A discontinuous residual-free bubble method for advection-diffusion problems

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Abstract

A discontinuous finite element method is presented for solving the linear advection-diffusion equation, based on the *Residual-Free Bubble* (RFB) finite-element formulation. After the *macro*-scales (usual piecewise-polynomials elements) are separated from the *micro*-scales (the *bubble* part), they are computed by a plain Galerkin formulation, while the bubble part is approximated by a discontinuous Galerkin method. The advantage of this approach, as compared to other implementations of the Residual-Free Bubble formulation, is that the macro-scales are computed accurately, at least for the model problem presently considered. Numerical tests are performed on a two-dimensional model problem to confirm the validity of the proposed approach.

keywords: advection-diffusion, finite element method, stabilization, SUPG, residual-free bubbles, multiscale.
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1 Introduction

In this paper, we present a numerical procedure based on the *Residual-Free Bubble* (RFB) *Finite Element Method* (FEM) for solving of the linear advection-diffusion equation. This simple model problem encompasses one of the difficulties encountered in the numerical simulation of *fluid flow* (e.g., [21,22]). It is well known that classical numerical methods—such as the central finite difference method or the plain Galerkin FEM—are inadequate when the diffusive term is *small* compared to the advective term. Typically in our model problem, but also in real fluid flow simulation, unphysical oscillations pollute the numerical solution in the whole domain, while the exact solution only shows boundary or internal *layers*.

To overcome this difficulty, so-called *stabilized methods* have been developed. In the framework of finite element methods, a simple modification consists of adding a suitable amount of *artificial* diffusion. This idea was developed by Hughes and collaborators in the eighties [12, 18, 19]. Their *Streamline-Upwind Petrov-Galerkin* (SUPG) method adds diffusion only in the *streamline* direction, that is, in the direction of the advection field, while preserving the *consistency* of the variational formulation. The SUPG technique performs better than the naive artificial diffusion technique, as shown by the theoretical analysis [12], and also confirmed by numerical tests (e.g. [22]). SUPG and its variants (such as the *Galerkin Least-Squares* method) have become the most popular numerical methods for this kind of problems among FEMs.

Despite the success of SUPG, there are areas for improvement. For example, because SUPG is not *monotone*, it does not preserve the positivity of the solution, which is unphysical in some applications. Another weakness of SUPG is that the amount of *streamline diffusion* has to be tuned depending on the problem at hand. For our simple model problem, an effective tuning of SUPG is available (see [12]), while in other cases, for example, in real-world fluid flow simulation, tuning of the method can be difficult. This difficulty has motivated the introduction of intrinsically stable methods. Examples include the *Variational Multiscale* method of Hughes and coworkers (see [17]), and the *Residual-Free Bubbles* (RFB) method of Brezzi and Russo (in [10]). These two methods are closely related, as discussed in [3]. A detailed discussion on the advantages and disadvantages of the methods can be found in references [4, 9, 17].

In particular, the Residual-Free Bubble (RFB) method is based on a local enrichment of the finite element space instead of a modification of the variational formulation. The idea is to add to the usual space of piecewise polynomials, referred to as *macro*-scales in this paper, the so-called *bubbles*, representing the *micro*-scales. *Bubbles* are functions whose support remains inside the elements of the triangulation. The numerical method turns out to be intrinsically stable (see, for example, [7] and [23]), at the price of having to solve local problems in order to approximate, and possibly eliminate, the infinite *bubble* degrees of freedom. In one dimension, the local problems can be solved analytically, and the final numerical scheme produces nodally exact numerical solutions (see [10]). In the multi-dimensional case, one can approximate analytically the bubble effect only in particular cases; for example, in [10] the case of linear elements is considered. In more general cases, procedures for dealing with the bubble degrees of freedom have been proposed, as will be discussed in §2.

In this paper, we propose to approximate the local problems for the bubble degrees of freedom of the Residual-Free Bubble (RFB) formulation by means of a discontinuous Galerkin method. This approach has the advantage of allowing us to accurately compute the effect of the bubbles on the *macro*-scales when using linear or higher order elements, in the advection-dominated case. In §2, we present the Residual-Free Bubble (RFB) idea and discuss the practical implementation, also including the new proposal. In §3, we present numerical tests, and in §4 we discuss the conclusions.

2 The RFB formulation and implementation

We consider the linear advection-diffusion problem

$$\mathcal{L}_\varepsilon u = f \quad \text{in } \Omega, \tag{1}$$

subject to the homogeneous Dirichlet boundary condition, where

$$\mathcal{L}_\varepsilon := -\varepsilon \Delta + \mathbf{c} \cdot \nabla, \tag{2}$$

∇ denotes the gradient operator, i.e., the vector of first derivatives, and Δ denotes the Laplace operator, i.e.,

$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}.$$

The unknown real-valued function u is defined on the convex polygonal domain $\Omega \subset \mathbb{R}^2$, ε is a strictly positive diffusivity coefficient, and $\mathbf{c}$ is the velocity field in Ω. As mentioned in the introduction, this model problem encompasses some of the difficulties encountered in the numerical simulations of fluid flow (see, e.g., [22]). The variational formulation of (1) can be stated as follows: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f, v \rangle, \quad \forall v \in H_0^1(\Omega),$$

where

$$a(w,v) := \varepsilon \int_\Omega \nabla w \cdot \nabla v \, d\mathbf{x} + \int_\Omega (\mathbf{c} \cdot \nabla w)\, v \, d\mathbf{x}, \tag{3}$$

and

$$\langle f, v \rangle := \int_\Omega f v \, d\mathbf{x}.$$

We shall assume that f belongs to $L^2(\Omega)$, and $\operatorname{div}(\mathbf{c}) \le 0$. This guarantees that the variational formulation of (1) is well posed (see [20]). Given a subset ω of the domain Ω (possibly the whole Ω itself), we follow the usual notation for the Lebesgue spaces $L^p(\omega)$ $(1 \le p \le \infty)$ and Sobolev space $H^1(\omega)$ of functions whose partial derivatives are in $L^2(\omega)$, and denote by $H_0^1(\omega)$ the subspace of $H^1(\omega)$ of all functions vanishing on the boundary $\partial\omega$ (see [20]). Moreover, we denote by $\partial\omega^-$, $\partial\omega^0$ and $\partial\omega^+$, respectively, the *inflow* boundary, the *characteristic* boundary, and the *outflow* boundary,

$$\begin{aligned} \partial\omega^- &:= \{\mathbf{x} \in \partial\omega \text{ such that } \mathbf{c}\cdot\mathbf{n} < 0\}, \\ \partial\omega^0 &:= \{\mathbf{x} \in \partial\omega \text{ such that } \mathbf{c}\cdot\mathbf{n} = 0\}, \\ \partial\omega^+ &:= \{\mathbf{x} \in \partial\omega \text{ such that } \mathbf{c}\cdot\mathbf{n} > 0\}, \end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector.

We shall deal with a family of partitions $\mathcal{T}_h$ of the domain Ω into open triangles, satisfying the usual conditions of *admissibility* (any two elements have disjoint closure, a vertex in common, or share a complete edge), and *shape regularity* (see [14]). The diameter of an element T will be denoted by h_T, and the maximum diameter of all elements in $\mathcal{T}_h$ will be denoted by h.

We also assume that $\mathbf{c}$ and f are piecewise constant on the triangulation $\mathcal{T}_h$. Consequently, the assumption $\operatorname{div}(\mathbf{c}) \le 0$ has to be accepted in the sense of distributions, i.e., $\mathbf{c}\cdot\mathbf{n}_{T_1} + \mathbf{c}\cdot\mathbf{n}_{T_2} \le 0$ on the common edge $\partial T_1 \cup \partial T_2$ of any two elements T_1, T_2 of $\mathcal{T}_h$, where $\mathbf{n}_{T_i}$ denotes the outward direction on ∂T_i. We shall focus our attention on the advection-dominated regime, where ε is large compared to $h_T \|\mathbf{c}_{|T}\|$ in each element $T \in \mathcal{T}_h$. This is indeed the regime where standard numerical methods are inadequate (see [22]).

Consider the usual conforming finite-dimensional space of order $k \ge 1$,

$$V_P \equiv V_P(\mathcal{T}_h, k) := \left\{ v \in H_0^1 \text{ such that } v_{|T} \in \mathbb{P}_k, \forall T \in \mathcal{T}_h \right\}, \tag{4}$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree k. The Streamline-Upwind Petrov-Galerkin (SUPG) method can be stated as follows: find $u_P^{SUPG} \in V_P$, such that

$$a\left(u_P^{SUPG}, v_P\right) + \sum_{T\in\mathcal{T}_h} \tau_T \int_T \mathcal{L}_\varepsilon u_P^{SUPG}\, \mathbf{c}\cdot\nabla v_P = \langle f, v_P \rangle + \sum_{T\in\mathcal{T}_h} \tau_T \int_T f\, \mathbf{c}\cdot\nabla v_P, \forall v_P \in V_P, \tag{5}$$

where τ_T is the artificial streamline diffusion parameter [12],

$$\tau_T := \frac{h_T}{2\|\mathbf{c}\|}, \quad \text{in } T \in \mathcal{T}_h. \tag{6}$$

The *Residual-Free Bubble* (RFB) approach was proposed by Brezzi and Russo [10], inspired by a different philosophy: taking the variational formulation of (1) without modification, the numerical solution is found in the enriched space V_E of functions that are piecewise polynomials on the boundaries of the elements,

$$V_E \equiv V_E(\mathcal{T}_h, k) := \left\{ v \in H_0^1 \text{ such that } v_{|\partial T} \in \mathbb{P}_k, \forall T \in \mathcal{T}_h \right\}. \tag{7}$$

The Residual-Free Bubble (RFB) formulation is stated as follow: find $u_E^{RFB} \in V_E$, such that

$$a\left(u_E^{RFB}, v_E\right) = \langle f, v_E \rangle, \quad \forall v_E \in V_E. \tag{8}$$

An error analysis for the Residual-Free Bubble (RFB) method was presented in references [8, 23]. Note that the stabilizing mechanism is intrinsically contained in the enrichment of the space. Contrary to the Streamline-Upwind Petrov-Galerkin (SUPG) formulation, there are no free parameters. Because of the presence of the bubbles, (8) is an infinite-dimensional variational formulation, and cannot be coded into a numerical algorithm. To develop an algorithm, we need to approximate (8) with a finite number of degrees of freedom. The aim of this paper is to propose and justify an original finite-dimensional approximation of (8) based on a non-conforming (discontinuous) approximation of V_E.

We shall denote by $\mathcal{L}_0 := \mathbf{c} \cdot \nabla(\cdot)$ the purely hyperbolic operator, which is the formal limit of $\mathcal{L}_\varepsilon$ when $\varepsilon \to 0$; $\mathcal{L}_\varepsilon^* := -\varepsilon\Delta(\cdot) - \operatorname{div}(\mathbf{c}(\cdot))$ and $\mathcal{L}_0^* := -\operatorname{div}(\mathbf{c}(\cdot))$ denote the adjoint of $\mathcal{L}_\varepsilon$ and $\mathcal{L}_0$, respectively. In particular, if we restrict our attention to (the interior of) an element $T \in \mathcal{T}_h$, where $\mathbf{c}$ is assumed to be constant, then $\mathcal{L}_\varepsilon^* := -\varepsilon\Delta(\cdot) - \mathbf{c} \cdot \nabla(\cdot)$ and $\mathcal{L}_0^* := \mathbf{c} \cdot \nabla(\cdot)$.

The key idea in developing an algorithm from (8) is the distinction between *macro*-scales, which are represented by piecewise polynomials, and *micro*-scales, or *bubbles*, which reside inside the elements. We therefore assume that any $v_E \in V_E$ admits a unique decomposition in

$$v_E = v_P + v_B, \text{ with } v_P \in V_P, v_B \in V_B, \tag{9}$$

where the *bubble* space is

$$V_B \equiv V_B(\mathcal{T}_h) := \left\{ v_B : v_{B|T} \in H_0^1(T), \forall T \in \mathcal{T}_h \right\}. \tag{10}$$

Note that, in order to have a unique splitting (9), namely $V_E = V_P \oplus V_B$, we must restrict the order of polynomials to $1 \leq k \leq 2$. Indeed, in a triangular element, we can have *bubbles* which are polynomials of order 3 or higher; an example is the product of the usual barycentric coordinates, i.e., of the distances from the edges of the element. Therefore, as usual, we split $u_E^{RFB} = u_P^{RFB} + u_B^{RFB}$, where $u_P^{RFB} \in V_P$ and $u_B^{RFB} \in V_B$, and test (8) using $v_P \in V_P$, $v_B \in V_B$, yielding

$$a\left(u_P^{RFB}, v_P\right) + a\left(u_B^{RFB}, v_P\right) = \langle f, v_P \rangle, \qquad \forall v_P \in V_P, \tag{11}$$

$$a\left(u_P^{RFB}, v_B\right) + a\left(u_B^{RFB}, v_B\right) = \langle f, v_B \rangle, \qquad \forall v_B \in V_B. \tag{12}$$

Equation (12) gives u_B^{RFB} from u_P^{RFB} and f: actually, u_B^{RFB} solves in each element, T, the boundary-value problem

$$\begin{cases} \mathcal{L}_\varepsilon u_B^{RFB} = f - \mathcal{L}_\varepsilon u_P^{RFB} & \text{in } T, \\ u_B^{RFB} = 0 & \text{on } \partial T. \end{cases}$$

Substituting u_B^{RFB} in (11), we obtain a closed-form problem for u_P^{RFB}. If $M(w)$ and $F(f)$ are , respectively, the solutions in each element, T, of the problems

$$\begin{cases} \mathcal{L}_\varepsilon M(w) = -\mathcal{L}_\varepsilon w & \text{in } T \\ M(w) = 0 & \text{on } \partial T, \end{cases} \tag{13}$$

and,

$$\begin{cases} \mathcal{L}_\varepsilon F(f) = f & \text{in } T \\ F(f) = 0 & \text{on } \partial T, \end{cases} \tag{14}$$

then the final variational formulation for u_P^{RFB}, after integrating by parts, is

$$a(u_P^{RFB}, v_P) + \sum_{T \in \mathcal{T}_h} \int_T M(u_P^{RFB})\, \mathcal{L}_\varepsilon^* v_P = \langle f, v_P \rangle - \sum_{T \in \mathcal{T}_h} \int_T F(f)\, \mathcal{L}_\varepsilon^* v_P, \quad \forall v_P \in V_P. \tag{15}$$

Although this is a finite-dimensional problem, it contains the terms $\int_T M(u_P^{RFB})\, \mathcal{L}_\varepsilon^* v_P$ and $\int_T F(f)\, \mathcal{L}_\varepsilon^* v_P$, which implicitly involve the solution of local infinite-dimensional problems. As proposed in [10], we can use the approximations

$$\begin{gathered} \sum_{T \in \mathcal{T}_h} \int_T M(u_P^{RFB})\, \mathcal{L}_\varepsilon^* v_P \approx \sum_{T \in \mathcal{T}_h} \int_T \widetilde{M}(u_P^{RFB})\, \mathcal{L}_0^* v_P, \\ \sum_{T \in \mathcal{T}_h} \int_T F(f)\, \mathcal{L}_\varepsilon^* v_P \approx \sum_{T \in \mathcal{T}_h} \int_T \widetilde{F}(f)\, \mathcal{L}_0^* v_P, \end{gathered} \tag{16}$$

where, in each element T, $\widetilde{M}(w)$ and $\widetilde{F}(f)$ are given, respectively, by

$$\begin{cases} \mathcal{L}_0 \widetilde{M}(w) = -\mathcal{L}_0 w & \text{in } T \\ \widetilde{M}(w) = 0 & \text{on } \partial T^-, \end{cases} \tag{17}$$

and

$$\begin{cases} \mathcal{L}_0 \widetilde{F}(f) = f & \text{in } T \\ \widetilde{F}(f) = 0 & \text{on } \partial T^-. \end{cases} \tag{18}$$

Roughly speaking, (16) are justified as $\varepsilon \ll h_T \|\mathbf{c}_{|T}\|$, and so $\widetilde{M}(w)$ and $\widetilde{F}(f)$ are accurate approximations of $M(w)$ and $F(f)$ in $L^2(T)$, for all element $T \in \mathcal{T}_h$. Indeed, by virtue of the asymptotic expansion techniques of [22], one may think of $M(w)$ (resp. $F(f)$) as the sum of $\widetilde{M}(w)$ (resp. $\widetilde{F}(f)$) and a negligible boundary layer. For $k = 1$, corresponding to linear elements being used as *macro*-scales, the solutions of (17) and (18) can be easily evaluated analytically, as shown in [10]. From (16), we can define an approximated *Static Condensation* of the *bubble* degrees of freedom, and search for u_P^{SC} such that

$$a(u_P^{SC}, v_P) + \sum_{T \in \mathcal{T}_h} \int_T \widetilde{M}(u_P^{SC})\, \mathcal{L}_0^* v_P = \langle f, v_P \rangle - \sum_{T \in \mathcal{T}_h} \int_T \widetilde{F}(f)\, \mathcal{L}_0^* v_P, \quad \forall v_P \in V_P. \tag{19}$$

Because of (16), we may expect

$$u_P^{SC} \approx u_P^{RFB}. \tag{20}$$

For $k = 1$, both (19) and the original scheme (15)) reduce to the Streamline-Upwind Petrov-Galerkin (SUPG) scheme (5), with a special choice of the streamline diffusion τ_T. the choice $k = 2$ leads to a different scheme; we refer to [10] or [1] for a more detailed analysis.

Instead of computing by hand the effect of the *micro*-scales on the *macro*-scales, a different approach involves using a suitable numerical method for the approximation of the *micro*-scales, namely for solving (12). Because (12) gives local problems in each element T with the same structure and difficulties of the original problem (1), we are led to use an *ad hoc* method. We shall consider here three typical approaches:

1. The *Sub-grid Viscosity* (SV) method of Brezzi *et al.* and Guermond in [5,16], or a similar two-level method of Franca *et al.* in [15], which involves using an artificial diffusion or a Streamline-Upwind Petrov-Galerkin (SUPG) method for (12) on a quasi-uniform sub-grid mesh $\mathcal{T}_h^{SV}$ in each element T.

2. The *Pseudo Bubble* (PB) method of Brezzi *et al.* in [6], which involves using a standard Galerkin scheme for (12) on a suitable distorted mesh $\mathcal{T}_h^{PB}$ in each element T.

3. A new method dealing with (17)–(18) instead of (13)–(14), coined the *Discontinuous Bubble* (DB) method. In this formulation we solve (17)–(18) by a standard Galerkin method, but the boundary conditions (on ∂T^-) are weakly imposed through the variational formulation. Specifically, we look for a discontinuous k-order polynomial u_D^{DB} which approximates u_B^{RFB} on the mesh $\mathcal{T}_h^{DB} \equiv \mathcal{T}_h$ which leads to a non-conforming approximation of V_B, due to the discontinuities across the element boundaries.

The different meshes used in these three approaches are shown in Figure 1.

Details of the first two methodologies can be found in the references; A difficulty encountered in these approaches is that one has to *tune* the stabilized method for solving local problems (for example, by choosing the amount of artificial diffusion in the *sub-grid viscosity* case or the shape of the sub-grid mesh in the *pseudo residual-free bubble* case) in order to accurately compute the RFB approximation, u_E^{RFB} or, at least, the *macro*-scale degrees of freedom, u_P^{RFB}. So, in a sense, the original difficulty of tuning a numerical method which depends on some parameters remains.

The proposed approach (DG) allows the accurate computation of the *macro*-scales u_P^{RFB}, in advection-dominated regime. We present now the idea in detail. Define the space of discontinuous piecewise polynomial functions,

$$V_D \equiv V_D\left(\mathcal{T}_h, k\right) := \left\{ v \in L^2(\Omega) : v_{|T} \in \mathbb{P}_k, \forall T \in \mathcal{T}_h \right\}, \tag{21}$$

and introduce $M_D(w) \in V_D$ and $F_D(w) \in V_D$, which are the discretizations of $\widetilde{M}(w)$ and $\widetilde{F}(f)$, satisfying

$$\int_T \mathbf{c} \cdot \nabla M_D(w)\, v_D - \int_{\partial T^-} M_D(w)\, v_D\, \mathbf{c} \cdot \mathbf{n} = -\int_T \mathbf{c} \cdot \nabla w\, v_D, \quad \forall v_D \in V_D, \tag{22}$$

and

$$\int_T \mathbf{c} \cdot \nabla F_D(w)\, v_D - \int_{\partial T^-} F_D(w)\, v_D\, \mathbf{c} \cdot \mathbf{n} = \int_T f\, v_D, \quad \forall v_D \in V_D. \tag{23}$$

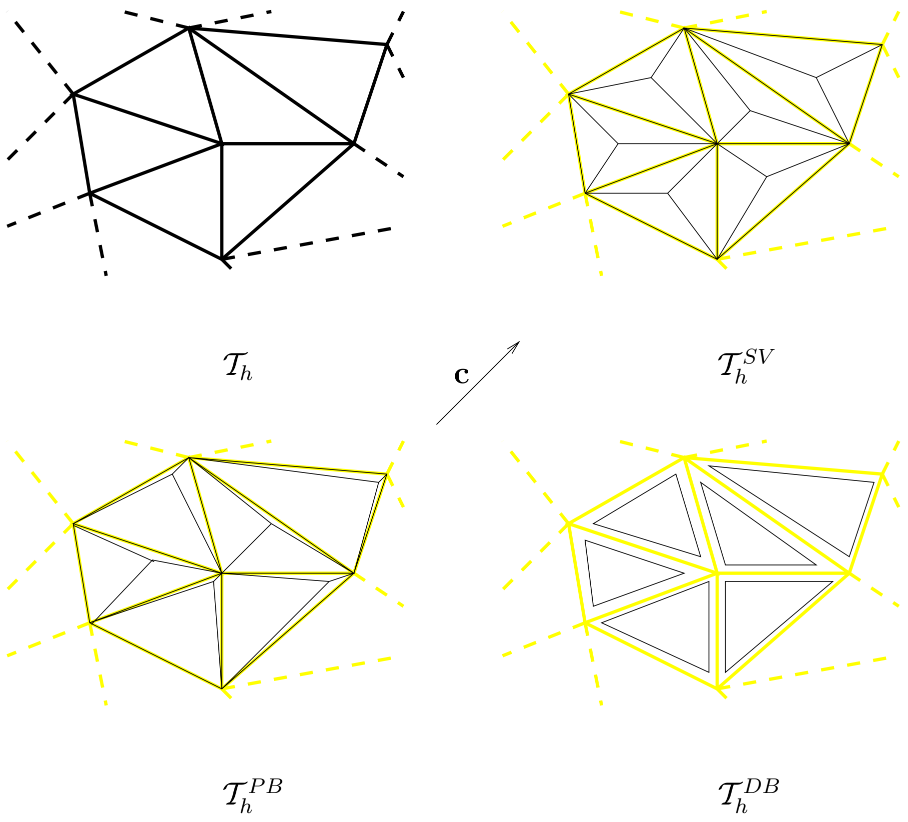


Figure 1: Examples of meshes involved in the approximation of *macro* and *micro*-scales.

Thus we define $u_P^{DB} \in V_P$ by

$$a(u_P^{DB}, v_P) + \sum_{T \in \mathcal{T}_h} \int_T M_D(u_P^{DB})\, \mathcal{L}_0^* v_P = \langle f, v_P \rangle - \sum_{T \in \mathcal{T}_h} \int_T F_D(f)\, \mathcal{L}_0^* v_P, \quad \forall v_P \in V_P, \tag{24}$$

and, since the *micro*-scales are approximated by $u_D^{DB} := M_D(u_P^{DB}) + F_D(f)$, we define $u^{DB} := u_P^{DB} + u_D^{DB}$.

Concerning the implementation, we recall that (22) and (23) describe local operators which we can invert at the element level; in other words, we can compute the local matrices which represent $M_D(\cdot)$ and $F_D(\cdot)$ at the first stage, and then use them in assembling the linear system for (24).

As mentioned earlier, we want to show that u_P^{DB}, given in (24), is equal to u_P^{SC}, whence our procedure is accurate in computing the *macro*-scales u_P^{RFB}, thanks to (20). Indeed, the formulations (19) and (24) are equivalent, because, for any $v_P \in V_P$, we have

$$\sum_{T \in \mathcal{T}_h} \int_T M_D(u_P^{DB})\, \mathcal{L}_0^* v_P = \sum_{T \in \mathcal{T}_h} \int_T \widetilde{M}(u_P^{DB})\, \mathcal{L}_0^* v_P$$

$$\sum_{T \in \mathcal{T}_h} \int_T F_D(f)\, \mathcal{L}_0^* v_P = \sum_{T \in \mathcal{T}_h} \int_T \widetilde{F}(f)\, \mathcal{L}_0^* v_P,$$

as a consequence of the following proposition.

Proposition 1. *Consider $T \in \mathcal{T}_h$, $k \geq 1$ and $\phi \in \mathbb{P}_{k-1}$; let $w \in H^1(T)$ such that*

$$\begin{cases} \mathcal{L}_0 w = \phi & \textit{in } T \\ w = 0 & \textit{on } \partial T^-, \end{cases} \tag{25}$$

and let $z \in \mathbb{P}_k$ be the solution of

$$\int_T \mathcal{L}_0 z \, v - \int_{\partial T^-} z \, v \, \mathbf{c} \cdot \mathbf{n} = \int_T \phi \, v, \quad \forall v \in \mathbb{P}_k. \tag{26}$$

Then

$$\int_T w \, \mathcal{L}_0^* v = \int_T z \, \mathcal{L}_0^* v, \quad \forall v \in \mathbb{P}_k. \tag{27}$$

Proof. The possible orientations of the element T with respect to the advection field $\mathbf{c}$ are shown in Figure 2. We denote by $\mathbf{x}^- \equiv \mathbf{x}^-(\mathbf{x}, \mathbf{c})$ the inflow point corresponding to $\mathbf{x}$, namely $\mathbf{x}^- \in \partial T^-$ and the direction from $\mathbf{x}^-$ to $\mathbf{x}$ is aligned with $\mathbf{c}$; similarly we define $\mathbf{x}^+ \equiv \mathbf{x}^+(\mathbf{x}, \mathbf{c})$ as the outflow point corresponding to $\mathbf{x}$.

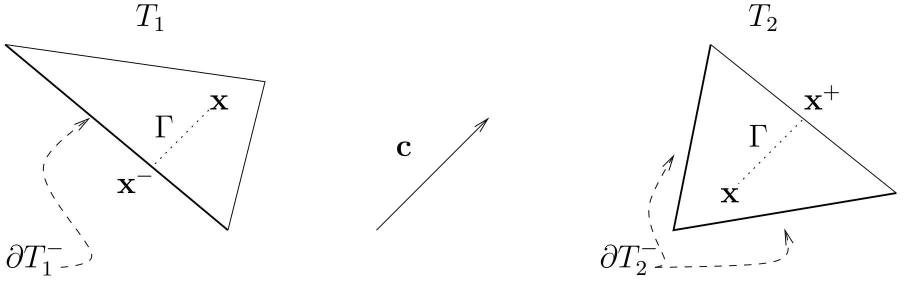


Figure 2: A triangle has either a single inflow edge, T_1, or a single outflow edge, T_2.

Consider the case of T_1: since we have

$$w(\mathbf{x}) = |\mathbf{c}|^{-1} \int_{\mathbf{x}^-}^{\mathbf{x}} \phi \, d\Gamma,$$

and since ∂T_1^- is a straight line, then $w \in \mathbb{P}_k$, and $w = z$, which in particular gives (27).

Consider now the element T_2, which has a single edge on the outflow boundary instead. In this case, $w \neq z$, but given $v \in \mathbb{P}_k$ the solution $\widetilde{v}$ of the dual problem

$$\begin{cases} \mathcal{L}_0^* \widetilde{v} = \mathcal{L}_0^* v & \text{in } T \\ \widetilde{v} = 0 & \text{on } \partial T^+, \end{cases} \tag{28}$$

which is

$$\widetilde{v}(\mathbf{x}) = -|\mathbf{c}|^{-1} \int_{\mathbf{x}^+}^{\mathbf{x}} \mathcal{L}_0^* v \, d\Gamma,$$

belongs to $\mathbb{P}_k$. Using $\widetilde{v}$ in (26), invoking (25) and integrating by parts we obtain

$$\begin{aligned}\int_T z\,\mathcal{L}_0^*\widetilde{v} + \int_{\partial T^+} z\,\widetilde{v}\,\mathbf{c}\cdot\mathbf{n} &= \int_T \mathcal{L}_0 z\,\widetilde{v} + \int_{\partial T^-} z\,\widetilde{v}\,\mathbf{c}\cdot\mathbf{n} \\ &= \int_T \phi\,\widetilde{v} \\ &= \int_T \mathcal{L}_0 w\,\widetilde{v} \\ &= \int_T w\,\mathcal{L}_0^*\widetilde{v} + \int_{\partial T} w\,\widetilde{v}\,\mathbf{c}\cdot\mathbf{n} \\ &= \int_T w\,\mathcal{L}_0^*\widetilde{v} + \int_{\partial T^+} w\,\widetilde{v}\,\mathbf{c}\cdot\mathbf{n}.\end{aligned}$$

Finally (28) gives (27). □

3 Numerical tests

In this section, we test the proposed numerical method for a simple model problem. In particular, we compare results of three approaches:

- The discontinuous approximation u^{DB}, which contains both the *macro*-scales u_P^{DB} and an approximation u_D^{DB} of the *micro*-scales u_B^{RFB} of the RFB formulation,
- The *macro*-scales u_P^{SC} only; these are still obtained invoking (24), since $u_P^{DB} \equiv u_P^{SC} \approx u_P^{RFB}$, as shown above,
- The u_P^{SUPG} approximation, given by (5).

For the sake of simplicity we restrict to linear elements (case $k = 1$), so that $u_P^{DB} \equiv u_P^{SC}$ and u_P^{SUPG} are both given by the Streamline-Upwind Petrov-Galerkin (SUPG) variational formulation (5), the only difference being the amount of *streamline diffusion* τ_T (see [10]): we recall that for u_P^{SUPG} we follow the notation of [12], where

$$\tau_T := \frac{h_T}{2\|\mathbf{c}\|}, \qquad \text{in } T \in \mathcal{T}_h.$$

We solve (1) in an L-shaped domain Ω, where the source term f and the advection field $\mathbf{c}$ are taken piecewise and discontinuous along the internal line Γ_1, as shown in Figure 3. Furthermore, we take $\varepsilon = 10^{-5}$.

The structure of the exact solution u is depicted in Figure 4; the behavior is typical of this class of problems:

- Near the outflow boundary, $\partial\Omega^+$, an exponential layer is present,
- Along the characteristic boundary, $\partial\Omega^0$, a parabolic layer is present,
- Two internal layers are present, one along Γ_1 (due to the discontinuity of f and $\mathbf{c}$), and the other along Γ_2 (which is due to the reentrant corner in $(1/2, 1/2)$).

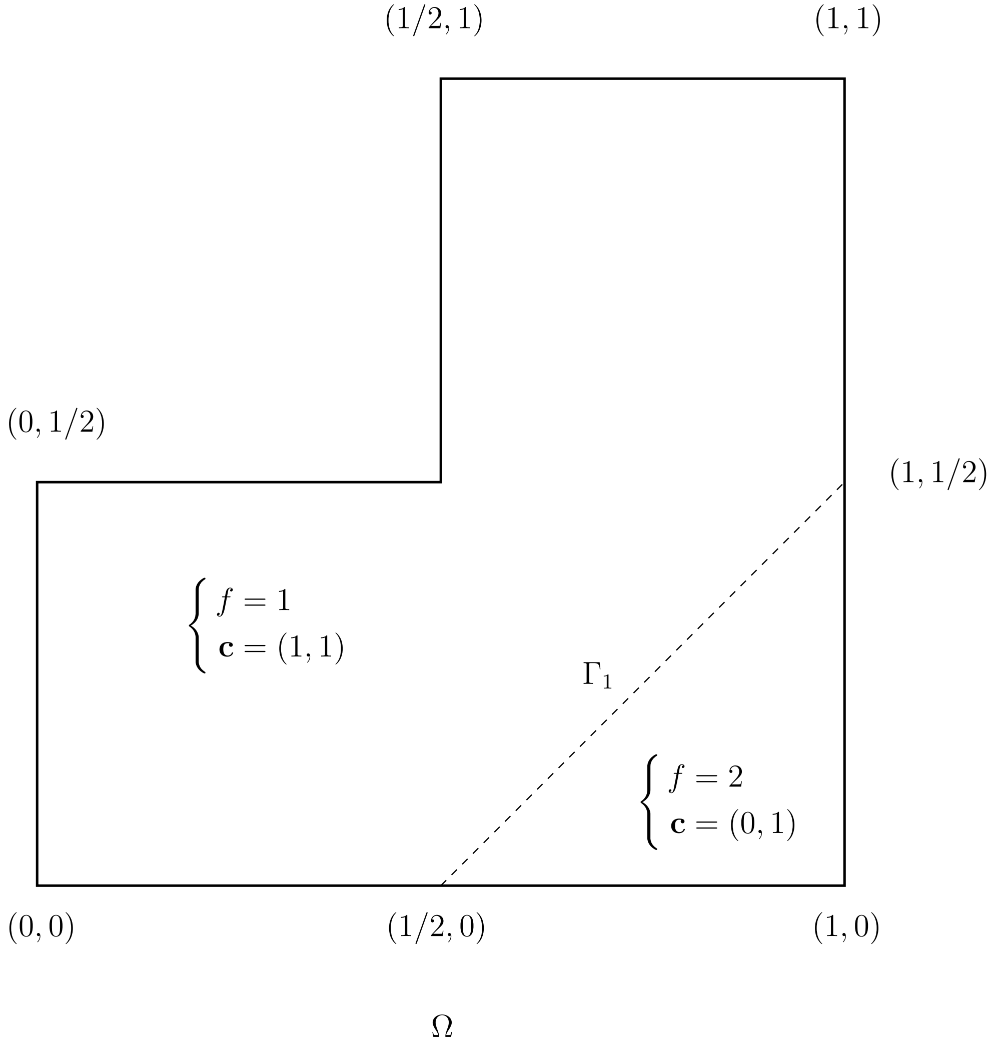


Figure 3: Domain and data for the numerical test.

The domain Ω is partitioned in a quasi-uniform, non-structured, Delaunay triangulation using the *Triangle* routine [25]. Each triangle is required to have angles that are larger than 30°.

As an example, we plot the numerical solutions obtained on one mesh; we give two different points of view on each numerical solution. The first one (from NW to SE) focuses on the internal discontinuities and structures while the second one (from NE to SW) focuses on the boundary layers. We plot u_P^{SUPG} in Figure 5(a), u_P^{SC} in Figure 5(b) and u^{DB} in Figure 5(c). In our tests, the CPU-time required for computing u_P^{SUPG}, u_P^{SC} and u^{DB}, on a given triangulation, is almost the same.

From the plots of Figure 5(a)–5(c) we see that the numerical methods capture the structure of the exact solution, even though small spurious oscillations appear near the layers. This is not surprising, because both SUPG and RFB are non-monotone numerical methods. Moreover, it seems that our complete approximation u^{DB}, which contains the discontinuous approximation of the *bubbles*, is not better than $u_P^{DB} \equiv u_P^{SC}$. On the other hand, in Figure 6 we show the numerical error of the methods in the L^2-norm (i.e., $\|u - u_P^{SUPG}\|_{L^2}$, $\|u - u_P^{SC}\|_{L^2}$ and

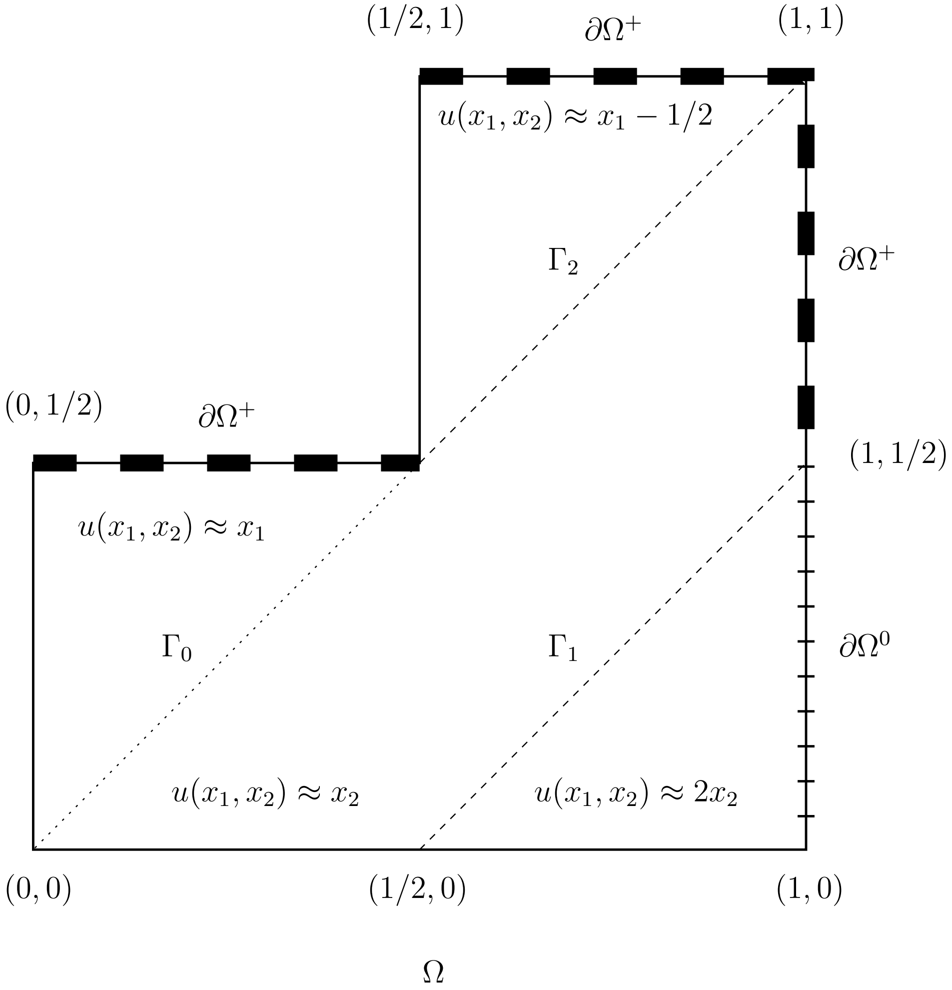
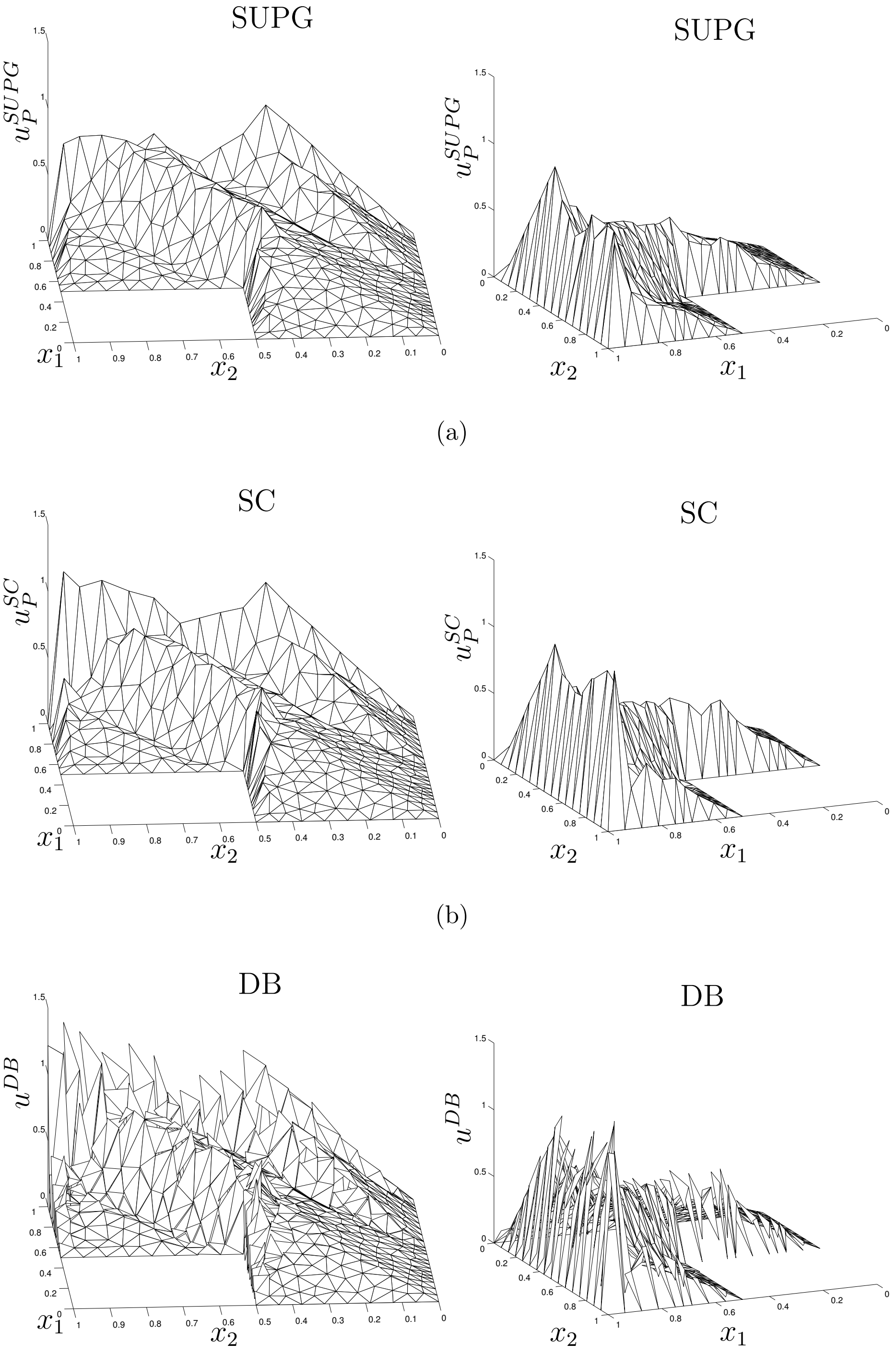


Figure 4: Structure of the exact solution u for the test problem.

$\|u - u^{DB}\|_{L^2}$), on the different meshes. It is clear that the presence of spurious oscillations does not affect the order of the methods—the optimal order of convergence in L^2-norm is 1/2, due to the presence of boundary layers. The discontinuous approximation of the *bubbles* degrees of freedom u_D^{DB} actually improves the accuracy when added to the *macro*-scales u_P^{DB}, since we see that DB is $20\% - 25\%$ better than SC (and SUPG), or, in other words, DB is as accurate as SC on a two-times finer mesh.

We have evaluated the numerical errors for the previous test-cases in different subregions of the domain. The results, not reported here, shows that the improvement of DB versus SC is maximal in exponential boundary layers, where actually DB gives a highly discontinuous numerical solution, while the two methods are nearly equivalent on internal layers. This reveals that the *bubble* part u_B^{RFB} of the RFB solution, which we approximate through u_D^{DB}, produces a proper contribution.



(a)

(b)

(c)

Figure 5: Plot of u^{SUPG} (a), u_P^{SC} (b), and u^{DB} (c) for the model problem.

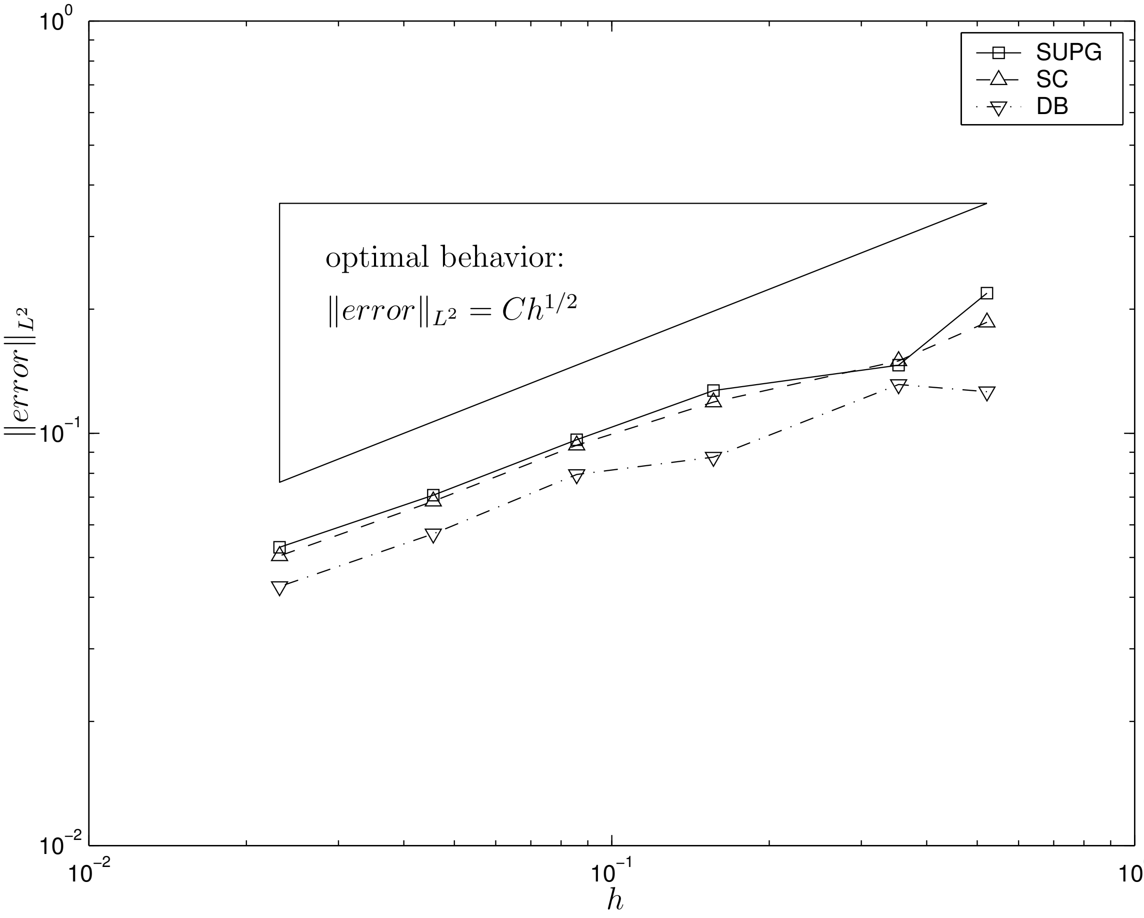


Figure 6: Convergence analysis with respect to the L^2-norm.

4 Conclusions

In this paper, we have proposed an implementation of the Residual-Free Bubble (RFB) method for the advection-diffusion linear problem in the advection-dominated regime. The Residual-Free Bubble (RFB) method is a general methodology for solving partial differential equations. From an abstract standpoint, it is based on a Finite Element formulation on an enriched space, in which the standard piecewise polynomial functions are enriched by means of bubbles, i.e., functions whose support remains inside the elements. The bubbles make the whole formulation intrinsically stable.

For a practical implementation of the Residual-Free Bubble (RFB) formulation, one has to approximate the infinitely many degrees of freedom of the bubble, to suitably approximate the local problems. We use for that purpose a discontinuous method, which has the advantage of computing in an accurate way the effect of the bubble on the *coarse*-scale, where the *coarse*-scale are piecewise linear or quadratic polynomials.

We test the procedure for linear elements at the *coarse*-scale level. Other studies have been devoted to the numerical testing of Residual-Free Bubble (RFB) based procedures, and we confirm here that the results are favorable in comparison to the popular Streamline-Upwind Petrov-Galerkin (SUPG) numerical results.

Interest in analyzing and developing the Residual-Free Bubble (RFB) methodology, as well as other *multiscale* methodologies, stems from the realization that these are a quite general methodologies whose validity in other contests has been confirmed in many recent investigations (see, e.g., [1, 2, 4, 11, 13, 24]).

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