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Analysis of the advection-diffusion operator using fractional order norms.

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Summary In this paper we obtain a family of optimal estimates for the linear advection-diffusion operator. More precisely we define norms on the domain of the operator, and norms on its image, such that it behaves as an isomorphism: it stays bounded as well as its inverse does, uniformly with respect to the diffusion parameter. The analysis makes use of the interpolation theory between function spaces. One motivation of the present work is our interest in the theoretical properties of stable numerical methods for this kind of problem: we will only give some hints here and we will take a deeper look in a further paper.

1 Introduction

We consider the linear advection-diffusion operator

$$\mathcal{L}_\varepsilon := -\varepsilon \Delta + \mathbf{c} \cdot \nabla, \tag{1}$$

and the related p.d.e. problem

$$\begin{cases} \mathcal{L}_\varepsilon u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where the unknown $u \equiv u_\varepsilon$ is a real function on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^n$) and f is the source term. The diffusion coefficient ε is assumed to be constant, while the advection velocity $\mathbf{c}$ is a vector field on Ω. When ε is strictly positive and with some assumptions on $\mathbf{c}$, the operator $\mathcal{L}_\varepsilon$ is elliptic—it fits into the Lax-Milgram framework—and in particular it is an algebraic and topological isomorphism from $H^1_0(\Omega)$ into $H^{-1}(\Omega) \equiv (H^1_0(\Omega))^*$. Nevertheless its inverse $\mathcal{L}_\varepsilon^{-1}$, which gives the solution to (2), is not uniformly bounded with respect to ε:

$$\|\mathcal{L}_\varepsilon^{-1}\|_{L(H^{-1};H^1_0)} := \sup_{\phi \in H^{-1}} \frac{\|\mathcal{L}_\varepsilon^{-1}\phi\|_{H^1_0}}{\|\phi\|_{H^{-1}}} \simeq \varepsilon^{-1}.$$

On the other hand one easily recognizes—for example by computing, when possible, the solutions of (2) related to regular source terms—that the problem is in some way well conditioned—also for very small values of ε, even though boundary layers or internal layers appear: actually the solution, as $\varepsilon \to 0^+$, approaches the solution of the purely hyperbolic problem, with $\varepsilon = 0$ and suitable boundary conditions.

This leads us to define a couple of Banach spaces W and V^* in such a way that $\mathcal{L}_\varepsilon$ behaves as a uniformly bounded isomorphism between the two spaces, with uniformly bounded inverse (with respect to ε); using notation (7), it will read:

$$\|\mathcal{L}_\varepsilon\|_{L(W;V^*)} \simeq \|\mathcal{L}_\varepsilon^{-1}\|_{L(V^*;W)} \simeq 1. \tag{3}$$

By introducing the bilinear form $a_\varepsilon : W \times V \to \mathbb{R}$ defined as

$$a_\varepsilon(w, v) := {}_{V^*}\langle \mathcal{L}_\varepsilon w, v \rangle_V$$

and by using the Banach's closed range theorem, (3) is equivalent to the conditions

$$\sup_{w \in W} \sup_{v \in V} \frac{a_\varepsilon(w, v)}{\|w\|_W \|v\|_V} \leq \kappa < +\infty; \tag{4}$$

$$\inf_{w \in W} \sup_{v \in V} \frac{a_\varepsilon(w, v)}{\|w\|_W \|v\|_V} \geq \gamma > 0; \tag{5}$$

$$\forall v \in V \backslash \{0\}, \exists w \in W \text{ such that } a_\varepsilon(w, v) \neq 0, \tag{6}$$

with κ and γ independent of ε. In this work we construct a family of spaces W and V that give (3): from the algebraic point of view we shall set $W \equiv V \equiv H^1_0(\Omega)$, but their norms $\|\cdot\|_W$ and $\|\cdot\|_V$ will take into account the anisotropic effect due to the convection term of the operator and the dependence on ε of the diffusive regularization. As a result, we are able to make the well posedness of the problem (2) precise, as one gets from (4)–(6) that a perturbation δf (in the source term f of (2)) yields a variation δu (of the solution u) which is smaller in relative magnitude (up to a multiplicative constant independent of ε):

$$\frac{\|\delta u\|_W}{\|u\|_W} \le \kappa\gamma^{-1}\frac{\|\delta f\|_{V^*}}{\|f\|_{V^*}}.$$

We obtain our results by means of the interpolation theory of function spaces (see [21] and [26]). Actually our result will be true for a more general class of operators, where a skew-symmetric part is *singularly* perturbed by a symmetric higher order term.

Different estimates for (1) have been proposed by many authors; we particularly refer the interested reader to [23] for a wide survey, to the very recent [14], where estimates in anisotropic Lebesgue spaces are proposed and applied to the analysis of numerical methods, and also to [3], where estimates are derived by interpolation. As far as we know, our work is the first one in which estimates like (3) have been derived.

There is a huge literature devoted to numerical methods for (2) (see, for example, [23] and the reference therein), which is indeed a model problem for more complex situations arising, for example, in computational fluid dynamic. Actually, our interest in the present analysis of the advection-diffusion differential operator is mainly motivated by its relevance for the construction and understanding of finite element methods for solving (2): in the last part of the present paper we shall briefly show how (3) is related to the classical *a priori* error theory. A deeper investigation of numerical aspects will be the subject of further works.

The outline of the paper is as follows: in §2 we present the notation and assumptions while §3 is devoted to the analysis. Finally in §4, as mentioned above, we briefly discuss some consequences for numerical methods.

2 Preliminaries

We denote by $L^p \equiv L^p(\Omega)$, the usual Lebesgue spaces endowed with the norm $\|\cdot\|_{L^p}$, and by $H^1 \equiv H^1(\Omega)$ the usual Sobolev space endowed with the norm $\|\cdot\|_{H^1}$ and seminorm $|\cdot|_{H^1} \equiv \|\nabla\cdot\|_{L^2}$; $H^1_0 \equiv H^1_0(\Omega)$ is the space of functions contained in $H^1(\Omega)$ with zero trace on $\partial\Omega$, endowed with the norm $|\cdot|_{H^1}$; finally $H^{-1} \equiv H^{-1}(\Omega)$ denotes the dual space of $H^1_0(\Omega)$ endowed with the dual norm $\|\cdot\|_{H^{-1}}$ and the usual pairing ${}_{H^{-1}}\langle\cdot,\cdot\rangle_{H^1_0}$. We shall make use of the interpolation theory of function spaces; more specifically we shall use the *K-method* and we refer to [26] for its definition, notation and properties.

In the sequel C denotes a generic constant whose value, possibly different at any occurrence, does not depend on any other mathematical quantity appearing in the analysis (e.g., ε, $\mathbf{c}$, θ, p, w, v, f). We also adopt the notational convention

$$\begin{aligned} \alpha \preceq \beta \quad &\Longleftrightarrow \quad \alpha \leq C\beta, \\ \alpha \simeq \beta \quad &\Longleftrightarrow \quad \alpha \preceq \beta \text{ and } \beta \preceq \alpha. \end{aligned} \tag{7}$$

We restrict ourselves to the case of smooth and skew-symmetric advection terms, that means

$$\operatorname{div}(\mathbf{c}) = 0; \tag{H1}$$

this also gives the coercivity of a_ε on $H_0^1 \times H_0^1$ (without uniformity with respect to ε). With small changes in the analysis, one could also take into account fields $\mathbf{c}$ with non-positive divergence.

$\mathcal{L}_0$ is an hyperbolic operator and therefore it has a different nature from $\mathcal{L}_\varepsilon$, $\varepsilon > 0$; the associated p.d.e. problem (compare with (2)) now reads

$$\begin{cases} \mathcal{L}_0 u_0 = f & \text{in } \Omega \\ \quad u_0 = 0 & \text{on } \partial\Omega^-, \end{cases} \tag{8}$$

where $\partial\Omega^-$ denotes the *inflow* boundary

$$\partial\Omega^- := \{\mathbf{x} \in \partial\Omega \text{ such that } \mathbf{c}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}, \tag{9}$$

and $\mathbf{n}$ denotes the outward normal unit vector defined almost everywhere on $\partial\Omega$. In some parts of our analysis we need problem (8) to be well posed too: this is in fact a very natural assumption, in order to have problem (2) well conditioned when ε is very small. Under the assumption:

$$\text{there exists a smooth } \eta : \Omega \to \mathbb{R} \text{ such that } \nabla\eta \cdot \mathbf{c} \geq C > 0, \tag{H2}$$

problem (8) turns out to be well posed for any $f \in L^2$; we refer to [15] for the details.

3 Main results

Let us define the following norms (and spaces):

$$\begin{aligned}
&\|w\|_{A_0} := \varepsilon|w|_{H^1} + \|\mathbf{c}\cdot\nabla w\|_{H^{-1}}, && \forall w \in A_0 := H_0^1 \\
&\|w\|_{A_1} := |w|_{H^1}, && \forall w \in A_1 := H_0^1 \\
&\|\phi\|_{B_0} := \|\phi\|_{A_1^*} = \sup_{v \in H_0^1} \frac{{}_{H^{-1}}\langle \phi, v\rangle_{H_0^1}}{\|v\|_{A_1}} && \forall \phi \in B_0 := H^{-1} \\
&\|\phi\|_{B_1} := \|\phi\|_{A_0^*} = \sup_{v \in H_0^1} \frac{{}_{H^{-1}}\langle \phi, v\rangle_{H_0^1}}{\|v\|_{A_0}} && \forall \phi \in B_1 := H^{-1}.
\end{aligned} \tag{10}$$

We prove now two unusual estimates for $\mathcal{L}_\varepsilon$, with respect to the norms defined above.

Proposition 1 *Under assumption (H1) we have*

$$\|w\|_{A_0} \simeq \|\mathcal{L}_\varepsilon w\|_{B_0}, \qquad \forall w \in H_0^1, \tag{11}$$
$$\|w\|_{A_1} \simeq \|\mathcal{L}_\varepsilon w\|_{B_1}, \qquad \forall w \in H_0^1. \tag{12}$$

Proof Let w be a generic function in H_0^1. The continuity of $\mathcal{L}_\varepsilon$ is trivial in both cases:

$$\|\mathcal{L}_\varepsilon w\|_{B_i} \preceq \|w\|_{A_i}, \qquad i = 1, 2.$$

Thanks to the coercivity of $\mathcal{L}_\varepsilon$

$${}_{H^{-1}}\langle \mathcal{L}_\varepsilon w, w\rangle_{H_0^1} = \varepsilon |w|_{H^1}^2,$$

one easily gets $\varepsilon |w|_{H^1} \leq \|\mathcal{L}_\varepsilon w\|_{H^{-1}} = \|\mathcal{L}_\varepsilon w\|_{B_0}$. Therefore

$$\begin{aligned}
\|\mathbf{c} \cdot \nabla w\|_{H^{-1}} &\leq \|\mathcal{L}_\varepsilon w\|_{H^{-1}} + \|\varepsilon \Delta w\|_{H^{-1}} \\
&\leq \|\mathcal{L}_\varepsilon w\|_{H^{-1}} + \varepsilon |w|_{H^1} \\
&\preceq \|\mathcal{L}_\varepsilon w\|_{H^{-1}},
\end{aligned}$$

and so

$$\|w\|_{A_0} \preceq \|\mathcal{L}_\varepsilon w\|_{B_0},$$

that gives (11).

An analogous estimate holds true for the transpose $\mathcal{L}_\varepsilon^* := -\varepsilon\Delta - \mathbf{c} \cdot \nabla$, and therefore (12) is obtained by a standard duality argument: given w, we define $\widetilde{w} \in H_0^1$ as the solution of $\mathcal{L}_\varepsilon^* \widetilde{w} = \Delta w$; integrating by parts we have

$$\begin{aligned}
\|\Delta w\|_{B_0} &= \sup_{v \in H_0^1} \frac{{}_{H^{-1}}\langle \Delta w, v\rangle_{H_0^1}}{|v|_{H^1}} \\
&= \sup_{v \in H_0^1} \frac{\int_\Omega \nabla w(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, d\mathbf{x}}{|v|_{H^1}} \\
&= |w|_{H^1} \\
&= \|w\|_{A_1},
\end{aligned} \tag{13}$$

which, together with $\|\widetilde{w}\|_{A_0} \preceq \|\mathcal{L}_\varepsilon^* \widetilde{w}\|_{B_0}$, gives

$$\begin{aligned}
\|w\|_{A_1}^2 &= {}_{H^{-1}}\langle \Delta w, w\rangle_{H_0^1} \\
&= {}_{H^{-1}}\langle \mathcal{L}_\varepsilon^* \widetilde{w}, w\rangle_{H_0^1} \\
&= {}_{H^{-1}}\langle \mathcal{L}_\varepsilon w, \widetilde{w}\rangle_{H_0^1} \\
&\le \|\mathcal{L}_\varepsilon w\|_{A_0^*} \|\widetilde{w}\|_{A_0} \\
&\preceq \|\mathcal{L}_\varepsilon w\|_{B_1} \|w\|_{A_1},
\end{aligned}$$

and (12) follows.

The previous proposition suggests two different choices for the spaces W and V in order to reach condition (3): in fact $\mathcal{L}_\varepsilon$ is a uniform isomorphism between A_0 and B_0, as well as between A_1 and B_1. Nevertheless both these choices are not fully satisfactory for our purposes. We easily recognize that the norm $\|\cdot\|_{A_0}$ appearing in the left hand side of (11) is very weak, because the part of it which is independent of ε—i.e., $\|\mathbf{c}\cdot\nabla(\cdot)\|_{H^{-1}}$—is only a seminorm; roughly speaking, for very small ε one can have solutions u with $\|u\|_{L^2}$ large and $\|u\|_{A_0}$ very small. On the other hand the norm $\|\cdot\|_{B_1}$ that appears in the left hand side of (12) is too strong: e.g., a constant source term $f = 1$ gives $\|f\|_{B_1} \simeq \varepsilon^{-1/2}$, as one expects in this case; indeed, because of the boundary layer, one also gets $\|u\|_{A_1} \simeq \varepsilon^{-1/2}$ for the related solution u, in agreement with (12).

Since the norms in (11) are too week, and the norms in (12) are too strong, we construct a family of intermediate estimates by means of the function spaces interpolation. We follow the notation and the definitions of [26]; for the reader's convenience, we recall the fundamental definition of *interpolated norm*: given $0 < \theta < 1$ and $1 \le p \le +\infty$ we define

$$\|w\|_{(A_0,A_1)_{\theta,p}} := \left[\int_0^{+\infty} \inf_{\substack{w_0\in A_0, w_1\in A_1,\\ w_0+w_1=w}} \left(t^{-\theta}\|w_0\|_{A_0} + t^{1-\theta}\|w_1\|_{A_1}\right)^p \frac{dt}{t}\right]^{\frac{1}{p}}. \tag{14}$$

Typically A_0 and A_1 are different from the algebraic point of view; in our case instead, given the definitions (10), w, w_0, w_1 belong to H_0^1 and (14) is just a new norm on H_0^1. Similarly here $\|w\|_{(B_0,B_1)}$ is a new norm on H^{-1}.

Proposition 2 *Assume (H1),* $0 < \theta < 1$ *and* $1 \leq p \leq +\infty$*; then we have*

$$\|w\|_{(A_0,A_1)_{\theta,p}} \simeq \|\mathcal{L}_\varepsilon w\|_{(A_0,A_1)^*_{1-\theta,p'}}, \qquad \forall w \in H^1_0, \tag{15}$$

where $1/p + 1/p' = 1$.

Proof By Proposition 1 $\mathcal{L}_\varepsilon$ is a linear operator uniformly bounded (with respect to ε) from A_i to B_i, $i = 1, 2$. Thanks to the interpolation theorem (see [26, §1.3.3]) $\mathcal{L}_\varepsilon$ is an uniformly bounded operator from $(A_0, A_1)_{\theta,p}$ into $(B_0, B_1)_{\theta,p}$, with $0 < \theta < 1$ and $1 \leq p \leq +\infty$. In the same way $\mathcal{L}_\varepsilon^{-1}$ turns out to be uniformly bounded from $(B_0, B_1)_{\theta,p}$ to $(A_0, A_1)_{\theta,p}$. Therefore

$$\|w\|_{(A_0,A_1)_{\theta,p}} \simeq \|\mathcal{L}_\varepsilon w\|_{(B_0,B_1)_{\theta,p}}. \tag{16}$$

Recall we are not constructing spaces with different regularity—$w \in (A_0, A_1)_{\theta,p}$ if and only if $w \in H^1_0$ as well as $\phi \in (B_0, B_1)_{\theta,p}$ if and only if $\phi \in H^{-1}$—as from the algebraic point of view $A_0 \equiv A_1 \equiv H^1_0$ and $B_0 \equiv B_1 \equiv H^{-1}$; we are just defining norms having a different dependence on the parameter ε, as we shall see in the sequel. Note moreover that the estimate (16) is uniform and independent of θ and p because we are using an *exact* interpolator functor (see [26, §1.2.2, Definition 2]).

When $p > 1$, the estimate (15) is equivalent to (16), thanks to [26, §1.11.2]. For $p = 1$ the referred theory gives $(B_0, B_1)_{\theta,1} \equiv [(A_0, A_1)^0_{1-\theta,+\infty}]^*$, where $(A_0, A_1)^0_{1-\theta,+\infty}$ denotes the closure of $A_0 \cap A_1$ in the topology of $(A_0, A_1)_{1-\theta,+\infty}$. Since in our case we have $A_0 + A_1 \equiv A_0 \cap A_1 \equiv H^1_0$ in the algebraic sense, as mentioned above, we get $(A_0, A_1)^0_{1-\theta,+\infty} \equiv (A_0, A_1)_{1-\theta,+\infty}$, and (15) follows.

In order to understand the structure of the norms appearing in Proposition 2, we separate the diffusive part (depending on ε) from the advective part. For this purpose, we introduce a new pair of spaces C_0 and C_1. We set $C_0 \equiv H^{-1}$, endowed with the natural norm $\|\phi\|_{C_0} := \|\phi\|_{H^{-1}}$. We denote by C_1 the set of functions $\phi = \mathcal{L}_0\Phi$, where Φ is a generic function belonging to H^1_0, and define $\|\phi\|_{C_1} := |\Phi|_{H^1}$; in other words $C_1 = \mathcal{L}_0(H^1_0)$, endowed with the norm which makes $\mathcal{L}_0$ an isometry between H^1_0 and C_1. The definition of C_1 makes sense under assumption (H2).

Proposition 3 *Assume (H1), (H2),* $0 < \theta < 1$ *and* $1 \leq p \leq +\infty$*; we have*

$$\|w\|_{(A_0,A_1)_{\theta,p}} \simeq \varepsilon^{1-\theta}|w|_{H^1} + \|\mathbf{c} \cdot \nabla w\|_{(C_0,C_1)_{\theta,p}}, \qquad \forall w \in H^1_0. \tag{17}$$

Proof We first prove $\varepsilon^{1-\theta}|w|_{H^1} + \|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{\theta,p}} \preceq \|w\|_{(A_0,A_1)_{\theta,p}}$. For any $w \in H^1_0$ we have

$$|w|_{H^1} \leq \varepsilon^{-1}\|w\|_{A_0},$$
$$|w|_{H^1} \leq \|w\|_{A_1},$$

therefore, as a simple consequence of the interpolation theorem (see [26, §1.3.3]) applied to the identity operator, we obtain $\varepsilon^{1-\theta}|w|_{H^1} \leq \|w\|_{(A_0,A_1)_{\theta,p}}$. We also have

$$\|\mathbf{c}\cdot\nabla w\|_{C_0} \leq \|w\|_{A_0},$$
$$\|\mathbf{c}\cdot\nabla w\|_{C_1} \leq \|w\|_{A_1},$$

which gives $\|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{\theta,p}} \preceq \|w\|_{(A_0,A_1)_{\theta,p}}$.

In order to complete the proof of (17) we directly deal with the definition of interpolation norm. For any $t > 0$ consider a generic splitting

$$w = w_0(t) + w_1(t), \text{ with } w_i(t) \in H^1_0, i = 1, 2, \tag{18}$$

and similarly

$$\mathbf{c}\cdot\nabla w = \phi_0(t) + \phi_1(t), \text{ with } \phi_0(t) \in C_0, \phi_1(t) \in C_1. \tag{19}$$

Let $\widetilde{w}_0(t) \in H^1_0$ be the solution of

$$\mathcal{L}_\varepsilon \widetilde{w}_0(t) = -\varepsilon\Delta w_0(t) + \phi_0(t)$$

and let $\widetilde{w}_1(t) \in H^1_0$ be the solution of

$$\mathcal{L}_\varepsilon \widetilde{w}_1(t) = -\varepsilon\Delta w_1(t) + \phi_1(t).$$

As

$$\mathcal{L}_\varepsilon(\widetilde{w}_0(t) + \widetilde{w}_1(t)) = -\varepsilon\Delta(w_0(t) + w_1(t)) + \phi_0(t) + \phi_1(t) = \mathcal{L}_\varepsilon w,$$

we have $w = \widetilde{w}_0(t) + \widetilde{w}_1(t)$. Moreover, thanks to (11), we also have

$$\|\widetilde{w}_0(t)\|_{A_0} \preceq \varepsilon|w_0(t)|_{H^1} + \|\phi_0(t)\|_{C_0},$$

and, using (12) and denoting by $\Phi_1(t) \in H_0^1$ the solution of $\mathcal{L}_0\Phi_1(t) = \phi_1(t)$, we obtain

$$\begin{aligned}
\|\widetilde{w}_1(t)\|_{A_1} &\preceq \sup_{v\in H_0^1} \frac{{}_{H^{-1}}\langle -\varepsilon\Delta w_1(t) + \phi_1(t), v\rangle_{H_0^1}}{|v|_{A_0}} \\
&\preceq \sup_{v\in H_0^1} \frac{{}_{H^{-1}}\langle -\Delta w_1(t), v\rangle_{H_0^1}}{|v|_{H^1}} \\
&\quad + \sup_{v\in H_0^1} \frac{{}_{H^{-1}}\langle \phi_1(t), v\rangle_{H_0^1}}{\|\mathbf{c}\cdot\nabla v\|_{H^{-1}}} \\
&\preceq \sup_{v\in H_0^1} \frac{\int_\Omega \nabla w_1(t)(\mathbf{x})\cdot\nabla v(\mathbf{x})\,d\mathbf{x}}{|v|_{H^1}} \\
&\quad + \sup_{v\in H_0^1} \frac{\int_\Omega \Phi_1(t)(\mathbf{x})\,\mathbf{c}\cdot\nabla v(\mathbf{x})\,d\mathbf{x}}{\|\mathbf{c}\cdot\nabla v\|_{H^{-1}}} \\
&\preceq |w_1(t)|_{H^1} + \|\phi_1(t)\|_{C_1}.
\end{aligned}$$

Therefore, by the triangle inequality and recalling [26, §1.3.2] we have

$$\begin{aligned}
\|w\|_{(A_0,A_1)_{\theta,p}} &\leq \left[\int_0^{+\infty} \left(t^{-\theta}\|\widetilde{w}_0(t)\|_{A_0} + t^{1-\theta}\|\widetilde{w}_1(t)\|_{A_1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&\preceq \Bigg[\int_0^{+\infty} \big(t^{-\theta}\varepsilon|w_0(t)|_{H^1} + t^{-\theta}\|\phi_0(t)\|_{C_0} \\
&\qquad\qquad + t^{1-\theta}|w_1(t)|_{H^1} + t^{1-\theta}\|\phi_1(t)\|_{C_1}\big)^p \frac{dt}{t}\Bigg]^{1/p} \\
&\preceq \left[\int_0^{+\infty} \left(t^{-\theta}\varepsilon|w_0(t)|_{H^1} + t^{1-\theta}|w_1(t)|_{H^1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&\quad + \left[\int_0^{+\infty} \left(t^{-\theta}\|\phi_0(t)\|_{C_0} + t^{1-\theta}\|\phi_1(t)\|_{C_1}\right)^p \frac{dt}{t}\right]^{1/p};
\end{aligned}$$

finally, taking the infimum over all w_0, w_1, ϕ_0 and ϕ_1 under the constraints given in (18)–(19) and using [26, 1.3.3.(f)], this yields $\|w\|_{(A_0,A_1)_{\theta,p}} \preceq \varepsilon^{1-\theta}|w|_{H^1} + \|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{\theta,p}}$.

At this point, one may ask what happens to the estimate (15) for $\varepsilon = 0$. Actually, (17) suggests that in this case we obtain a useless relation between suitable norms of $\mathbf{c}\cdot\nabla w$. In other words we are not investigating the structure of problem (8)—though it has a very simple structure in this model case—whereas we are only investigating the coupling between diffusive and advective terms. On the opposite side, when $\varepsilon \geq \mathrm{diam}(\Omega)\|\mathbf{c}\|_{L^\infty}$ one has $\|w\|_{(A_0,A_1)_{\theta,p}} \simeq \varepsilon^{1-\theta}|w|_{H^1}$, and we recover from (15) the classical elliptic estimate; notice that in this case the value of p has no effect.

In the following proposition we prove a useful statement that holds true for our norm $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$: it will turn out that, for our purposes, the parameter p has a minor effect on the norm, even for small ε.

Proposition 4 *Assume (H1), $0 < \theta < 1$, $1 \leq p < q \leq +\infty$ and $\varepsilon < 1$; then we have for all $w \in H^1_0$*

$$\|w\|_{(A_0,A_1)_{\theta,p}} \leq \Big\{ (p-\theta p)^{-1/p} + (\theta p)^{-1/p} + |\log\varepsilon|^{1/p-1/q} \Big\} \|w\|_{(A_0,A_1)_{\theta,q}}. \tag{20}$$

Proof We have, by definition and by the triangle inequality,

$$\begin{aligned}
\|w\|_{(A_0,A_1)_{\theta,p}} &\leq \left[\int_0^{+\infty} \left(t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&\leq \left[\int_0^{\varepsilon} \left(t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&\quad + \left[\int_\varepsilon^{1} \left(t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&\quad + \left[\int_1^{+\infty} \left(t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1}\right)^p \frac{dt}{t}\right]^{1/p} \\
&= I + II + III
\end{aligned}$$

for any $w_0(t)$ and $w_1(t)$ with $w = w_0(t) + w_1(t)$, $w_i(t) \in H^1_0, i = 1, 2$ and $0 < t < +\infty$. Taking $w_0(t) = 0$ and $w_1(t) = w$ for $0 < t < \varepsilon$ we have

$$\begin{aligned}
I &\leq \left[\int_0^{\varepsilon} t^{(1-\theta)p-1} dt\right]^{1/p} \|w\|_{A_1} \\
&\leq [(1-\theta)p]^{-1/p}\, \varepsilon^{1-\theta}\|w\|_{A_1}.
\end{aligned}$$

In a very similar way we deal with the third term, taking $w_0(t) = w$ and $w_1(t) = 0$ for $t > \varepsilon$ instead:

$$\begin{aligned}
III &\leq \left[\int_1^{+\infty} t^{-(1+\theta p)} dt\right]^{1/p} \|w\|_{A_0} \\
&\leq (\theta p)^{-1/p}\, \|w\|_{A_0}.
\end{aligned}$$

Assume for a moment $q \neq +\infty$. Thanks to the Hölder inequality we have

$$\begin{aligned}\int_\varepsilon^1 & \left(t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1}\right)^p \frac{dt}{t} \\ & \leq \left[\int_\varepsilon^1 \frac{1}{t}\, dt\right]^{(q-p)/q} \\ & \quad \cdot \left[\int_\varepsilon^1 \left(t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1}\right)^q \frac{dt}{t}\right]^{p/q} \\ & \leq \Bigg\{ [-\log(\varepsilon)]^{1/p-1/q} \\ & \qquad \cdot \left[\int_\varepsilon^1 \left(t^{-\theta}\|w_0(t)\|_{A_0} + t^{1-\theta}\|w_1(t)\|_{A_1}\right)^q \frac{dt}{t}\right]^{1/q} \Bigg\}^p\end{aligned} \tag{21}$$

that holds true for any choice of $w_0(t)$ and $w_1(t)$ on $\varepsilon < t < 1$; taking the infimum on w_0, w_1 we obtain

$$II \leq [-\log(\varepsilon)]^{1/p-1/q}\, \|w\|_{(A_0,A_1)_{\theta,q}}.$$

The case $q = +\infty$ is similar.

Finally, recalling

$$\varepsilon^{1-\theta}\|w\|_{A_1} \leq \|w\|_{(A_0,A_1)_{\theta,q}}$$

and

$$\|w\|_{A_0} \leq \|w\|_{(A_0,A_1)_{\theta,q}},$$

(20) follows from the previous estimates.

Estimate (20) is likely non-optimal when $\theta \to 0$ or $\theta \to 1$; this is not so relevant in the sequel, since the more interesting case is indeed $\theta = 1/2$, for which (20)reduces to

$$\|w\|_{(A_0,A_1)_{1/2,p}} \preceq |\log\varepsilon|^{1/p-1/q}\|w\|_{(A_0,A_1)_{1/2,q}}, \quad \forall w \in H_0^1, \tag{22}$$

because the terms $(p-\theta p)^{-1/p}\,|_{\theta=1/2}$ and $(\theta p)^{-1/p}\,|_{\theta=1/2}$ are uniformly bounded with respect to p. It is worth recalling that, as a general result of the interpolation theory (see, e.g., [26, 1.3.3.d]), we also have

$$\|w\|_{(A_0,A_1)_{\theta,q}} \preceq \|w\|_{(A_0,A_1)_{\theta,p}}, \quad \forall w \in H_0^1, \tag{23}$$

for $0 < \theta < 1$ and $1 \leq p < q \leq +\infty$. More generally, for a fixed θ the norms $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ and $\|\cdot\|_{(A_0,A_1)_{\theta,q}}$ are equivalent up to a $(\log\varepsilon)$-like factor.

As mentioned above, one reason to use interpolation is that we are interested in constructing a norm that is neither too weak (as $\|\cdot\|_{A_0}$ is) nor too strong (as $\|\cdot\|_{A_1}$ is). The following proposition states a Poincaré-like inequality that gives the answer.

Proposition 5 *Under assumptions (H1) and (H2), there exists a constant $\gamma = \gamma(\mathbf{c}, \Omega)$ (i.e., only dependent on $\mathbf{c}$ and Ω) such that*

$$\|w\|_{L^2} \le \gamma \|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{1/2,1}}, \qquad \forall w \in H^1_0. \tag{24}$$

Proof Let η be the solution of (8) for the source term $f \equiv 1$ (i.e., $\mathbf{c}\cdot\nabla\eta = 1$ with $\eta|_{\partial\Omega^-} = 0$). Given $w \in H^1_0$, integrating by parts, using the Cauchy-Schwartz inequality and (H1) we have

$$\begin{aligned}\|w\|^2_{L^2} &= \int_\Omega \mathbf{c}(\mathbf{x})\cdot\nabla\eta(\mathbf{x})\, w^2(\mathbf{x})\, d\mathbf{x}\\ &= -2\int_\Omega \eta(\mathbf{x}) w(\mathbf{x})\, \mathbf{c}(\mathbf{x})\cdot\nabla w(\mathbf{x})\, d\mathbf{x}\\ &\preceq \|\mathbf{c}\cdot\nabla w\|_{H^{-1}} |\eta w|_{H^1}.\end{aligned} \tag{25}$$

Using the classical Poincaré inequality we easily obtain

$$|\eta w|_{H^1} \preceq (\|\eta\|_{L^\infty} + \|\nabla\eta\|_{(L^\infty)^2})|w|_{H^1}. \tag{26}$$

Actually, thanks to (H2), we have $\|\eta\|_{L^\infty} + \|\nabla\eta\|_{(L^\infty)^2} < +\infty$ (e.g., see [15, Theorem 3.2]), whence

$$\|w\|^2_{L^2} \le \gamma \|\mathbf{c}\cdot\nabla w\|_{H^{-1}} |w|_{H^1},$$

which gives

$$\|w\|_{L^2} \le \gamma \|\mathbf{c}\cdot\nabla w\|^{1/2}_{C_0} \|\mathbf{c}\cdot\nabla w\|^{1/2}_{C_1}, \tag{27}$$

where γ depends on η, namely on $\mathbf{c}$ and Ω. Finally (27) yields (24) by means of a classical theorem of Lions and Peetre (see [22]), applied to the linear operator $\mathcal{L}_0^{-1} : C_1 \to H^1_0$.

Hypothesis (H2) is essential in proving (24). Moreover the estimate is optimal in the sense that we can replace the right hand side of (24) only by $\|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{\theta,p}}$ with $\theta > 1/2$, as stated in the following corollary.

Corollary 1 *Under assumptions of Proposition 5, we also have*

$$\begin{aligned}&\|w\|_{L^2} \le \gamma \|w\|_{(A_0,A_1)_{\theta,p}}, \forall w \in H^1_0\\ &\qquad \Leftrightarrow \theta > 1/2 \text{ or } (\theta, p) = (1/2, 1),\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\|\phi\|_{(A_0,A_1)^*_{\theta,p}} \le \gamma \|\phi\|_{L^2}, \forall \phi \in L^2\\ &\qquad \Leftrightarrow \theta > 1/2 \text{ or } (\theta, p) = (1/2, 1).\end{aligned} \tag{29}$$

Proof The two conditions

$$\|w\|_{L^2} \leq \gamma \|w\|_{(A_0,A_1)_{\theta,p}}, \forall w \in H_0^1 \tag{30}$$

and

$$\|\phi\|_{(A_0,A_1)^*_{\theta,p}} \leq \gamma \|\phi\|_{L^2}, \forall \phi \in L^2 \tag{31}$$

are equivalent—for a given (θ, p) they are both true or false—because the norms in (31) are the dual of those in (30). Further, (30) is equivalent to

$$\|w\|_{L^2} \leq \gamma \|\mathbf{c} \cdot \nabla w\|_{(C_0,C_1)_{\theta,p}}, \forall w \in H_0^1; \tag{32}$$

indeed, the left hand side of (30) is independent of ε, so we can take as a right hand side the $\inf_{\varepsilon>0} \gamma \|w\|_{(A_0,A_1)_{\theta,p}} = \gamma \|\mathbf{c} \cdot \nabla w\|_{(C_0,C_1)_{\theta,p}}$.

¿From (24), using [26, Theorem 1.3.3.e], one easily gets (32) for each $\theta > 1/2$.

Assume in the sequel of the proof that $\theta < 1/2$, or $\theta = 1/2$ and $p > 1$. Then it remains to show that (32) fails; one can restrict to the case $\theta = 1/2$ and $p > 1$ only, because of [26, Theorem 1.3.3.e]. For the sake of simplicity, we just give a sketch of the construction of a sequence $\{w_n\}$ of functions which verify

$$\|w_n\|_{L^2([0,1]^2)} \to C > 0 \tag{33}$$

and

$$\left\| \frac{\partial}{\partial x} w_n \right\|_{(C_0([0,1]^2), C_1([0,1]^2))_{1/2,p}} \to 0, \tag{34}$$

which is a counterexample to (32) in the very simple case of $\Omega = [0,1]^2$, and $\mathbf{c} = [1\ 0]$. We start from a sequence of functions $\{\bar{w}_n\}$ on $[0,1]$ which verifies the one-dimensional counterpart of (33)– (34):

$$\|\bar{w}_n\|_{L^2(0,1)} \to 1 \quad \text{and} \quad \left\| \frac{d}{dx} \bar{w}_n \right\|_{(H^{-1}(0,1), L_0^2(0,1))_{1/2,p}} \to 0, \tag{35}$$

where $L_0^2(0,1) \subset L^2(0,1)$ is the subspace of zero mean value functions, and is indeed the one-dimensional counterpart of C_1 (see [25]). We can take, for example, a sequence $\{\bar{w}_n\}$ of functions in $C_0^\infty(0,1)$ converging to the constant 1 in the space $(L^2(0,1), H^1(0,1))_{1/2,p}$ (see [24, §2.2.4] and recall that now $p > 1$) which is not restrictive to assume symmetric, in the sense that $\bar{w}_n(x) = \bar{w}_n(1-x)$. Therefore

$\|\bar{w}_n\|_{L^2(0,1)} \to 1$, and

$$\begin{aligned}\left\|\frac{d}{dx}\bar{w}_n\right\|_{(H^{-1}(0,1),L^2_0(0,1))_{\theta,p}} &= \left\|\frac{d}{dx}(\bar{w}_n - 1)\right\|_{(H^{-1}(0,1),L^2_0(0,1))_{1/2,p}} \\ &\preceq \|\bar{w}_n - 1\|_{(L^2_\sharp(0,1),H^1_\sharp(0,1))_{1/2,p}} \\ &= \|\bar{w}_n - 1\|_{(L^2(0,1),H^1(0,1))_{1/2,p}} \\ &\to 0,\end{aligned}$$

using [25, Proposition 1] and the symmetry of $\bar{w}_n$, respectively; we used the notation $L^2_\sharp(0,1)$ and $H^1_\sharp(0,1)$ to refer to spaces of periodic functions, see [25] for all the details. Finally one can easily check that $w_n(x,y) = y(1-y)\bar{w}_n(x)$ satisfy (33)–(34).

As a consequence of (15), (17), (28) and (29), the choice $\theta = 1/2$ turns out to be the most interesting. For example we have for all $w \in H^1_0$

$$\begin{aligned}\varepsilon^{1/2}|w|_{H^1} + \|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{1/2,+\infty}} &\simeq \|\mathcal{L}_\varepsilon w\|_{(A_0,A_1)^*_{1/2,1}} \\ &\leq \gamma\|\mathcal{L}_\varepsilon w\|_{L^2},\end{aligned} \tag{36}$$

as well as

$$\begin{aligned}\gamma^{-1}\|w\|_{L^2} &\leq \varepsilon^{1/2}|w|_{H^1} + \|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{1/2,1}} \\ &\simeq \|\mathcal{L}_\varepsilon w\|_{(A_0,A_1)^*_{1/2,+\infty}};\end{aligned} \tag{37}$$

the case with $p = 2$ is also of special interest as $\|\cdot\|_V \equiv \|\cdot\|_W$ are Hilbertian norms, and (15) becomes

$$\begin{aligned}&\left[\varepsilon|w|^2_{H^1} + \|\mathbf{c}\cdot\nabla w\|^2_{(C_0,C_1)_{1/2,2}}\right]^{1/2} \\ &\simeq \sup_{v\in H^1_0} \frac{{}_{H^{-1}}\langle\mathcal{L}_\varepsilon w, v\rangle_{H^1_0}}{\left[\varepsilon|v|^2_{H^1} + \|\mathbf{c}\cdot\nabla v\|^2_{(C_0,C_1)_{1/2,2}}\right]^{1/2}}.\end{aligned} \tag{38}$$

Roughly speaking, in the present framework we can not derive estimates like (3) for a norm $\|\cdot\|_W$ which is stronger than the usual $\|\cdot\|_{L^2}$ and such that $\|u\|_W$ remains bounded with respect to ε even in presence of internal or boundary layers; nevertheless we are very close to this (up to $|\log\varepsilon|$ factors) when $\theta = 1/2$.

4 Connections to numerical methods

First of all we recall the classical theory due to Babuška and Brezzi (see [1,4]): let $a_{\varepsilon,h}$ be a *discretization* of a_ε on $W \times V_h$, where V_h denotes a finite dimensional subspace of V; similarly, in the sequel, W_h will denote a finite dimensional subspace of W. The related numerical method reads:

$$\begin{cases} \text{find } u_h \in W_h \text{ such that} \\ a_{\varepsilon,h}(u_h, v_h) = {}_{V_h'}\langle f_h, v_h\rangle_{V_h}, \quad \forall v_h \in V_h, \end{cases} \tag{39}$$

which we assume to be *consistent*, i.e., $a_{\varepsilon,h}(u - u_h, v_h) = 0$ for any $v_h \in V_h$ and for u and u_h given by (2) and (39), respectively. If the discrete counterpart of (4)–(5) holds true, i.e.,

$$\sup_{w\in W} \sup_{v_h\in V_h} \frac{a_{\varepsilon,h}(w, v_h)}{\|w\|_W \|v_h\|_V} \le \widetilde{\kappa} < +\infty, \tag{40}$$

$$\inf_{w_h\in W_h} \sup_{v_h\in V_h} \frac{a_{\varepsilon,h}(w_h, v_h)}{\|w_h\|_W \|v_h\|_V} \ge \widetilde{\gamma} > 0, \tag{41}$$

then the method is *quasi-optimal*, as one gets from (40)–(41):

$$\|u - u_h\|_W \le (\widetilde{\kappa}\widetilde{\gamma}^{-1} + 1) \inf_{w_h\in W_h} \|u - w_h\|_W. \tag{42}$$

The error estimate (42) is of practical interest when $\widetilde{\kappa}$ and $\widetilde{\gamma}$ do not depend on ε. Moreover the norm $\|\cdot\|_W$, which appears in (42), should allow boundary layers without being too weak, as we discussed in details in the previous section.

The standard Galerkin formulation—given by $a_{\varepsilon,h} \equiv a_\varepsilon$—is unstable for small values of ε. In the eighties effective improvements have been proposed by Hughes and coworkers (see [12, 18, 19]) with *the Streamline-Upwind Petrov-Galerkin* (SUPG) and *Galerkin Least-Squares* (GaLS) methods. Even though they are quite satisfactory for practical purposes, there is no proof, up to now, that they fit into the classical framework (40)–(41), for suitable choices of the norms $\|\cdot\|_W$ and $\|\cdot\|_V$, e.g., for norms that give (4)–(5). Actually the analysis of this kind of methods follows a slightly different argument: roughly speaking, (40) and (41) are proved with the same $\|\cdot\|_V$ but with different $\|\cdot\|_W$. The final error estimates obtained in this way are based on unrealistic regularity assumptions on the exact solution u.

In the last few years further improvements or reinterpretations of previous methods have been proposed, e.g.,

- *enriched* formulations (*Bubbles*, e.g., *RFB* or *subgrid viscosity* methods) by Brezzi *et al.* (see [2, 5–11]),

– *multi-scale* methods by Hughes *et al.* (see [16, 17]),
– *least squares* formulations by Lazarov *et al.* (see [20]),
– methods with a *negative order stabilization* by Canuto *et al.* (see [3]).

The error analysis proposed for these methods is similar to the analysis for SUPG and GaLS mentioned before, but nevertheless these methods are, roughly speaking, closer to the framework (40)–(41). Among them, the *negative order stabilization* technique is the most interesting one from this point of view, and we shall come back to it in a moment.

The analysis proposed in the previous section, and in particular (38), suggests in some sense how a *stabilization by improved coercivity* should work. Consider for example the "ideal" least squares formulation, whose numerical solution $u_h \in V_h$ is given by

$$(\mathcal{L}_\varepsilon u_h, \mathcal{L}_\varepsilon v_h)_{(A_0,A_1)^*_{1/2,2}} = (f, \mathcal{L}_\varepsilon v_h)_{(A_0,A_1)^*_{1/2,2}}, \quad \forall v_h \in V_h, \tag{43}$$

where $(\cdot,\cdot)_{(A_0,A_1)^*_{1/2,2}}$ denotes the scalar product related to the Hilbert norm $\|\cdot\|_{(A_0,A_1)^*_{1/2,2}}$; the method fits into the framework (39) with the definitions $a^{LS}_{\varepsilon,h}(w,v) := (\mathcal{L}_\varepsilon w, \mathcal{L}_\varepsilon v)_{(A_0,A_1)^*_{1/2,2}}$ and ${}_{V_h'}\langle f^{LS}_h, v_h\rangle_{V_h} := (f, \mathcal{L}_\varepsilon v_h)_{(A_0,A_1)^*_{1/2,2}}$. The bilinear form $a^{LS}_{\varepsilon,h}(\cdot,\cdot)$ easily verifies (40) and (41), with respect to $\|\cdot\|_W \equiv \|\cdot\|_V := \|\cdot\|_{(A_0,A_1)_{1/2,2}}$, thanks to Proposition 2; e.g., (41) follows by coercivity:

$$a^{LS}_{\varepsilon,h}(v,v) = \|\mathcal{L}_\varepsilon v\|^2_{(A_0,A_1)^*_{1/2,2}} \simeq \|v\|^2_{(A_0,A_1)_{1/2,2}}. \tag{44}$$

Therefore this "ideal" formulation turns out to be quasi-optimal.

Following a similar reasoning, one can consider an "ideal" SUPG discretization, which we shall describe for the advection-dominated case (i.e., we assume $\varepsilon/\|\mathbf{c}\|$ smaller than the mesh-size h). Recall that our bilinear form $a_\varepsilon(\cdot,\cdot)$ verifies (4)–(6) if we take $\|w\|^2_W \equiv \|w\|^2_V := \|w\|_{(A_0,A_1)_{1/2,2}} \simeq \varepsilon|w|^2_{H^1} + \|\mathbf{c}\cdot\nabla w\|^2_{(C_0,C_1)_{1/2,2}}$, while the standard Galerkin numerical method is unstable because the discrete inf-sup condition (41) does not hold true for usual choices of W_h and V_h. In fact by means of coercivity ($a_\varepsilon(w,w) = \varepsilon|w|^2_{H^1}$) one can only prove a very weak discrete inf-sup condition, without any control on $\|\mathbf{c}\cdot\nabla w\|_{(C_0,C_1)_{1/2,2}}$. We would like the numerical bilinear form $a^{SUPG}_{\varepsilon,h}(\cdot,\cdot)$ to be coercive with respect to $\varepsilon|\cdot|^2_{H^1} + \|\mathbf{c}\cdot\nabla(\cdot)\|^2_{(C_0,C_1)_{1/2,2}}$, as in the previous example (see (44)), without losing consistency. Therefore we set:

$$a^{SUPG}_{\varepsilon,h}(w,v) := a_\varepsilon(w,v) + (\mathcal{L}_\varepsilon w, \mathbf{c}\cdot\nabla v)_{(C_0,C_1)_{1/2,2}}; \tag{45}$$

and accordingly

$$ {}_{V_h'}\langle f_h^{SUPG}, v_h\rangle_{V_h} := {}_{V'}\langle f, v_h\rangle_V + (f, \mathbf{c}\cdot\nabla v_h)_{(C_0,C_1)_{1/2,2}}, \quad \forall v_h \in V_h, \tag{46} $$

where $(\cdot,\cdot)_{(C_0,C_1)_{1/2,2}}$ denotes the scalar product related to the norm $\|\cdot\|_{(C_0,C_1)_{1/2,2}}$, and the definition (45)–(46) is intended for w and v regular enough. With a minor modification of the previous norms, i.e.,

$$ \begin{aligned} \|w\|_W^2 &:= \varepsilon|w|_{H^1}^2 + \|\mathbf{c}\cdot\nabla w\|^2_{(C_0,C_1)_{1/2,2}} + \|\varepsilon\Delta w\|^2_{(C_0,C_1)_{1/2,2}} \\ \|v\|_V^2 &:= \varepsilon|v|_{H^1}^2 + \|\mathbf{c}\cdot\nabla v\|^2_{(C_0,C_1)_{1/2,2}}, \end{aligned} $$

and assuming the inverse inequality

$$ 2\|\varepsilon\Delta w_h\|^2_{(C_0,C_1)_{1/2,2}} \le \varepsilon|w_h|^2_{H^1} + \|\mathbf{c}\cdot\nabla w_h\|^2_{(C_0,C_1)_{1/2,2}} $$

—which requires regular enough elements and small enough ε— we recover the discrete continuity (40) and the discrete infsup (41). As a result, the method is quasi optimal, as stated in (42).

The main drawback of those "ideal" formulation is that the practical computation of the unusual scalar products $(\cdot,\cdot)_{(A_0,A_1)^*_{1/2,2}}$ and $(\cdot,\cdot)_{(C_0,C_1)_{1/2,2}}$ is difficult. Nevertheless a lot has been done in this direction by Bertoluzza, Canuto and Tabacco in their work [3] mentioned above. They propose a wavelet element method (WEM) for a Galerkin formulation stabilized by adding $(\mathcal{L}_\varepsilon w, \mathcal{L}_\varepsilon v)_{(H^{-1},L^2)_{1/2,2}}$. Their stabilization acts as $(\mathbf{c}\cdot\nabla w, \mathbf{c}\cdot\nabla v)_{(H^{-1},L^2)_{1/2,2}}$ at the discrete level, and is based on the scalar product $(\cdot,\cdot)_{(H^{-1},L^2)_{1/2,2}}$, which is of order $-1/2$. To make a comparison, our "ideal" SUPG formulation is stabilized by the term $(\mathbf{c}\cdot\nabla w, \mathbf{c}\cdot\nabla v)_{(C_0,C_1)_{1/2,2}}$, and the stronger scalar product $(\cdot,\cdot)_{(C_0,C_1)_{1/2,2}}$ has order $-1/2$ only in the $\mathbf{c}$ direction, and order 0 in the orthogonal directions. Recently Canuto and Tabacco (in [13]) have also proposed and implemented (for a constant advection field $\mathbf{c}$) a formulation with non-isotropic stabilizing scalar products, which turn out to be very similar to the "ideal" formulation mentioned above.

In conclusion we think that the analysis of the previous section can lead to a deeper understanding of the numerical discretization of the advection-diffusion problem (2), it can suggest new numerical methods—e.g., based on the "ideal" formulations described above and implemented by means of WEM—and, finally, our technique could improve the theoretical analysis of the more popular numerical methods for (2)—see, for example, our work [25].

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