_K \widetilde{M}(w)\mathcal{L}_0^* v, \tag{2.22}$$

$$\widetilde{L}(v) := \langle f, v\rangle - \sum_{K\in\mathcal{T}_h}\int_K \widetilde{F}(f)\mathcal{L}_0^* v. \tag{2.23}$$

We can now define $\widetilde{H}(w) := w + \widetilde{M}(w)$ for any w regular enough; integrating by parts and using $\mathcal{L}_0\widetilde{H}(w)_{|K} = 0$, for all $K \in \mathcal{T}_h$, we also have

$$\begin{aligned}\widetilde{B}(w,v) &= \kappa\int_\Omega \nabla w\cdot\nabla v \,-\, \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widetilde{M}_{|K}(w)\, v\,\mathbf{a}\cdot\mathbf{n}_K \\ &= \kappa\int_\Omega \nabla w\cdot\nabla v \,-\, \sum_{e\in\mathcal{E}_h}\int_e [\![\widetilde{M}(w)\mathbf{a}]\!]\, v \\ &= \kappa\int_\Omega \nabla w\cdot\nabla v \,+\, \sum_{e\in\mathcal{E}_h}\int_e [\![\mathbf{a}]\!]\, wv - \sum_{e\in\mathcal{E}_h}\int_e [\![\widetilde{H}(w)\mathbf{a}]\!]\, v.\end{aligned} \tag{2.24}$$

The formulation (2.21) is suitable for a numerical algorithm, as we shall see in §4. We try now to discuss the derivation of (2.21), i.e., to see whether the *approximated* RFB method is close to the *exact* RFB method, or not. This actually depends on the magnitude of the coefficients κ, $\mathbf{a}$ and σ, but it is difficult to estimate the error $u_R - \widetilde{u}_R$ directly; on the other hand one can heuristically evaluate the effect of the perturbation $B - \widetilde{B}$ on $\widetilde{B}$ itself for some typical cases for which the theory of asymptotic expansions[10] gives us qualitative informations on the structure of the operator $\mathcal{L}_\kappa$.

The most favorable case here is when the advection is dominant with respect to both the diffusion and the reaction. This is expressed by the conditions

$$\kappa \ll |\mathbf{a}_{|K}| h_K \tag{2.25}$$

and

$$\kappa_{|K}\sigma_{|K} \ll |a_{|K}|^2, \tag{2.26}$$

respectively. Indeed we have:

$$\begin{aligned}B(w_R,v_R) - \widetilde{B}(w_R,v_R) &= \sum_{K\in\mathcal{T}_h}\int_K M(w_R)\,\mathcal{L}_\kappa^* v_R - \int_K \widetilde{M}(w_R)\,\mathcal{L}_0^* v_R \\ &= \sum_{K\in\mathcal{T}_h}\int_K \Big(M(w_R) - \widetilde{M}(w_R)\Big)\,\mathcal{L}_0^* v_R \\ &\qquad - \kappa\sum_{K\in\mathcal{T}_h}\int_K M(w_R)\,\Delta v_R \\ &= I + II.\end{aligned}$$

Actually, $M(w_R) - \widetilde{M}(w_R)$, in the generic element K, represents the boundary layer of $M(w_R)$ which we have neglected. Therefore, when (2.25) and (2.26) hold true, $M(w_R) - \widetilde{M}(w_R)$ is null outside of a layer near to the outflow boundary ∂K^+ which extend in a $\kappa_{|K}|\mathbf{a}_{|K}|^{-1}$ wide region (also assuming, for the sake of simplicity, that the element K has no edges aligned with $\mathbf{a}$); the variation of $M(w_R) - \widetilde{M}(w_R)$ trough the layer equals the jump of $\widetilde{M}(w_R)$ trough the corresponding edge. Then one can heuristically expect that

$$I = \sum_{K\in\mathcal{T}_h} \int_K \Big(M(w_R) - \widetilde{M}(w_R) \Big) \, \sigma v_R$$
$$- \sum_{K\in\mathcal{T}_h} \int_K \Big(M(w_R) - \widetilde{M}(w_R) \Big) \, \mathbf{a}\cdot\nabla v_R$$

is a very small perturbation of $\widetilde{B}$. Further, using (2.25) and inverse inequalities, one can heuristically expect that II is a small perturbation of $\widetilde{B}$, too. This is not surprising at the end, because what we are doing here is the natural extension of what has been done by many authors for the advection-diffusion (i.e., $\sigma = 0$) advection-dominated (i.e., condition (2.25)) problem (see, for example, [4,8]).

When the reaction is dominant with respect to the other terms, i.e.,

$$\kappa \ll \sigma_{|K} h_K^2, \tag{2.27}$$

and

$$\kappa_{|K}\sigma_{|K} \gg |a_{|K}|^2 \tag{2.28}$$

then $M(w_R)$ has a layer of wide $\kappa_{|K}^{1/2}\sigma_{|K}^{-1/2}$ near to the whole element boundary ∂K. It turns out that B and $\widetilde{B}$ are quite different now: this does not imply that the *approximated* RFB method does not work, of course, but at least our philosophical justification of that method fails. For the limiting case $\mathbf{a} = \mathbf{0}$, a different suitable approximation of the RFB formulation has been proposed and analyzed by Russo and Sangalli[13], indeed.

The case of a dominant diffusion term is also not accordant to the approximation we are presenting here, as one can easily understand.

In conclusion we do not expect our *approximated* RFB method covers for all the cases, and our interest here is mainly focused on the advection dominant regime. An effective approximation of the RFB formulation for *any* regime has been proposed[5] for the one-dimensional model problem.

In the following section we prove independently *a priori* error estimates for the two formulations (2.12) and (2.21): this is, beside the previous heuristic discussion, the theoretical confirmation of the effectiveness of our approach.

3. Error analysis

This section is devoted to the theoretical error analysis of both the *exact* RFB method and the *approximated* RFB method. The analysis is based on the classical Céa's argument, and is developed having in mind the singularly perturbed case (2.25) and (2.26), for which we expect a close relation between the two methods, as discussed above.

We first analyze the *exact* RFB method.

Lemma 1 *The* exact *RFB method is consistent:*

$$B(u - u_R, v_R) = 0, \qquad \forall v_R \in V_R. \tag{3.29}$$

Proof. From (2.12)–(2.14) we get

$$\begin{aligned} B(u - u_R, v_R) &= \langle f, v_R \rangle + a(M(u), v_R) - B(u_R, v_R) \\ &= \langle f, v_R \rangle - a(F(f), v_R) - B(u_R, v_R) \\ &= 0. \qquad \square \end{aligned}$$

Since $\mathcal{L}_\kappa H(w)_{|K} = 0$, we also have:

$$B(w, v_U) = 0, \qquad \forall v_U \in V_U. \tag{3.30}$$

We introduce now the energy norm related to B:

$$\begin{aligned} \|\!|w|\!\|_B^2 &:= \kappa |H(w)|^2_{H^1(\Omega)} + \sum_{e \in \mathcal{E}_h} [\![\mathbf{a}]\!]_{|e} \|w\|^2_{L^2(e)} + \sum_{K \in \mathcal{T}_h} \sigma_{|K} \|H(w)\|^2_{L^2(K)} \\ &\equiv \kappa |H(w)|^2_{H^1(\Omega)} + \|[\![\mathbf{a}]\!]^{1/2} w\|^2_{L^2(\Gamma_h)} + \|\sigma^{1/2} H(w)\|^2_{L^2(\Omega)}. \end{aligned}$$

For the case $\sigma = 0$ and $[\![\mathbf{a}]\!] = 0$, it has been shown[6] that for a piecewise linear w_R the norm $\kappa^{1/2} |H(w_R)|_{H^1(\Omega)}$ is equivalent to the usual *SUPG* norm.

Lemma 2 *The bilinear form B is coercive:*

$$\|\!|w|\!\|_B^2 \preceq B(w, w), \qquad \forall w \in H^1_0(\Omega). \tag{3.31}$$

Proof. Using (2.15) and (3.30) $B(w, w) = a(H(w), H(w))$; then (3.31) follows by integrating by parts the advective term on each element K. $\square$

Lemma 3 *The bilinear form B verifies the continuity estimate: $\forall w, v \in H^1_0(\Omega)$*

$$\begin{aligned} B(w, v) \preceq \|\!|w|\!\|_B \Bigg\{ &\kappa^{1/2} |v|_{H^1(\Omega)} \\ &+ \left[\sum_{K \in \mathcal{T}_h} \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma_{|K}^{-1/2} \right) \|v\|^2_{L^2(\partial K)} \right]^{1/2} \Bigg\}. \end{aligned} \tag{3.32}$$

Proof. Thanks to Corollary 2.3 in [14] (applied with $\varepsilon := \kappa(|\mathbf{a}_{|K} + \kappa^{1/2} \sigma_{|K}^{1/2}|)^{-1}$) we construct a function $\hat{v}$ such that $v - \hat{v} \in V_U$ and

$$\begin{aligned} \kappa |\hat{v}|^2_{H^1(K)} + &\kappa^{-1} \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma_{|K}^{1/2} \right)^2 \|\hat{v}\|^2_{L^2(K)} \\ &\preceq \kappa |v|^2_{H^1(K)} + \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma_{|K}^{1/2} \right) \|v\|^2_{L^2(\partial K)}, \end{aligned} \tag{3.33}$$

in any $K \in \mathcal{T}_h$. Therefore, thanks to (3.30), (2.15) and the Cauchy-Schwartz's inequality,

$$
\begin{aligned}
B(w,v) &= B(w,\hat{v}) \\
&= \kappa \int_\Omega \nabla H(w) \cdot \nabla \hat{v} + \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{a} \cdot \nabla H(w) + \sigma H(w))\, \hat{v} \\
&\preceq \|w\|_B \left\{ \sum_{K \in \mathcal{T}_h} \left[\kappa |\hat{v}|^2_{H^1(K)} + \kappa^{-1} \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma^{1/2}_{|K} \right)^2 \|\hat{v}\|^2_{L^2(K)} \right] \right\}^{1/2} \\
&\preceq \|w\|_B \left[\kappa |v|^2_{H^1(\Omega)} + \sum_{K \in \mathcal{T}_h} \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma^{1/2}_{|K} \right) \|v\|^2_{L^2(\partial K)} \right]^{1/2},
\end{aligned}
$$

which gives (3.32) □

We can now state the a priori error estimate for the *exact* RFB method.

Theorem 1 *Let u be the solution of* (1.2)*, let u_R be its numerical approximation by the* exact *RFB method* (2.12)–(2.14)*, and $0 < s \le p$. Then*

$$
\|u - u_R\|_B \le C(s) \left\{ \sum_{K \in \mathcal{T}_h} \left[\kappa h_K^{2s} + \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma^{1/2}_{|K} \right) h^{2s+1} \right] |u|^2_{H^{s+1}(K)} \right\}^{1/2}. \tag{3.34}
$$

Proof. Let $u_I \in V_R$ be the nodal interpolant of u, which verifies

$$
h_K^{2s} |u - u_I|^2_{H^1(T)} + h_K^{2s+1} \|u - u_I\|^2_{L^2(\partial K)} \le C(s) |u|^2_{H^{s+1}(K)}. \tag{3.35}
$$

Applying moreover (3.31), (3.29) and (3.32) we obtain

$$
\begin{aligned}
&\|u - u_R\|_B^2 \\
&\preceq B(u - u_R, u - u_R) \\
&= B(u - u_R, u - u_I) \\
&\preceq \|u - u_R\|_B \left[\kappa |u - u_I|^2_{H^1(\Omega)} + \sum_{K \in \mathcal{T}_h} \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma^{1/2}_{|K} \right) \|u - u_I\|^2_{L^2(\partial K)} \right]^{1/2} \\
&\le C(s) \|u - u_R\|_B \left\{ \sum_{K \in \mathcal{T}_h} \left[\kappa h_K^{2s} + \left(|\mathbf{a}_{|K}| + \kappa^{1/2} \sigma^{1/2}_{|K} \right) h^{2s+1} \right] |u|^2_{H^{s+1}(K)} \right\}^{1/2},
\end{aligned}
$$

which gives (3.34) □

Now we turn the attention on the *approximated* RFB formulation, which is the one that we will use for the numerical computations. This formulation is not consistent: the consistency error is evaluated in the following lemma.

Lemma 4 *Let u be the solution of* (1.2)*, $\widetilde{u}_R$ be its numerical approximation by the* approximated *RFB method* (2.21)–(2.23). *Then*

$$\widetilde{B}(u-\widetilde{u}_R, v_R) \preceq \kappa |v_R|_{H^1(\Omega)} \left[\sum_{K\in\mathcal{T}_h} (h_K^2+1)\|\Delta u\|^2_{L^2(K)}\right]^{1/2}, \quad \forall v_R \in H^1_0(\Omega). \tag{3.36}$$

Proof. From (2.21)–(2.23) we get

$$\begin{aligned}
\widetilde{B}(u-\widetilde{u}_R, v_R) =&\ \langle f, v_R\rangle + \sum_{K\in\mathcal{T}_h}\int_K \widetilde{M}(u)\,\mathcal{L}_0^* v_R - \widetilde{B}(\widetilde{u}_R, v_R)\\
=&\ \langle f, v_R\rangle - \sum_{K\in\mathcal{T}_h}\int_K \widetilde{F}(f)\,\mathcal{L}_0^* v_R - \widetilde{B}(\widetilde{u}_R, v_R)\\
&+ \sum_{K\in\mathcal{T}_h}\int_K (\widetilde{M}(u)+\widetilde{F}(f))\mathcal{L}_0^* v_R\\
=& \sum_{K\in\mathcal{T}_h}\int_K (\widetilde{M}(u)+\widetilde{F}(f))\sigma v_R - \sum_{K\in\mathcal{T}_h}\int_K (\widetilde{M}(u)+\widetilde{F}(f))\mathbf{a}\cdot\nabla v_R\\
=&\ I + II.
\end{aligned}$$

The function $\phi := \widetilde{M}(u) + \widetilde{F}(f)$ verifies, in each element K, the b.v.p.

$$\begin{cases} \mathcal{L}_0\phi = -\kappa\Delta u & \text{in } K\\ \phi = 0 & \text{on } \partial K^-, \end{cases} \tag{3.37}$$

Using (3.37) and the Cauchy-Schwartz's inequality we have

$$\begin{aligned}
\sigma\|\phi\|^2_{L^2(K)} &\le \frac{1}{2}\int_{\partial K^+}\phi^2\mathbf{a}\cdot\mathbf{n}_K + \sigma\int_K\phi^2\\
&= \frac{1}{2}\int_{\partial K}\phi^2\mathbf{a}\cdot\mathbf{n}_K + \sigma\int_K\phi^2\\
&= \frac{1}{2}\int_K \mathbf{a}\cdot\nabla(\phi^2) + \sigma\int_K\phi^2\\
&= \int_K \phi\,\mathbf{a}\cdot\nabla\phi + \int_K\sigma\phi^2\\
&= \int_K (\mathcal{L}_0\phi)\,\phi\\
&= -\kappa\int_K \Delta u\,\phi\\
&\le \kappa\|\Delta u\|_{L^2(K)}\|\phi\|_{L^2(K)},
\end{aligned}$$

which gives, after dividing by $\|\phi\|_{L^2(K)}$,

$$\|\phi\|_{L^2(K)} \le \sigma^{-1}\kappa\|\Delta u\|_{L^2(K)}. \tag{3.38}$$

In a similar way, we obtain

$$
\begin{aligned}
\|\mathbf{a}\cdot\nabla\phi\|^2_{L^2(K)} &\le \int_K (\mathbf{a}\cdot\nabla\phi)^2 + \frac{\sigma}{2}\int_{\partial K^+} \phi^2\mathbf{a}\cdot\mathbf{n}_K \\
&= \int_K (\mathbf{a}\cdot\nabla\phi)^2 + \frac{\sigma}{2}\int_{\partial K} \phi^2\mathbf{a}\cdot\mathbf{n}_K \\
&= \int_K (\mathbf{a}\cdot\nabla\phi)^2 + \frac{\sigma}{2}\int_{K} \mathbf{a}\cdot\nabla(\phi^2) \\
&= \int_K (\mathbf{a}\cdot\nabla\phi)^2 + \int_{K} \sigma\phi\,\mathbf{a}\cdot\nabla\phi \\
&= \int_K (\mathcal{L}_0\phi)\,\mathbf{a}\cdot\nabla\phi \\
&= -\kappa\int_K \Delta u\,\mathbf{a}\cdot\nabla\phi \\
&\le \kappa\|\Delta u\|_{L^2(K)}\|\mathbf{a}\cdot\nabla\phi\|_{L^2(K)},
\end{aligned}
$$

which gives, after dividing by $\|\mathbf{a}\cdot\nabla\phi\|_{L^2(K)}$,

$$
\|\mathbf{a}\cdot\nabla\phi\|_{L^2(K)} \le \kappa\|\Delta u\|_{L^2(K)},
$$

and, thanks to the Poincaré inequality,

$$
\|\phi\|_{L^2(K)} \le h_K\kappa|\mathbf{a}|^{-1}\|\Delta u\|_{L^2(K)}. \tag{3.39}
$$

Using (3.38), (3.39) and the Cauchy-Schwartz's inequality, we have

$$
I \le \kappa\sum_{K\in\mathcal{T}_h}\|\Delta u\|_{L^2(K)}\|v_R\|_{L^2(K)}
$$

and

$$
II \le \kappa\sum_{K\in\mathcal{T}_h} h_K\|\Delta u\|_{L^2(K)}|v_R|_{H^1(K)}.
$$

Using again the Cauchy-Schwartz's inequality and the Poincaré inequality we end with (3.36) □

We now introduce the energy norm related to $\widetilde{B}$:

$$
\begin{aligned}
|\!|\!|w|\!|\!|^2_{\widetilde{B}} := {}& \kappa|w|^2_{H^1(\Omega)} + \sum_{e\in\mathcal{E}_h}[\![\mathbf{a}]\!]_{|e}\|w\|^2_{L^2(e)} \\
&+ \sum_{e\in\mathcal{E}_h}(\!|\mathbf{a}|\!)^{-1}_{|e}\|[\![\widetilde{H}(w)\mathbf{a}]\!]\|^2_{L^2(e)} + \sum_{K\in\mathcal{T}_h}\sigma_{|K}\|\widetilde{H}(w)\|^2_{L^2(K)} \\
= {}& \kappa|w|^2_{H^1(\Omega)} + \|[\![\mathbf{a}]\!]^{1/2}w\|^2_{L^2(\Gamma_h)} \\
&+ \|(\!|\mathbf{a}|\!)^{-1/2}[\![\widetilde{H}(w)\mathbf{a}]\!]\|^2_{L^2(\Gamma_h)} + \|\sigma^{1/2}\widetilde{H}(w)\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Lemma 5 *The bilinear form $\widetilde{B}$ is coercive:*

$$
|\!|\!|w|\!|\!|^2_{\widetilde{B}} \preceq \widetilde{B}(w,w), \qquad \forall w\in H^1_0(\Omega). \tag{3.40}
$$

Proof. We have from (2.24)

$$\begin{aligned}\widetilde{B}(w,w) &= \kappa|w|^2_{H^1(\Omega)} + \sum_{e\in\mathcal{E}_h} [\![\mathbf{a}]\!]_{|e}\|w\|^2_{L^2(e)} - \sum_{e\in\mathcal{E}_h}\int_e [\![\widetilde{H}(w)\mathbf{a}]\!]\, w \\ &= \kappa|w|^2_{H^1(\Omega)} + I + II,\end{aligned} \tag{3.41}$$

where we have introduced the two terms

$$I := \frac{1}{2}\sum_{e\in\mathcal{E}_h} [\![\mathbf{a}]\!]_{|e}\|w\|^2_{L^2(e)},$$

and

$$II := \frac{1}{2}\sum_{e\in\mathcal{E}_h} [\![\mathbf{a}]\!]_{|e}\|w\|^2_{L^2(e)} - \sum_{e\in\mathcal{E}_h}\int_e [\![\widetilde{H}(w)\mathbf{a}]\!]\, w.$$

Since $\mathcal{L}_0\widetilde{H}(w)_{|K} = 0$, for any element K, we also have

$$\begin{aligned}0 &= \sum_{K\in\mathcal{T}_h}\int_K (\mathcal{L}_0\widetilde{H}(w))\,\widetilde{H}(w) \\ &= \sum_{K\in\mathcal{T}_h}\left[\int_K \widetilde{H}(w)\,\mathbf{a}\cdot\nabla\widetilde{H}(w) + \int_K \sigma\widetilde{H}(w)^2\right] \\ &= \sum_{K\in\mathcal{T}_h}\left[\frac{1}{2}\int_K \mathbf{a}\cdot\nabla(\widetilde{H}(w)^2) + \sigma\int_K \widetilde{H}(w)^2\right] \\ &= \sum_{K\in\mathcal{T}_h}\left[\frac{1}{2}\int_{\partial K} \widetilde{H}(w)^2\mathbf{a}\cdot\mathbf{n}_K + \sigma\int_K \widetilde{H}(w)^2\right] \\ &= \frac{1}{2}\sum_{e\in\mathcal{E}_h}\int_e [\![\widetilde{H}(w)^2\mathbf{a}]\!] + \sum_{K\in\mathcal{T}_h}\sigma_{|K}\|\widetilde{H}(w)\|^2_{L^2(K)}.\end{aligned} \tag{3.42}$$

Therefore adding $\frac{1}{2}\sum_{e\in\mathcal{E}_h}\int_e [\![\widetilde{H}(w)^2\mathbf{a}]\!] + \sum_{K\in\mathcal{T}_h}\sigma_{|K}\|\widetilde{H}(w)\|^2_{L^2(K)}$ to II does not change its value:

$$\begin{aligned}II &= \frac{1}{2}\sum_{e\in\mathcal{E}_h}\int_e \left[[\![\mathbf{a}]\!]w^2 - 2[\![\widetilde{H}(w)\mathbf{a}]\!]w + [\![\widetilde{H}(w)^2\mathbf{a}]\!]\right] + \sum_{K\in\mathcal{T}_h}\sigma_{|K}\|\widetilde{H}(w)\|^2_{L^2(K)} \\ &= \frac{1}{2}\sum_{e\in\mathcal{E}_h}\int_e [\![(\widetilde{H}(w)-w)^2\mathbf{a}]\!] + \sum_{K\in\mathcal{T}_h}\sigma_{|K}\|\widetilde{H}(w)\|^2_{L^2(K)}.\end{aligned}$$

In order to complete the proof of (3.40), it is enough to show that

$$\sum_{e\in\mathcal{E}_h} (\!|\mathbf{a}|\!)^{-1}_{|e}\|[\![\widetilde{H}(w)\mathbf{a}]\!]\|^2_{L^2(e)} + \sum_{K\in\mathcal{T}_h}\sigma_{|K}\|\widetilde{H}(w)\|^2_{L^2(K)} \preceq \frac{1}{2}I + II,$$

or, in particular, that on any edge $e\in\mathcal{E}_h$ we have

$$(\!|\mathbf{a}|\!)^{-1}_{|e}\|[\![\widetilde{H}(w)\mathbf{a}]\!]\|^2_{L^2(e)} \preceq [\![\mathbf{a}]\!]_{|e}\|w\|^2_{L^2(e)} + 2\int_e [\![(\widetilde{H}(w)-w)^2\mathbf{a}]\!]. \tag{3.43}$$

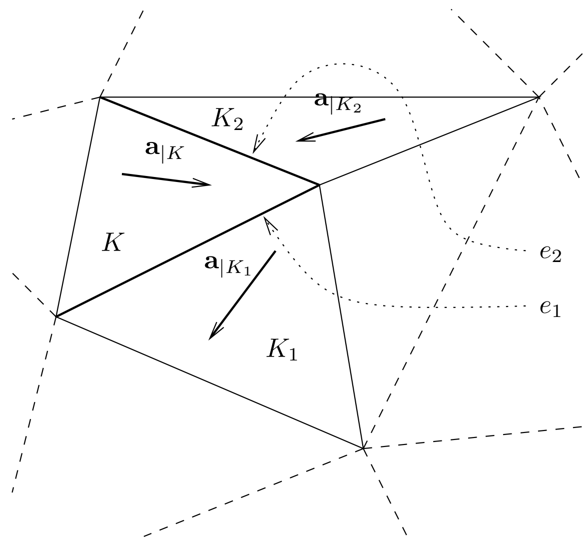

Figure 1: The element K has outflow boundary $\partial K^+ = e_1 \cup e_2$; e_1 also belongs to the inflow boundary ∂K_1^-, while e_2 also belongs to the outflow boundary ∂K_2^+

This is actually obvious when $e \subset \partial\Omega$. For internal edges, we distinguish two typical situations, which are shown in Figure 1.

The edge e_1 is an outflow edge for K and an inflow edge for K_1; since $\widetilde{H}(w)_{|K_1}$ and $w_{|K_1}$ have the same trace on e_1, we get $[\![(\widetilde{H}(w)-w)^2\mathbf{a}]\!] = (\widetilde{H}(w)_{|K}-\widetilde{H}(w)_{|K_1})^2\mathbf{a}_{|K}\cdot \mathbf{n}_K$, and therefore

$$\begin{aligned}
(\!|\mathbf{a}|\!)_{|e_1}^{-1}[\![\widetilde{H}(w)\mathbf{a}]\!]^2 &= (\!|\mathbf{a}|\!)_{|e_1}^{-1}(\widetilde{H}(w)_{|K}\mathbf{a}_{|K}\cdot\mathbf{n}_{|K} + \widetilde{H}(w)_{|K_1}\mathbf{a}_{|K_1}\cdot\mathbf{n}_{|K_1})^2 \\
&= (\!|\mathbf{a}|\!)_{|e_1}^{-1}[(\widetilde{H}(w)_{|K} - \widetilde{H}(w)_{|K_1})\mathbf{a}_{|K}\cdot\mathbf{n}_{|K} \\
&\qquad\qquad + \widetilde{H}(w)_{|K_1}(\mathbf{a}_{|K_1}\cdot\mathbf{n}_{|K_1} + \mathbf{a}_{|K}\cdot\mathbf{n}_{|K})]^2 \\
&\preceq 2(\widetilde{H}(w)_{|K} - \widetilde{H}(w)_{|K_1})^2\mathbf{a}_{|K}\cdot\mathbf{n}_K \\
&\qquad + (\!|\mathbf{a}|\!)_{|e_1}^{-1}\widetilde{H}(w)_{|K_1}^2(\mathbf{a}_{|K_1}\cdot\mathbf{n}_{|K_1} + \mathbf{a}_{|K}\cdot\mathbf{n}_{|K})^2 \\
&\preceq 2[\![(\widetilde{H}(w)-w)^2\mathbf{a}]\!] + [\![\mathbf{a}]\!]w_{|K}^2.
\end{aligned}$$

Consider now the edge e_2, which is an outflow edge for both K and K_2. In this case both $\mathbf{a}_{|K} \cdot \mathbf{n}_{|K}$ and $\mathbf{a}_{|K_1} \cdot \mathbf{n}_{|K_1}$ are non-negative, whence

$$\begin{aligned}
(\!|\mathbf{a}|\!)^{-1}_{|e_2} [\![\widetilde{H}(w)\mathbf{a}]\!]^2 &= (\!|\mathbf{a}|\!)^{-1}_{|e_2} (\widetilde{H}(w)_{|K}\mathbf{a}_{|K} \cdot \mathbf{n}_{|K} + \widetilde{H}(w)_{|K_2}\mathbf{a}_{|K_2} \cdot \mathbf{n}_{|K_2})^2 \\
&= (\!|\mathbf{a}|\!)^{-1}_{|e_2} [(\widetilde{H}(w)_{|K} - w_{|K})\mathbf{a}_{|K} \cdot \mathbf{n}_{|K} \\
&\qquad + (\widetilde{H}(w)_{|K_2} - w_{|K_2})\mathbf{a}_{|K_2} \cdot \mathbf{n}_{|K_2} + [\![\mathbf{a}]\!] w]^2 \\
&\preceq (\widetilde{H}(w)_{|K} - w_{|K})^2 \mathbf{a}_{|K} \cdot \mathbf{n}_{|K} \\
&\quad + (\widetilde{H}(w)_{|K_2} - w_{|K_2})^2 \mathbf{a}_{|K_2} \cdot \mathbf{n}_{|K_2} + [\![\mathbf{a}]\!] w^2 \\
&\preceq 2[\![(\widetilde{H}(w) - w)^2 \mathbf{a}]\!] + [\![\mathbf{a}]\!] w^2,
\end{aligned}$$

which gives (3.43) for any internal edge e, and ends the proof.□

Lemma 6 *The bilinear form $\widetilde{B}$ verifies the continuity estimate:*

$$\widetilde{B}(w,v) \preceq |\!|\!| w |\!|\!|_{\widetilde{B}} \left[\kappa^{1/2} |v|_{H^1(\Omega)} + \left(\sum_{K \in \mathcal{T}_h} |\mathbf{a}_{|K}| \|v\|^2_{L^2(\partial K)} \right)^{1/2} \right], \qquad \forall w, v \in H^1_0(\Omega). \tag{3.44}$$

Proof. Use (2.24) and the Cauchy-Schwartz's inequality □

We can now state the a priori error estimate for the *approximated* RFB method. We assume, for the sake of simplicity, $h_K \leq 1$, for all $K \in \mathcal{T}_h$ in order to simplify (3.36) and derive a simpler estimate.

Theorem 2 *Let u be the solution of* (1.2)*, $\widetilde{u}_R$ be its numerical approximation by the* approximated *RFB method* (2.21)–(2.23)*, $h_K \leq 1, \forall K$ and $0 < s \leq p$. Then*

$$|\!|\!| u - \widetilde{u}_R |\!|\!|_{\widetilde{B}} \leq C(s) \left\{ \sum_{K \in \mathcal{T}_h} \left[\kappa h_K^{2s} + |\mathbf{a}_{|K}| h^{2s+1} \right] |u|^2_{H^{s+1}(K)} + \kappa |u|^2_{H^2(K)} \right\}^{1/2} \tag{3.45}$$

Proof. By means of (3.40) we obtain

$$\begin{aligned}
|\!|\!| u - \widetilde{u}_R |\!|\!|^2_{\widetilde{B}} &\leq \widetilde{B}(u - \widetilde{u}_R, u - \widetilde{u}_R) \\
&= \widetilde{B}(u - \widetilde{u}_R, u - u_I) + \widetilde{B}(u - \widetilde{u}_R, u_I - \widetilde{u}_R) \\
&= I + II,
\end{aligned} \tag{3.46}$$

when u_I is the nodal interpolant of u and verifies (3.35). Using (3.44) and (3.35), with $0 < s \leq p$, we have

$$\begin{aligned}
I &\preceq |\!|\!| u - \widetilde{u}_R |\!|\!|_{\widetilde{B}} \left[\kappa^{1/2} |u - u_I|_{H^1(\Omega)} + \left(\sum_{K \in \mathcal{T}_h} |\mathbf{a}_{|K}| \|u - u_I\|^2_{L^2(\partial K)} \right)^{1/2} \right] \\
&\leq C(s) |\!|\!| u - \widetilde{u}_R |\!|\!|_{\widetilde{B}} \left[\sum_{K \in \mathcal{T}_h} \left(\kappa h_K^{2s} + |\mathbf{a}_{|K}| h^{2s+1} \right) |u|^2_{H^{s+1}(K)} \right]^{1/2},
\end{aligned} \tag{3.47}$$

Using (3.36) and $h_K \leq 1, \forall K$, we get

$$\begin{aligned} II &\preceq \kappa^{1/2}|u_I - \widetilde{u}_R|_{H^1(\Omega)} \cdot \kappa^{1/2}|u|_{H^2(\Omega)} \\ &\preceq \left[\kappa^{1/2}|u - \widetilde{u}_R|_{H^1(\Omega)} + \kappa^{1/2}|u_I - u|_{H^1(\Omega)}\right] \kappa^{1/2}|u|_{H^2(\Omega)} \\ &\preceq \|u - \widetilde{u}_R\|_{\widetilde{B}} \cdot \kappa^{1/2}|u|_{H^2(\Omega)} + \kappa C(s)|u|_{H^2(\Omega)}. \end{aligned} \tag{3.48}$$

Finally (3.46)–(3.48) give (3.45) □

Remark 1 *The consistency error in the* approximated *RFB method, represented by the term* $\{\sum_{K\in\mathcal{T}_h} \kappa|u|^2_{H^2(K)}\}^{1/2}$ *in* (3.45)*, has a minor effect when the diffusion coefficient* κ *is small compared to the mesh-size. Moreover the result can be improved in some circumstances: for example, when* $\sigma \leq |\mathbf{a}|$ *we can change* (3.36) *in*

$$\widetilde{B}(u - \widetilde{u}_R, v_R) \preceq \kappa^{1/2}|v_R|_{H^1(\Omega)} \left[\sum_{K\in\mathcal{T}_h} \kappa h_K^2 \|\Delta u\|^2_{L^2(K)}\right]^{1/2}$$

and the final consistency error is smaller than the other terms: we get, for example, the error estimate:

$$\|u - \widetilde{u}_R\|_{\widetilde{B}} \leq C \left\{\sum_{K\in\mathcal{T}_h} \left[\kappa h_K^2 + |\mathbf{a}_{|K}|h^3\right] |u|^2_{H^2(K)}\right\}^{1/2} \tag{3.49}$$

for $1 \leq s \leq p$*. Note moreover that the consistency error disappears when using* $\mathcal{L}^*_\kappa v_R$ *instead of* $\mathcal{L}^*_0 v_R$ *in* (2.19) *and* (2.20)*. This does not increase the numerical complexity of the final algorithm at all, so that it can be done without extra difficulties for problems where* κ *is not so small.*

Remark 2 *We have proposed a non rigorous justification (in §2) of the* approximated *RFB formulation, which is valid when the advection term is dominant with respect to the diffusion term (condition* (2.25)*) and with respect to the reaction term (condition* (2.26)*). In the mentioned case one can recognize that* $\|\cdot\|_B \approx \|\cdot\|_{\widetilde{B}}$*, and that the two error estimates given in Theorem 1 and 2 are quite similar.*

Remark 3 *As usual in this context, Theorem 1 and Theorem 2 give useless error estimates when the exact solution u present layers, which is typical of a very small diffusion coefficient* κ*. On the other hand, one may think to derive a* local *analogous of our estimates—using known techniques*[11] *—which guarantee the robustness of the methods.*

Remark 4 *In the error analysis we have never used the assumption that the sum between* V_R *and* V_U *is direct (i.e., the restriction* $k \leq 2$ *when defining* V_R*). Consider now higher order elements: the* exact *RFB variational formulation (2.12) is indeed well posed just for the degrees of freedom of* u_R *on the* Γ_h*, and accordingly the error estimate given in Theorem 1 takes into accounts just those degrees of freedom, because* $\|\cdot\|_B$ *is in essence a norm on* Γ_h*. In order to have a complete numerical scheme we should give an algorithm for recovering accurately the degrees of freedom* *of u_R which live inside the elements,* after *the computation of the degrees of freedom on Γ_h by means of (2.12). In a similar way, one should think to the* approximated *RFB formulation (2.21) as an algorithm for the computation of the degrees of freedom on Γ_h; note that in this case the error estimate in Theorem 2 also gives a control on the degrees of freedom inside the elements, because $\|\cdot\|_{\widetilde{B}}$ contains the term $\kappa|\cdot|_{H^1(\Omega)}$, but this is a very weak control, and again a* post-processing *for the internal degrees of freedom is required. We do not develop further this point in the present paper.*

4. Numerical tests

In this section we describe the numerical algorithm derived from (2.21) and some numerical experiments.
The solution of (2.17) is the solution of an ordinary differential equation in the streamline direction, and, under the assumption of piecewise constant data, the solution for $\sigma \neq 0$ are simply[1]

$$\widetilde{M}(w_R)_{|K}(\mathbf{x}) = -w_R(\mathbf{x}) + w_R(\mathbf{x}^-)\, E(\mathbf{x}), \tag{4.50}$$

Similarly for (2.18)

$$\widetilde{F}(f)_{|K}(\mathbf{x}) = \frac{f}{\sigma}\left[1 - E(\mathbf{x})\right], \tag{4.51}$$

where $\mathbf{x}^-$ is the coordinate at the inflow boundary obtained by passing a line through $\mathbf{x}$ in the direction of $\mathbf{a}$ and $E(\mathbf{x}) = e^{-\frac{\sigma}{|\mathbf{a}|^2}\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}^-)}$. For simplicity, we will set $w_R^-(\mathbf{x}) = w_R(\mathbf{x}^-)$. Note that if $\mathbf{x} \in \partial K^-$, $\mathbf{x}^- \equiv \mathbf{x}$ and $E(\mathbf{x}) \equiv 1$.
We are now in a position to write (2.21) as

$$\begin{aligned} a(u_R, v_R) \quad &+ \sum_{K\in\mathcal{T}_h} (u_R - u_R^- E, \mathbf{a}\cdot\nabla v_R - \sigma v_R)_K \\ &= (f, v_R) + \sum_{K\in\mathcal{T}_h} (\frac{f}{\sigma}\left[1-E\right], \mathbf{a}\cdot\nabla v_R - \sigma v_R)_K, \forall v_R \in V_R. \end{aligned} \tag{4.52}$$

A further simplification is immediate by noting that the zero order term from $a(u_R, v_R)$ cancels with a term in the additional modification in the left-hand side, and (f, v_R) cancels with a term in the additional modification in the right-hand side, reducing the method to:

$$\begin{aligned} &\kappa(\nabla u_R, \nabla v_R) + (\mathbf{a}\cdot\nabla u_R, v_R) \\ &\quad + \sum_{K\in\mathcal{T}_h} (u_R, \mathbf{a}\cdot\nabla v_R)_K - \sum_{K\in\mathcal{T}_h} (u_R^- E, \mathbf{a}\cdot\nabla v_R - \sigma v_R)_K \\ &\quad = \sum_{K\in\mathcal{T}_h} (\frac{f}{\sigma}\left[1 - E\right], \mathbf{a}\cdot\nabla v_R)_K + \sum_{K\in\mathcal{T}_h} (f\, E, v_R)_K \quad \forall v_R \in V_R. \end{aligned} \tag{4.53}$$

Last term in the left-hand side can be simplified[1] integrating by parts and using the fact that $\mathbf{a} \cdot \nabla E + \sigma E = 0$ in K and that r_R^- is constant along $\mathbf{a}$

$$(u_R^- E, \mathbf{a} \cdot \nabla v_R - \sigma v_R)_K = \int_{\partial K} u_R^- v_R E \mathbf{a} \cdot \mathbf{n} \tag{4.54}$$

and also the right-hand side

$$(\frac{f}{\sigma}[1 - E], \mathbf{a} \cdot \nabla v_R)_K + (f E, v_R)_K = \int_{\partial K^+} \frac{f}{\sigma}[1 - E]\, v_R \mathbf{a} \cdot \mathbf{n}. \tag{4.55}$$

Then we end up with the following expression used for programming

$$\begin{aligned} &\kappa(\nabla u_R, \nabla v_R) + (\mathbf{a} \cdot \nabla u_R, v_R) \\ &\quad + \sum_{K \in \mathcal{T}_h} (u_R, \mathbf{a} \cdot \nabla v_R)_K - \sum_{K \in \mathcal{T}_h} \int_{\partial K} u_R^- E v_R \mathbf{a} \cdot \mathbf{n} \\ &\quad = \sum_{K \in \mathcal{T}_h} \int_{\partial K^+} \frac{f}{\sigma}[1 - E]\, v_R \mathbf{a} \cdot \mathbf{n} \end{aligned} \tag{4.56}$$

If $\sigma = 0$ this algorithm reduces to[1]

$$\begin{aligned} &\kappa(\nabla u_R, \nabla v_R) + (\mathbf{a} \cdot \nabla u_R, v_R) \\ &\quad + \sum_{K \in \mathcal{T}_h} (u_R, \mathbf{a} \cdot \nabla v_R)_K - \sum_{K \in \mathcal{T}_h} \int_{\partial K} u_R^- v_R \mathbf{a} \cdot \mathbf{n} \\ &\quad = \sum_{K \in \mathcal{T}_h} \int_{\partial K^+} \frac{f}{|\mathbf{a}|^2} \mathbf{a} \cdot (\mathbf{x} - \mathbf{x}^-) \mathbf{a} \cdot \nabla v_R. \end{aligned} \tag{4.57}$$

We present a numerical experiment concerning to the propagation of a layer into a unit square domain, comparing the residual-free method with piecewise linear and quadratic elements in two dimensions.
We take $\kappa = 10^{-4}$, constant velocity field $\mathbf{a} = (1, 3)$ and $\sigma = f$ vary with values 0, 1 and 10^3. Boundary conditions are displayed in Fig. 2, and mesh in Fig. 3.
We present the elevation plots for linear and quadratic finite elements in Fig. 4, 5 and 6. We have also solved this problem with Streamline Up-wind Petrov-Galerkin method linear and quadratic elements in Fig. 7, 8 and 9.

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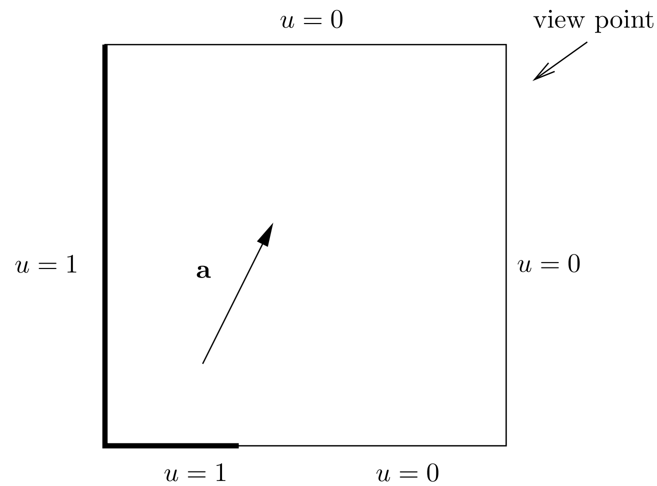

Figure 2: Boundary conditions.

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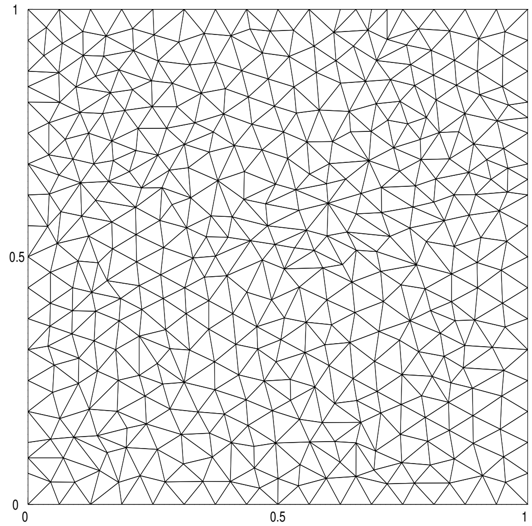

Figure 3: Mesh (804 triangles).

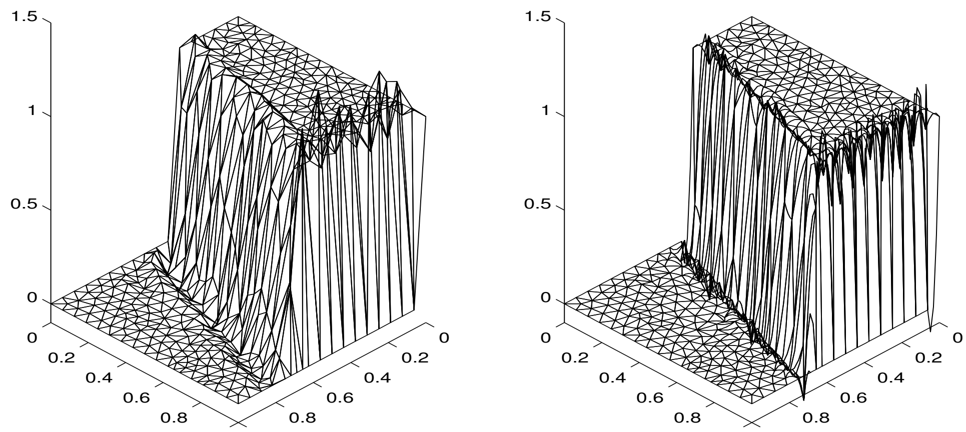

Figure 4: $\sigma = f = 0$, P1 (left) and P2 (right), RFB.

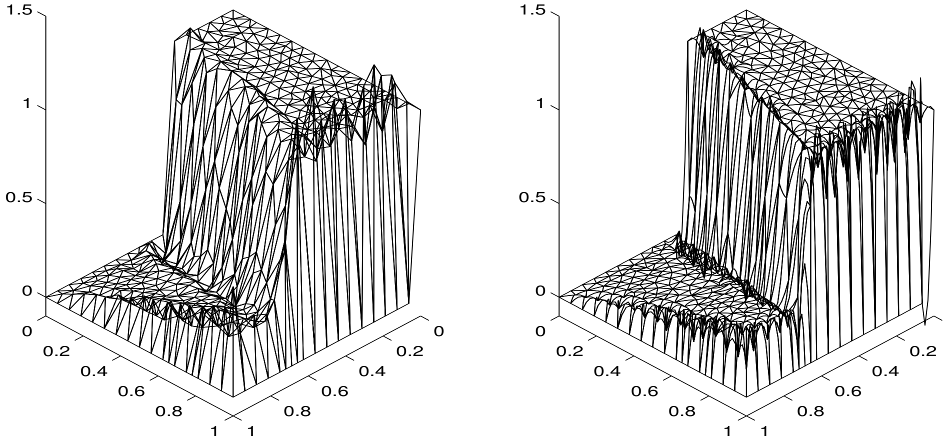

Figure 5: $\sigma = f = 1$, P1 (left) and P2 (right), RFB.

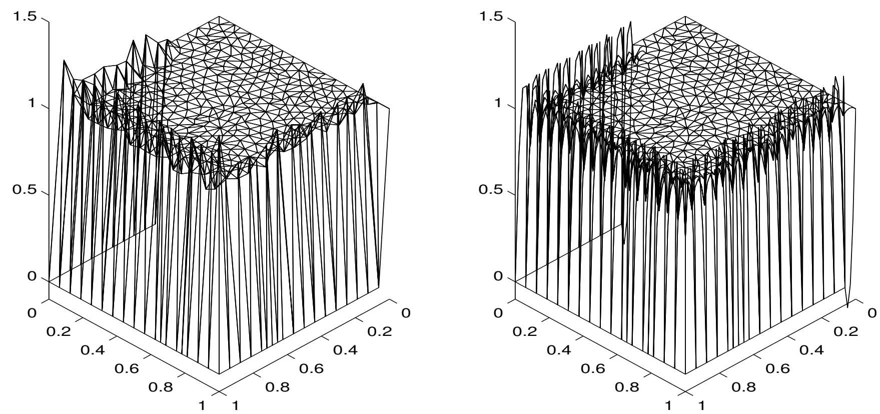

Figure 6: $\sigma = f = 10^3$, P1 (left) and P2 (right), RFB.

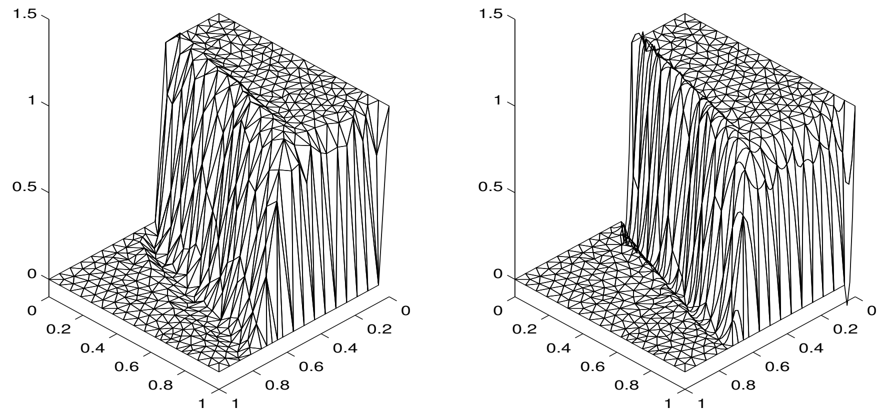

Figure 7: $\sigma = f = 0$, P1 (left) and P2 (right), SUPG.

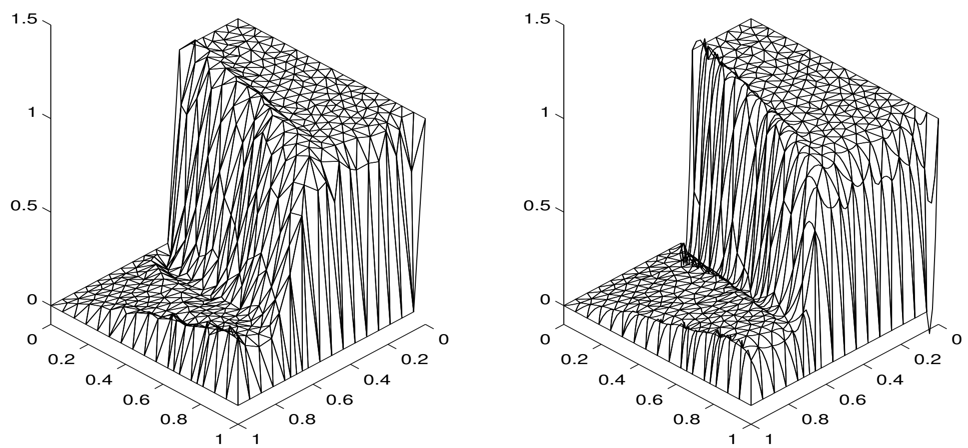

Figure 8: $\sigma = f = 10$, P1 (left) and P2 (right), SUPG.

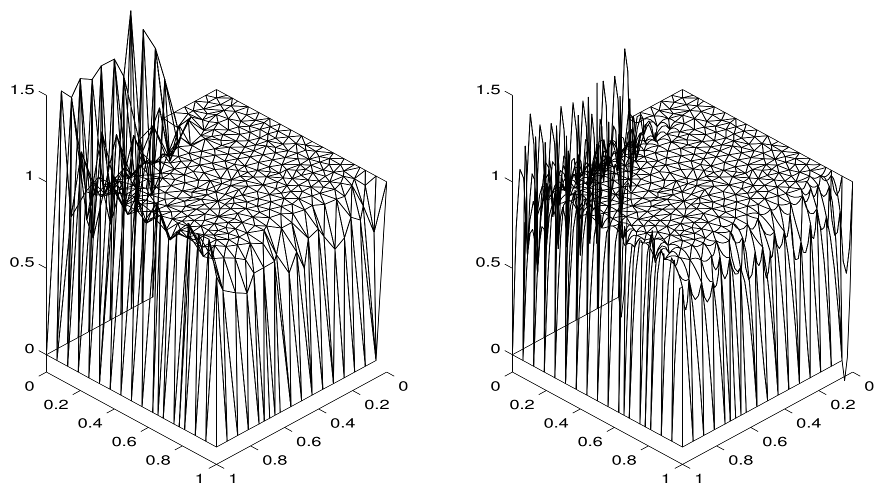

Figure 9: $\sigma = f = 10^3$, P1 (left) and P2 (right), SUPG.