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Some estimates for $h-p-k$−refinement in Isogeometric Analysis

L. Beirão da Veiga[1], **A. Buffa**[2], **J. Rivas**[3], **G. Sangalli**[4]

[1] Dipartimento di Matematica “F. Enriques”; Via Saldini, 50; 20133 Milano; Italy.
[2] Istituto di Matematica Applicata e Tecnologie Informatiche del CNR; Via Ferrata, 1; 27100 Pavia; Italy.
[3] Departamento de Matemáticas, Universidad del País Vasco-Euskal Herriko Unibertsitatea; Barrio Sarriena s/n; 48940 Leioa (Bizkaia); Spain.
[4] Dipartimento di Matematica, Università di Pavia; Via Ferrata, 1; 27100 Pavia; Italy.

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Summary In this paper, we propose a theoretical study of the approximation properties of NURBS spaces, which are used in Isogeometric Analysis. We obtain error estimates that are explicit in terms of the mesh-size h, the degree p and the global regularity, measured by the parameter k. Our approach covers the approximation with global regularity from C^0 up to C^{k-1}, with $2k-1 \leq p$. Notice that the interesting case of higher regularity, up to $k = p$, is still open. However, our results give an indication of the role of the smoothness k in the approximation properties, and offer a first mathematical justification of the potential of Isogeometric Analysis based on globally smooth NURBS.

1 Introduction

Isogeometric Analysis was introduced in [16] with the aim of improving the connection between numerical simulation and Computer Aided Design (CAD). Its potential has been shown in the rich recent engineering literature on this topic (e.g., [1,15,2,4,7–9,5,18]). The main idea of Isogeometric Analysis is to use directly the geometry provided by CAD in terms of non-uniform rational B-splines (NURBS) (see e.g., [14]) and to approximate the unknown solutions of differential equations by the same type of functions. Isogeometric

Analysis offers several advantages when compared to the finite elements method. First of all, complicated geometries are represented more accurately, and some common geometries as circles or ellipses are described exactly. In particular, the description of the geometry, taken directly from the CAD system, is incorporated exactly at the coarsest mesh level. In fact, this eliminates the necessity of further communication with the CAD when mesh refinement is carried out. Mesh refinement does not modify the geometry. Another advantage is that, apart from the standard h- and p-refinements, in [16] the authors introduced the possibility of k-refinement. By k-refinement, Isogeometric Analysis provides smoother functions than finite element methods. This has been proved to be superior in various applications, such as in the approximation of the spectrum of second order differential operators (see, for example, [17]) or in turbulent flow simulations (see [4]). More generally, k-refinement grants improved accuracy. A numerical study of the approximation properties by k-refinement is performed in [17,13] and various benchmark problems are considered in [7]. However, the approximation theory for NURBS-based Isogeometric Analysis is still incomplete. In fact, the only available results are in [3] and they focus on h-refinement: explicit approximation error analysis in terms of the mesh-size h is obtained there, but the estimates are implicit with respect to the degree p and the regularity of the approximation spaces.

In this paper, we initiate a theoretical study of the approximation properties of NURBS spaces smoother than C^0. We obtain error estimates that are explicit in terms of h, p and the global regularity, measured by the parameter k. Our approach is restricted to C^{k-1} approximation, with $2k-1 \leq p$. Thus, the interesting case of higher regularity, up to $k = p$, is still open. However, our results give an indication of the role of the smoothness k in the approximation properties, and offer a first mathematical justification of the potential of Isogeometric Analysis based on globally smooth NURBS.

The approximation results for NURBS presented in this paper are obtained from analogous results for splines on the parametric domain, that is, piecewise polynomials. In particular, we extend the results of [20], where only the cases $k = 0, 1$ are considered. Other estimates on the accuracy of smooth splines approximation are proposed in [12], where an optimal L^∞-error bound is given for the approximation of functions f whose derivative of order $p+1$ is bounded, by an interpolating spline of degree p and regularity C^m, with $m = [\frac{p+1}{2}]$; and also in [21], where error bounds in Sobolev norms for the Hermite interpolant, $p = 2k-1$ in our case, are given.

The outline of the paper is as follows. In Section 2 we give an overview of splines, NURBS, and Isogeometric Analysis, and present the notation. In Section 3, after proving some preliminary results on Legendre polynomials, we construct a projector on univariate splines which is defined element by element, and is interpolatory at the knots up to the derivative of order $k-1$. We prove error estimates for functions in Sobolev spaces, and analytic functions. We also give numerical evidence that our estimate of the approximation error for an analytic function is sharp. We then extend the projector, and the analysis, to the bivariate (two-dimensional tensor-product) case. In Section 5, we present our approximation estimates for NURBS. Eventually, in Section 6 we draw our conclusions.

2 An overview of NURBS-based Isogeometric Analysis

2.1 B-splines

Given two positive integers p and n, we introduce on the parametric interval (or one-dimensional *patch*) $(0,1)$, the (ordered) knot vector

$$\Xi := \{0 = \xi_1, \xi_2, \ldots, \xi_{n+p+1} = 1\}, \tag{1}$$

where we allow repetition of knots, that is, we only assume $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}$. We also introduce the vector $\{\zeta_1, \ldots, \zeta_m\}$ of knots without repetitions, and the vector $\{r_1, \ldots, r_m\}$ of their corresponding multiplicities, such that

$$\Xi = \{\underbrace{\zeta_1, \ldots, \zeta_1}_{r_1 \text{ times}}, \underbrace{\zeta_2, \ldots, \zeta_2}_{r_2 \text{ times}}, \ldots, \underbrace{\zeta_m, \ldots, \zeta_m}_{r_m \text{ times}}\},$$

with $\sum_{i=1}^{m} r_i = n+p+1$. The maximum multiplicity we allow is $p+1$. In the following we will only work with *open* knot vectors, which means that $r_1 = r_m = p+1$, that is, the first $p+1$ knots in Ξ are equal to 0, and the last $p+1$ are equal to 1. Notice that this implies $n \geq p+1$.

Through the iterative procedure detailed, for example, in [10,19] we construct p-degree (that is, $(p+1)$-order) B-spline basis functions, denoted by B_i, for $i = 1, \ldots, n$. These basis functions are piecewise polynomials of degree p on the subdivision $\{\zeta_1, \ldots, \zeta_m\}$. At ζ_i, they have $p - r_i$ continuous derivatives. We define $k_i := p - r_i + 1$ as a measure of the smoothness of the spline functions at ζ_i. Therefore, $0 \leq k_i \leq p$. The parameter k_i represents the number of matching constraints that are associated to ζ_i: the maximum multiplicity allowed, $r_i = p+1$, gives no matching conditions ($k_i = 0$) which stands for a discontinuity at ζ_i. The vector $\boldsymbol{k} = \{k_1, \ldots, k_m\}$ collects the regularity of the basis functions at the internal knots, with $k_1 = k_m = 0$ for the boundary knots, because of the open knot vector structure. Observe that the present definition of the smoothness parameter k_i is consistent with [20], but is not consistent with part of the literature on Isogeometric Analysis, where the number of continuous derivatives is directly taken as measure of the global smoothness of the discrete space (e.g., [3]). Each basis function B_i is non-negative and supported in the interval $[\xi_i, \xi_{i+p+1}]$. Moreover, these B-spline functions constitute a partition of unity, that is

$$\sum_{i=1}^{n} B_i(x) = 1 \quad \forall x \in (0,1). \tag{2}$$

The space of B-splines spanned by the basis functions B_i will be denoted by

$$\mathcal{S}^p_{\boldsymbol{k}} := \mathrm{span}\{B_i\}_{i=1}^n. \tag{3}$$

When the regularity at the inter-element points ζ_i is uniform, that is $k_i = k$ for all $i = 2, \ldots, m-1$, we shall denote (3) simply as $\mathcal{S}^p_k$.

An example of quadratic B-splines constructed from the open knot vector

$$\Xi = \{0, 0, 0, 1/5, 2/5, 3/5, 3/5, 4/5, 1, 1, 1\}$$

is presented in Figure 1. In this case $\boldsymbol{k} = \{0, 2, 2, 1, 2, 0\}$. Notice that, since the knot $\xi_6 = \xi_7 = \zeta_4 = 3/5$ has multiplicity $r_4 = 2$, the fourth, fifth and sixth functions are only continuous ($k_4 = 1$) at that point.

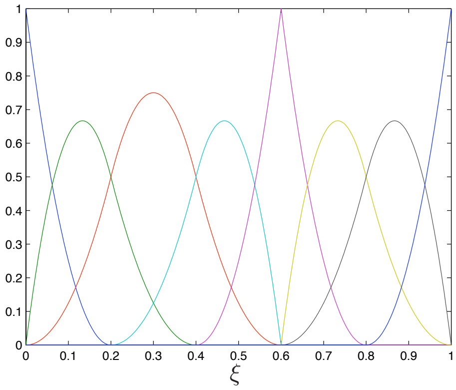

Fig. 1. Quadratic B-splines basis functions constructed from the open knot vector $\Xi = \{0, 0, 0, 1/5, 2/5, 3/5, 3/5, 4/5, 1, 1, 1\}$.

The previous definition of the B-splines space is extended to the two-dimensional framework in the following way. Let us consider the square $\widehat{\Omega}=[0,1]^2$, which will be referred to as a (two-dimensional) *patch*. Given integers p_d and n_d, with $d=1,2$, we introduce the knot vectors $\Xi_d=\{\xi_{1,d},\xi_{2,d},\dots,\xi_{n_d+p_d+1,d}\}$ and the associated vectors $\{\zeta_{1,d},\dots,\zeta_{m_d,d}\}$, $\{r_{1,d},\dots,r_{m_d,d}\}$ and $\{k_{1,d},\dots,k_{m_d,d}\}$, as in the one-dimensional case. Associated with these knot vectors there is a *mesh* $\mathcal{Q}_h$ on the patch, that is, a partition of $(0,1)^2$ into rectangles:

$$\mathcal{Q}_h=\{Q=\bigotimes_{d=1,2}(\zeta_{i_d,d},\zeta_{i_d+1,d}),\ 1\le i_d\le m_d-1\}. \tag{4}$$

Given an element $Q\in\mathcal{Q}_h$, we set by $h_Q=\operatorname{diam}(Q)$, while $h=\max\{h_Q,\ Q\in\mathcal{Q}_h\}$ represents the global mesh size.

We associate to the two given knot vectors Ξ_d, $d=1,2$, the p_d-degree univariate B-splines basis functions $B_{i_d,d}$, with $i_d=1,\dots,n_d$. Then, on the associated mesh $\mathcal{Q}_h$, we define the tensor-product B-spline basis functions as

$$B_{i_1i_2}:=B_{i_1,1}\otimes B_{i_2,2},\quad i_1=1,\dots,n_1,\ i_2=1,\dots,n_2\,. \tag{5}$$

Then, the tensor product B-spline space is defined as the space spanned by these basis functions, namely

$$\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\equiv\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}(\mathcal{Q}_h):=\mathcal{S}^{p_1}_{\boldsymbol{k}_1}\otimes\mathcal{S}^{p_2}_{\boldsymbol{k}_2}=\operatorname{span}\{B_{i_1i_2}\}_{i_1=1,i_2=1}^{n_1,n_2}\,. \tag{6}$$

Notice that the space $\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}(\mathcal{Q}_h)$ is fully characterized by the mesh $\mathcal{Q}_h$ and by p_1, p_2, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, as our notation reflects.

The minimum degree of the space $\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}(\mathcal{Q}_h)$ is denoted $p=\min\{p_d, d=1,2\}$. The minimum and maximum regularity of the space in the d-direction are denoted by $k_{\min,d}:=\min\{k_{i,d}: i=2,\dots,m_d-1\}$ and $k_{\max,d}:=\max\{k_{i,d}: i=2,\dots,m_d-1\}$, respectively.

2.2 NURBS and the geometry of the physical domain

Rational B-splines are typically used in CAD systems to parametrize geometrical entities. Here, we briefly introduce two-dimensional NURBS domains, which are constructed by means of a projective transformation of three-dimensional Spline surfaces, and the associated NURBS basis functions.

A Spline surface in $\mathbb{R}^3$ is the image $\widetilde{\Omega}$ of a map $\widetilde{\mathbf{F}}:\widehat{\Omega}\to\widetilde{\Omega}$ where

$$\widetilde{\mathbf{F}}=\sum_{i_1=1,i_2=1}^{n_1,n_2}\widetilde{\mathbf{C}}_{i_1i_2}B_{i_1i_2}. \tag{7}$$

The coefficients $\widetilde{\mathbf{C}}_{i_1 i_2} \in \mathbb{R}^3$ in (7) are referred as *control points* for the surface $\widetilde{\Omega}$. Let (x_1, x_2, x_3) denote the coordinates of points in $\mathbb{R}^3$, We assume that all $\widetilde{\mathbf{C}}_{i_1 i_2}$ lie in the semi-space $x_3 \geq C$, for a given constant $C > 0$. Consider now the projective transformation (see [14]) onto the plane $x_3 = 1$ defined by $(x_1, x_2, x_3) \mapsto (x_1/x_3, x_2/x_3, 1)$. This projective transformation maps the surface $\widetilde{\Omega}$ into the planar surface Ω, which is then identified with the corresponding two-dimensional domain $\Omega \in \mathbb{R}^2$, the physical domain of interest. Therefore Ω is parametrized by the map $\mathbf{F} : \widehat{\Omega} \to \Omega$ given by

$$\mathbf{F} = \frac{\sum_{i_1=1, i_2=1}^{n_1, n_2} \mathbf{C}_{i_1 i_2} B_{i_1 i_2}}{\sum_{i_1=1, i_2=1}^{n_1, n_2} w_{i_1 i_2} B_{i_1 i_2}} \tag{8}$$

where $\mathbf{C}_{i_1 i_2} \in \mathbb{R}^2$ and $w_{i_1 i_2}$ are referred as control points and weights of the NURBS parametrization, and they are associated to the previous control points in $\mathbb{R}^3$ by the relation $(\mathbf{C}_{i_1 i_2}, w_{i_1 i_2}) = \widetilde{\mathbf{C}}_{i_1 i_2}$.

We now introduce the *weighting function*

$$w := \sum_{i_1=1, i_2=1}^{n_1, n_2} w_{i_1 i_2} B_{i_1 i_2}, \tag{9}$$

which, due to the properties of B-spline bases, is strictly greater than zero and is smooth on each element, along with its reciprocal. We also define NURBS basis functions on $\widehat{\Omega}$ by

$$R_{i_1 i_2} = \frac{w_{i_1 i_2} B_{i_1 i_2}}{w}, \tag{10}$$

and, accordingly, the NURBS space on the patch, denoted by $\mathcal{N}$, is

$$\mathcal{N}_{\boldsymbol{k}_1, \boldsymbol{k}_2}^{p_1, p_2} \equiv \mathcal{N}_{\boldsymbol{k}_1, \boldsymbol{k}_2}^{p_1, p_2}(\mathcal{Q}_h, w) := \operatorname{span}\{R_{i_1 i_2}\}_{i_1=1, i_2=1}^{n_1, n_2}. \tag{11}$$

Notice that the NURBS geometrical map $\mathbf{F}$ is then given by

$$\mathbf{F} = \sum_{i_1=1, i_2=1}^{n_1, n_2} \mathbf{C}_{i_1 i_2} R_{i_1 i_2}. \tag{12}$$

We assume that $\mathbf{F}$ is invertible, with smooth inverse, on each element $Q \in \mathcal{Q}_h$.

The space $\mathcal{V}$ of NURBS on Ω is the *push-forward* of the space $\mathcal{N}$ of NURBS on the patch $\widehat{\Omega}$

$$\mathcal{V}_{\boldsymbol{k}_1, \boldsymbol{k}_2}^{p_1, p_2} \equiv \mathcal{V}_{\boldsymbol{k}_1, \boldsymbol{k}_2}^{p_1, p_2}(\mathcal{Q}_h, w, \mathbf{F}) := \operatorname{span}\{R_{i_1 i_2} \circ \mathbf{F}^{-1}\}_{i_1=1, i_2=1}^{n_1 n_2}. \tag{13}$$

On Ω there is a natural mesh, denoted by $\mathcal{K}_h$, which is the image of the mesh $\mathcal{Q}_h$ through $\mathbf{F}$.

2.3 Isogeometric method and $h-p-k-$refinement

The Isogeometric Analysis is a discretization method for partial differential equations that uses $\mathcal{V}_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{p_1,p_2}$ as approximation space for unknown fields.

Clearly, the accuracy of an isogeometric approach depends on the approximation properties of the space $\mathcal{V}_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{p_1,p_2}$. Three kinds of refinement are possible, which are summarized below:

- $h-$refinement: a finer mesh is constructed, maintaining the degrees p_1, p_2 of the space. That is, h is reduced, and the global regularity of the space is maintained;
- $p-$refinement: on the same mesh, a higher degree is used, keeping the same regularity at the mesh lines. That is, p_1, p_2 are increased;
- $k-$refinement: both mesh refinement and degree elevation are performed, with highest regularity at the new mesh interfaces. That is, first h is reduced, then p_1, p_2 are increased.

When $k_{i,d} \leq 1$, for all i and d, $h-$refinement and $p-$refinement are the usual ones of finite element analysis. All refinements are constructed by knot insertion (with possible repetitions) and order elevation, see [10,16]. The interplay of the three refinement procedures allows one to obtain, from an initial coarse space, refined spaces with various mesh-size, order, and global regularity. The purpose of this paper is to construct a projector from suitable Sobolev spaces onto $\mathcal{V}_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{p_1,p_2}(\mathcal{Q}_h, w, \mathbf{F})$ and to study its approximation order in terms of h, p_1, p_2, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ for some cases.

In Isogeometric Analysis, $\mathbf{F} : \hat{\Omega} \longrightarrow \Omega$ and the weight function $w : \hat{\Omega} \longrightarrow \mathbb{R}$ are given from the CAD description of the geometry, typically on a coarse mesh. For that reason, we assume that both w and $\mathbf{F}$ are defined at the coarsest level of discretization, on the coarsest mesh $\mathcal{Q}_{h_0}$, that is

$$w \in \mathcal{S}_{\boldsymbol{k}_{1,0},\boldsymbol{k}_{2,0}}^{p_{1,0},p_{2,0}}(\mathcal{Q}_{h_0}), \qquad \mathbf{F} \in \left(\mathcal{N}_{\boldsymbol{k}_{1,0},\boldsymbol{k}_{2,0}}^{p_{1,0},p_{2,0}}(\mathcal{Q}_{h_0}, w)\right)^2, \tag{14}$$

and that the approximation space $\mathcal{N}_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{p_1,p_2}(\mathcal{Q}_h, w, \mathbf{F})$ is obtained from the initial space $\mathcal{N}_{\boldsymbol{k}_{1,0},\boldsymbol{k}_{2,0}}^{p_{1,0},p_{2,0}}(\mathcal{Q}_{h_0}, w)$ with refinements of the kind described above.

3 Error estimates for splines in one dimension

We will construct a projection operator on the space of splines of degree p and continuous derivatives up to the order $k-1$, with the assumption that $2k-1 \le p$. The definition of this operator follows the idea used by Schwab, in [20], in the case of C^0 finite elements. It consists in approximating the k-th derivative of the given function by a scaled Legendre truncated series and then integrate it.

Therefore, we start this section with a review on some properties of Legendre polynomials, that can be consulted in [20], and the derivation of orthogonality properties of their primitives of order k.

After defining the projector in the reference interval $[-1,1]$ and obtaining the corresponding error estimates, we will be in the position to define the spline interpolation.

3.1 Preliminaries on Legendre polynomials

The Legendre polynomial of degree i is defined by the Rodrigues formula

$$L_i(x) = \frac{1}{i!\,2^i}\frac{d^i}{dx^i}\big((x^2-1)^i\big), \quad i = 0,1,\dots$$

They satisfy the Legendre differential equation

$$\big((1-x^2)L_i'(x)\big)' + i(i+1)L_i(x) = 0, \quad i = 0,1,\dots \tag{15}$$

and form an orthogonal basis of the space of square-integrable functions in $\Lambda = (-1,1)$, $L^2(\Lambda)$. More precisely, for $i,j = 0,1,\dots$

$$\int_{-1}^{1} L_i(x)L_j(x)\,dx = \frac{2}{2i+1}\delta_{ij},$$

where δ_{ij} is Kronecker's delta.

This fact allows to expand any given function, $\varphi \in L^2(\Lambda)$, into a Legendre series of the form

$$\varphi(x) = \sum_{i=0}^{\infty} \hat{\varphi}_i L_i(x),$$

with

$$\hat{\varphi}_i = \frac{2i+1}{2}\int_{-1}^{1}\varphi(x)L_i(x)\,dx, \quad i = 0,1,\dots$$

and

$$\|\varphi\|^2_{L^2(\Lambda)} = \sum_{i=0}^{\infty} |\hat{\varphi}_i|^2 \frac{2}{2i+1}.$$

If we denote $\pi_p\varphi(x) = \sum_{i=0}^{p}\hat{\varphi}_i L_i(x)$, where p is a non-negative integer, then $\pi_p\varphi$ is the best approximation of φ in the $L^2(\Lambda)$ norm by polynomials of degree p.

We give in the following lemma a generalization of the Legendre differential equation (15) satisfied by higher order derivatives of Legendre polynomials.

Lemma 1 *Let L_i be the Legendre polynomial of degree i, for $i = 0, 1 \ldots$. Then, for any $n \geq 1$*

$$(1-x^2)L_i^{(n+1)}(x)-2nxL_i^{(n)}(x)+(i+n)(i-n+1)L_i^{(n-1)}(x)=0. \quad (16)$$

Proof The result follows by an induction argument on n. If $n = 1$, (16) is just the Legendre differential equation (15). Suppose (16) is true for $n-1$, this is,

$$(1-x^2)L_i^{(n)}(x)-2(n-1)xL_i^{(n-1)}(x)+(i+n-1)(i-n+2)L_i^{(n-2)}(x)=0.$$

Derivating with respect to x one gets

$$\begin{aligned} &-2xL_i^{(n)}(x)+(1-x^2)L_i^{(n+1)}(x)-2(n-1)L_i^{(n-1)}(x) \\ &\qquad -2(n-1)xL_i^{(n)}(x)+(i+n-1)(i-n+2)L_i^{(n-1)}(x)=0, \end{aligned}$$

and rearranging terms,

$$\begin{aligned} &(1-x^2)L_i^{(n+1)}(x)-2x(n-1+1)L_i^{(n)}(x) \\ &\qquad +\big((i+n-1)(i-n+2)-2(n-1)\big)L_i^{(n-1)}(x)=0. \end{aligned}$$

Since $(i+n-1)(i-n+2)-2(n-1)=(i+n)(i-n+1)$, the result follows for any $n \in \mathbf{N}$.

Along this paper, we will also need primitives of Legendre polynomials. We will show that these functions form also an orthogonal basis of a weighted L^2 space. We start by giving their precise definition.

Definition 1 *Let L_i be the Legendre polynomial of degree i, $i \geq 0$. Given a non-negative integer n, we define its n-th primitive $\Psi_{i,n}$ in a recursive way,*

$$\Psi_{i,0}(x)=L_i(x), \quad \Psi_{i,n}(x)=\int_{-1}^{x}\Psi_{i,n-1}(\xi)\,d\xi, \quad n=1,2,\ldots \quad (17)$$

The following lemma relates primitives and derivatives of Legendre polynomials.

Lemma 2 *Let i, n be non-negative integers and $\Psi_{i,n}$ be defined as in (17). Then, for any $n = 0, 1, \ldots$*

$$\Psi_{i,n}(x)=(-1)^n\frac{(i-n)!}{(i+n)!}(1-x^2)^nL_i^{(n)}(x), \quad \forall\, i \geq n. \quad (18)$$

Proof The proof is again based on induction. When $n = 0$, (18) is true by definition. Let us assume that

$$\Psi_{i,n-1}(x) = (-1)^{n-1}\frac{(i-n+1)!}{(i+n-1)!}(1-x^2)^{n-1}L_i^{(n-1)}(x).$$

We want to prove that, for $i \geq n$,

$$(-1)^n\frac{(i+n)!}{(i-n)!}\Psi_{i,n}(x) = (1-x^2)^n L_i^{(n)}(x).$$

Taking into account the definition of $\Psi_{i,n}$ followed by the induction hypothesis, one gets

$$\begin{aligned}(-1)^n\frac{(i+n)!}{(i-n)!}\Psi_{i,n}(x) &= (-1)^n\frac{(i+n)!}{(i-n)!}\int_{-1}^{x}\Psi_{i,n-1}(\xi)\,d\xi\\ &= (-1)^n\frac{(i+n)!}{(i-n)!}(-1)^{n-1}\frac{(i-n+1)!}{(i+n-1)!}\int_{-1}^{x}(1-\xi^2)^{n-1}L_i^{(n-1)}(\xi)\,d\xi\\ &= -(i+n)(i-n+1)\int_{-1}^{x}(1-\xi^2)^{n-1}L_i^{(n-1)}(\xi)\,d\xi.\end{aligned}$$

Therefore, we want to prove that

$$(1-x^2)^n L_i^{(n)}(x) = -(i+n)(i-n+1)\int_{-1}^{x}(1-\xi^2)^{n-1}L_i^{(n-1)}(\xi)\,d\xi.$$

Since both sides of the equality cancel at $x = -1$, it is enough if we prove that their derivatives coincide, that is, we need

$$\begin{aligned}&-2nx(1-x^2)^{n-1}L_i^{(n)}(x) + (1-x^2)^n L_i^{(n+1)}(x)\\ &\qquad = -(i+n)(i-n+1)(1-x^2)^{n-1}L_i^{(n-1)}(x),\end{aligned}$$

which is just (16) multiplied by $(1-x^2)^{n-1}$, and thus the result follows.

It is known that the n-th derivatives of Legendre polynomials are orthogonal with respect to the weight $(1-x^2)^n$. In fact, the following holds for any $n \geq 0$,

$$\int_{-1}^{1}(1-x^2)^n L_i^{(n)}(x)L_j^{(n)}(x)\,dx = \frac{2}{2i+1}\frac{(i+n)!}{(i-n)!}\delta_{ij}, \quad i,j = 0,1,\ldots \tag{19}$$

We can therefore derive the orthogonality of primitives of Legendre polynomials.

Corollary 1 *Let n be a non-negative integer. The family $\{\Psi_{i,n}\}_{i=n}^{\infty}$ is orthogonal with respect to the weight $(1-x^2)^n$. More precisely,*

$$\int_{-1}^{1} \frac{\Psi_{i,n}(x)\Psi_{j,n}(x)}{(1-x^2)^n}\,dx = \frac{2}{2i+1}\frac{(i-n)!}{(i+n)!}\delta_{ij}, \quad i,j = n, n+1, \dots \tag{20}$$

where δ_{ij} is Kronecker's delta.

Due to the orthogonality of the derivatives of order n of Legendre polynomials with respect to the weight $(1-x^2)^n$ in the reference element $\Lambda = (-1,1)$, it will be easier to obtain estimates in terms of certain weighted seminorms instead of the classical Sobolev seminorms. Given non-negative integers $s_1 \le s_2$, we denote

$$V_{s_1}^{s_2}(\Lambda) = \{\varphi \in L^2(\Lambda)\colon\ |\varphi|^2_{V_{s_1}^{s_2}} = \sum_{j=s_1}^{s_2} \int_{-1}^{1} (1-x^2)^j |\varphi^{(j)}(x)|^2\,dx < +\infty\}.$$

In particular,

$$|\varphi|^2_{V_s^s} = \int_{-1}^{1} (1-x^2)^s |\varphi^{(s)}(x)|^2\,dx,$$

and taking into account (19), one can rewrite this integral in terms of the Legendre coefficients of φ.

Lemma 3 *Let $\varphi \in V_s^s(\Lambda)$, for some integer $s \ge 0$, and let $\varphi(x) = \sum_{i=0}^{\infty} a_i L_i(x)$ be its Legendre expansion. Then,*

$$|\varphi|^2_{V_s^s} = \sum_{i=s}^{\infty} |a_i|^2 \frac{2}{2i+1}\frac{(i+s)!}{(i-s)!}. \tag{21}$$

We finish this section with an estimate for some quotients of factorials that is a consequence of Stirling formula (see e.g. [20], p. 72).

Lemma 4 *Let m, n be non-negative integers, with $n \le m$. Then, there exists a positive constant, C, independent of m, n such that*

$$\frac{(m-n)!}{(m+n)!} \le C\left(\frac{e}{2}\right)^{2n} m^{-2n}. \tag{22}$$

3.2 Polynomial approximation on the reference element

Let $\Lambda = (-1,1)$ be the reference element. We denote by $\mathcal{S}^p(\Lambda)$ the space of polynomials of degree up to p on Λ.

Definition 2 *Let k, p be non-negative integers. We define the projection operator $\widehat{\pi}_{p,k} \colon H^k(\Lambda) \to \mathcal{S}^p(\Lambda)$ as follows:*

$$(\widehat{\pi}_{p,k} u)^{(k)}(x) = \pi_{p-k} u^{(k)}(x), \ x \in \Lambda, \tag{23}$$

$$(\widehat{\pi}_{p,k} u)^{(j)}(-1) = u^{(j)}(-1), \ j = 0, 1, \ldots, k-1, \tag{24}$$

where π_{p-k} is the classical L^2-orthogonal projection on $S^{p-k}(\Lambda)$.

Therefore, if $u^{(k)}(x) = \sum_{i=0}^{\infty} \alpha_i L_i(x)$, with

$$\alpha_i = \frac{2i+1}{2} \int_{-1}^{1} u^{(k)}(x) L_i(x)\, dx, \quad i = 0, 1, \ldots, \tag{25}$$

then

$$(\widehat{\pi}_{p,k} u)^{(k)}(x) = \sum_{i=0}^{p-k} \alpha_i L_i(x)$$

and integrating k times and taking into account the boundary conditions (24), one has

$$\widehat{\pi}_{p,k} u(x) = \sum_{i=0}^{p-k} \alpha_i \Psi_{i,k}(x) + \sum_{\ell=0}^{k-1} u^{(\ell)}(-1) \frac{(x+1)^\ell}{\ell!}.$$

Lemma 5 *Let p, k be non-negative integers and $u \in H^k(\Lambda)$. Then, if $p \geq 2k-1$,*

$$(\widehat{\pi}_{p,k} u)^{(j)}(1) = u^{(j)}(1), \ j = 0, 1, \ldots, k-1. \tag{26}$$

Proof Given a non-negative integer ℓ and a function $\varphi \in C^\ell(\Lambda)$, its Taylor expansion of order ℓ reads as follows,

$$\varphi(x) = \sum_{i=0}^{\ell-1} \frac{\varphi^{(i)}(-1)(x+1)^i}{i!} + \int_{-1}^{x} \frac{(x-y)^{\ell-1}}{(\ell-1)!} \varphi^{(\ell)}(y)\, dy.$$

For $j = 0, \ldots, k-1$, we write the Taylor expansions of both $u^{(j)}$ and $(\widehat{\pi}_{p,k} u)^{(j)}$ for $\ell = k - j$, to yield

$$\begin{aligned} u^{(j)}(x) &= \sum_{i=0}^{k-j-1} \frac{u^{(j+i)}(-1)(x+1)^i}{i!} + \int_{-1}^{x} \frac{(x-y)^{k-j-1}}{(k-j-1)!} u^{(k)}(y)\, dy \\ &= \sum_{i=j}^{k-1} \frac{u^{(i)}(-1)(x+1)^{i+j}}{(i+j)!} + \int_{-1}^{x} \frac{(x-y)^{k-j-1}}{(k-j-1)!} u^{(k)}(y)\, dy, \end{aligned}$$

and similarly

$$(\widehat{\pi}_{p,k}u)^{(j)}(x) = \sum_{i=j}^{k-1} \frac{(\widehat{\pi}_{p,k}u)^{(i)}(-1)(x+1)^{i+j}}{(i+j)!} + \int_{-1}^{x} \frac{(x-y)^{k-j-1}}{(k-j-1)!}(\widehat{\pi}_{p,k}u)^{(k)}(y)\,dy.$$

Taking $x=1$, subtracting the expressions above and recalling (24), we get

$$u^{(j)}(1) - (\widehat{\pi}_{p,k}u)^{(j)}(1) = \int_{-1}^{1} \frac{(1-y)^{k-j-1}}{(k-j-1)!}\big(u^{(k)}(y) - (\widehat{\pi}_{p,k}u)^{(k)}(y)\big)\,dy.$$

Finally, since $(\widehat{\pi}_{p,k}u)^{(k)} = \pi_{p-k}u^{(k)}$, the term inside the parenthesis is orthogonal to any polynomial of degree less than or equal to $p-k$, which is the case for $(1-y)^{k-j-1}$ because we have assumed that $p \geq 2k-1$ and therefore $k-j-1 \leq p-k$ for $j=0,\ldots,k-1$. Consequently, the integral above is null and the claim of the lemma follows.

Theorem 1 *Let p,k,s be non-negative integers, $p \geq 2k-1$, $\kappa = p-k+1$ and $u\colon \Lambda \to \mathbb{R}$ such that $u^{(k)} \in V_s^s$. Then, if $s \leq \kappa$,*

$$\left\| \frac{(u-\widehat{\pi}_{p,k}u)^{(k-\ell)}}{(1-x^2)^{\ell/2}} \right\|_{L^2(\Lambda)}^2 \leq \frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-\ell)!}{(\kappa+\ell)!}|u^{(k)}|_{V_s^s}^2, \tag{27}$$

for $\ell = 0,1,\ldots,k$.

Proof From the definition of $\widehat{\pi}_{p,k}u$ it follows that

$$u^{(k)}(x) - (\widehat{\pi}_{p,k}u)^{(k)}(x) = \sum_{i=\kappa}^{\infty} \alpha_i L_i(x),$$

where α_i is the i-th Legendre coefficient of $u^{(k)}$, defined in (25). Integrating ℓ times over the interval $[-1,x]$, as the derivatives of u and $\widehat{\pi}_{p,k}u$ coincide at $x=-1$, it yields that

$$u^{(k-\ell)}(x) - (\widehat{\pi}_{p,k}u)^{(k-\ell)}(x) = \sum_{i=\kappa}^{\infty} \alpha_i \Psi_{i,\ell}(x),$$

with $\Psi_{i,\ell}$ is defined in (17). Since $p \geq 2k-1$, applying (20), the functions $\{\Psi_{i,\ell}\}$ appearing in the previous summation are orthogonal with respect to the weight $(1-x^2)^{-\ell}$. Therefore,

$$\left\| \frac{(u-\widehat{\pi}_{p,k}u)^{(k-\ell)}}{(1-x^2)^{\ell/2}} \right\|_{L^2(\Lambda)}^2 = \sum_{i=\kappa}^{\infty} |\alpha_i|^2 \frac{2}{2i+1}\frac{(i-\ell)!}{(i+\ell)!}. \tag{28}$$

We now take into account (21) to rewrite the above equality as

$$\begin{aligned}\left\|\frac{(u-\widehat{\pi}_{p,k}u)^{(k-\ell)}}{(1-x^2)^{\ell/2}}\right\|^2_{L^2(\Lambda)} &= \sum_{i=\kappa}^{\infty}|\alpha_i|^2\frac{2}{2i+1}\frac{(i-\ell)!}{(i+\ell)!}\frac{(i-s)!}{(i+s)!}\frac{(i+s)!}{(i-s)!}\\ &\le \frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-\ell)!}{(\kappa+\ell)!}\sum_{i=\kappa}^{\infty}|\alpha_i|^2\frac{2}{2i+1}\frac{(i+s)!}{(i-s)!}\\ &\le \frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-\ell)!}{(\kappa+\ell)!}|u^{(k)}|^2_{V^s_s}\end{aligned}$$

since we have assumed $s \le \kappa$.

Corollary 2 *Let p, k, s be non-negative integers, $p \ge 2k-1$, $\kappa = p-k+1$ and $u\colon \Lambda \to \mathbb{R}$ such that $u^{(k)} \in H^s(\Lambda)$. Then, if $s \le \kappa$,*

$$\|u^{(j)} - (\widehat{\pi}_{p,k}u)^{(j)}\|^2_{L^2(\Lambda)} \le \frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-(k-j))!}{(\kappa+(k-j))!}|u^{(k)}|^2_{H^s(\Lambda)}, \qquad (29)$$

for $j = 0, \ldots, k$.

Proof Taking $j = k-\ell$ in (27), one gets

$$\left\|\frac{u^{(j)}-(\widehat{\pi}_{p,k}u)^{(j)}}{(1-x^2)^{(k-j)/2}}\right\|^2_{L^2(\Lambda)} \le \frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-(k-j))!}{(\kappa+(k-j))!}|u^{(k)}|^2_{V^s_s}.$$

Since $1-x^2 \le 1$ when $x \in \Lambda$, on the one hand, for $j \le k$,

$$\left\|\frac{u^{(j)}-(\widehat{\pi}_{p,k}u)^{(j)}}{(1-x^2)^{(k-j)/2}}\right\|^2_{L^2(\Lambda)} \ge \|u^{(j)} - (\widehat{\pi}_{p,k}u)^{(j)}\|^2_{L^2(\Lambda)},$$

and on the other hand, if $s \ge 0$, then

$$|u^{(k)}|^2_{V^s_s} = \int_{-1}^{1}\left|\left(u^{(k)}\right)^{(s)}(x)\right|^2(1-x^2)^s\,dx \le |u^{(k)}|^2_{H^s(\Lambda)},$$

from where (29) follows.

3.3 Spline approximation on the reference domain [0, 1]

We consider now the reference domain $[0,1]$, non-negative integers k and p, where $p \ge 2k-1$, and the mesh on $[0,1]$ formed by the vector of distinct knots $\{\zeta_1, \ldots, \zeta_m\}$. We denote $I_i = (\zeta_i, \zeta_{i+1})$, $i = 1, \ldots, m-1$. We intend to approximate a function $u \in H^k(0,1)$ by a piecewise polynomial in $\mathcal{S}^p_k$.

The first step is defining the projection operator $\pi_{p,k} : H^k(0,1) \to \mathcal{S}^p_k$ and then generalize Corollary 2.

Definition 3 *Let $T_i : \Lambda \to I_i$ be the linear mapping from Λ to I_i. We define $\pi_{p,k}\colon H^k(0,1) \to S_k^p$ as follows: given $u \in H^k(0,1)$,*

$$(\pi_{p,k}u)\circ T_i = \widehat{\pi}_{p,k}(u\circ T_i) \ , \qquad i=1,\dots,m-1.$$

Note that, thanks to the Definition 2 of $\widehat{\pi}_{p,k}$, the operator $\pi_{p,k}$ maps $H^k(0,1)$ onto the space of splines $\mathcal{S}_k^p$. Indeed, $(\pi_{p,k}u\circ T_i)^{(j)}(\pm 1) = (u\circ T_i)^{(j)}(\pm 1)$, $j=0,\dots,k-1$ implies that, for any $u\in H^k(0,1)$ we have

$$(\pi_{p,k}u)^{(j)}(\zeta_i) = u^{(j)}(\zeta_i) \qquad i=1,\dots,m$$

and this means that $\pi_{p,k}u$ has $k-1$ continuous derivatives at each knot ζ_i.

As standard, the following scaling holds: let $\varphi\colon I_i\to\mathbb{R}$ and $\phi = \varphi\circ T_i$. Then, for any $n\geq 0$,

$$\|\varphi^{(n)}\|^2_{L^2(I_i)} = \Big(\frac{h_i}{2}\Big)^{-2n+1}\|\phi^{(n)}\|^2_{L^2(\Lambda)}, \tag{30}$$

where $h_i = \zeta_{i+1}-\zeta_i$ is the length of I_i, for $i=1,\dots,m-1$.

We now state the mentioned generalization of Corollary 2.

Corollary 3 *Given the subdivision $\{0=\zeta_1,\dots,\zeta_m=1\}$ of the reference domain $(0,1)$, let $I_i=(\zeta_i,\zeta_{i+1})$, $h_i=\zeta_{i+1}-\zeta_i$, $i=1,\dots,m-1$, k, p non-negative integers with $p\geq 2k-1$ and $u^{(k)}\in H^s(0,1)$ for some $0\leq s\leq\kappa=p-k+1$. The following holds for all $i=1,\dots,m-1$*

$$\|u^{(j)}-(\pi_{p,k}u)^{(j)}\|^2_{L^2(I_i)} \leq \Big(\frac{h_i}{2}\Big)^{2(s+k-j)}\frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-(k-j))!}{(\kappa+(k-j))!}|u^{(k)}|^2_{H^s(I_i)} \tag{31}$$

where $h_i=\zeta_{i+1}-\zeta_i$.

Proof Applying (30) with $j=n$ and $\varphi = u-\pi_{p,k}u$,

$$\|u^{(j)}-(\pi_{p,k}u)^{(j)}\|^2_{L^2(I_i)} = \Big(\frac{h_i}{2}\Big)^{-2j+1}\|(u\circ T_i)^{(j)}-(\widehat{\pi}_{p,k}(u\circ T_i))^{(j)}\|^2_{L^2(\Lambda)}.$$

We make use of estimate (29) to get

$$\|u^{(j)}-(\pi_{p,k}u)^{(j)}\|^2_{L^2(I_i)} \leq \Big(\frac{h_i}{2}\Big)^{-2j+1}\frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-(k-j))!}{(\kappa+(k-j))!}\|(u\circ T_i)^{(k+s)}\|^2_{L^2(\Lambda)}.$$

Applying again (30), we end up with

$$\begin{aligned}&\|u^{(j)}-(\pi_{p,k}u)^{(j)}\|^2_{L^2(I_i)}\\ &\quad\leq \Big(\frac{h_i}{2}\Big)^{-2j+1}\cdot\frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-(k-j))!}{(\kappa+(k-j))!}\Big(\frac{h_i}{2}\Big)^{2s+2k-1}|u^{(k)}|^2_{H^s(I_i)},\end{aligned}$$

which is the desired bound.

Then, we have the following theorem.

Theorem 2 *Under the assumptions of Corollary 3, with* $h = \max\{h_i : i = 1, \ldots, m-1\}$, *we have*

$$\|u^{(j)} - (\pi_{p,k}u)^{(j)}\|_{L^2(0,1)} \le Ch^{\sigma-j}(p-k+1)^{-(\sigma-j)}|u|_{H^\sigma(0,1)}, \quad (32)$$

for $k \le \sigma \le p+1$, $j = 0, \ldots, k$, *and* C *independent of* σ, j, h, p *and* k. *In particular, when* $p = 2k-1$,

$$\|u^{(j)} - (\pi_{p,k}u)^{(j)}\|_{L^2(0,1)} \le Ch^{\sigma-j}\left(\frac{p+1}{2}\right)^{-(\sigma-j)}|u|_{H^\sigma(0,1)}, \quad j = 0, \ldots, k. \quad (33)$$

Proof Let $\kappa = p - k + 1$. Summing (31) over the elements, we have:

$$\|u^{(j)} - (\pi_{p,k}u)^{(j)}\|^2_{L^2(0,1)} \le \sum_{i=1}^{m-1}\left(\frac{h_i}{2}\right)^{2(s+k-j)}\frac{(\kappa-s)!}{(\kappa+s)!}\frac{(\kappa-(k-j))!}{(\kappa+(k-j))!}|u^{(k)}|^2_{H^s(I_i)}. \quad (34)$$

Now, take $s = \sigma - k$:

$$\|u^{(j)} - (\pi_{p,k}u)^{(j)}\|^2_{L^2(0,1)} \le \sum_{i=1}^{m-1}\left(\frac{h_i}{2}\right)^{2(\sigma-j)}\frac{(\kappa-(\sigma-k))!}{(\kappa+(\sigma-k))!}\frac{(\kappa-(k-j))!}{(\kappa+(k-j))!}|u^{(k)}|^2_{H^{\sigma-k}(I_i)}.$$

Taking into account the definition of h and (22), there exists a positive constant C such that

$$\begin{aligned}\|u^{(j)} - (\pi_{p,k}u)^{(j)}\|^2_{L^2(0,1)} &\le C\left(\frac{h}{2}\right)^{2(\sigma-j)}\left(\frac{e}{2}\right)^{2(\sigma-k)}\kappa^{-2(\sigma-k)}\left(\frac{e}{2}\right)^{2(k-j)}\kappa^{-2(k-j)}|u|^2_{H^\sigma(0,1)}\\ &\le Ch^{2(\sigma-j)}\left(\frac{e}{4}\right)^{2(\sigma-j)}\kappa^{-2(\sigma-j)}|u|^2_{H^\sigma(0,1)},\end{aligned}$$

so that (32) follows from the definition of κ and the fact that $e/4 \le 1$.

Besides, if $p = 2k-1$, then $\kappa = k = \dfrac{p+1}{2}$ and a direct substitution in (32) yields (33).

3.3.1 Smoothness and order of accuracy It is interesting to study how a different regularity k affects the interpolation error $u - \pi_{p,k}u$, according to the estimate (32). For the sake of simplicity, we consider a uniform mesh $\{\zeta_1, \dots, \zeta_m\}$, of mesh-size h, on $(0, 1)$ and we focus on the L^2 norm of the error. Recall that the dimension of $\mathcal{S}_k^p$ is

$$N = (m-1)(p+1) - k(m-2) = (m-1)(p-k+1) + k. \qquad (35)$$

We are interested in studying the interpolation error for different regularity k, when N is kept the same, or approximately the same. Since, from (35),

$$\frac{p-k+1}{h} = N - k, \qquad (36)$$

for $j = 0$ the estimate (32) reduces to

$$\|u - \pi_{p,k}u\|_{L^2(0,1)} \leq C(N-k)^{-\sigma}|u|_{H^\sigma(0,1)} \qquad (37)$$

First, we compare low-regular interpolation to high-regular interpolation when the degree p is the same. Then, we select a common p, odd for simplicity, and set $\sigma = p + 1$. The minimum and maximum regularity cases allowed by our framework are $k = k_{\text{low}} = 1$ and $k = k_{\text{high}} = (p + 1)/2$, respectively. Taking the same N in both cases, the mesh-size h will be different, according to (36). However, the error estimate in terms of N is approximately the same in the two cases: indeed we have

$$\begin{aligned}
\|u - \pi_{p,1}u\|_{L^2(0,1)} &\leq C(N-1)^{-(p+1)}|u|_{H^{p+1}(0,1)} \\
&\approx CN^{-(p+1)}|u|_{H^{p+1}(0,1)} \\
\|u - \pi_{p,(p+1)/2}u\|_{L^2(0,1)} &\leq C(N-(p+1)/2)^{-(p+1)}|u|_{H^{p+1}(0,1)} \\
&\approx CN^{-(p+1)}|u|_{H^{p+1}(0,1)}.
\end{aligned}$$

We notice that there is no deterioration of the order of approximation by smooth interpolation. In fact, numerical tests show that increasing the regularity k, while keeping N and p fixed, improves the approximation properties of the space $\mathcal{S}_k^p$ (see, e.g., [13, Section 5.5]), with higher improvement the higher p is. Our estimate, however, does not show this behavior for the interpolation error $u - \pi_{p,k}u$.

The other interesting comparison between low-regular and high-regular interpolation is on the same mesh, i.e., with equal mesh-size h. For example, select a p_{low} and take $k_{\text{low}} = 1$ for the low-regular interpolation, select then $p_{\text{high}} = 2p_{\text{low}} - 1$ and $k_{\text{high}} = (p_{\text{high}} + 1)/2$ for the high-regular case. We have from (36)

$$\begin{aligned} N_{\text{low}} - 1 = N_{\text{low}} - k_{\text{low}} &= \frac{p_{\text{low}}}{h} \\ N_{\text{high}} - k_{\text{high}} = \frac{p_{\text{high}} - k_{\text{high}} + 1}{h} &= \frac{p_{\text{low}}}{h}. \end{aligned} \tag{38}$$

With a little abuse of notation, we denote $N = N_{\text{low}} - k_{\text{low}} = N_{\text{high}} - k_{\text{high}}$; N approximately represents the dimension of both spaces $\mathcal{S}^{p_{\text{low}}}_{k_{\text{low}}}$ and $\mathcal{S}^{p_{\text{high}}}_{k_{\text{high}}}$, that is, $N \approx N_{\text{low}} \approx N_{\text{high}}$ when N is large. Using (37) and selecting the maximum σ yields

$$\|u - \pi_{p_{\text{low}},1} u\|_{L^2(0,1)} \le C N^{-(p_{\text{low}}+1)} |u|_{H^{p_{\text{low}}+1}(0,1)} \tag{39}$$

for the low-regularity case ($k_{\text{low}} = 1$), and

$$\begin{aligned} \|u - \pi_{p_{\text{high}},k_{\text{high}}} u\|_{L^2(0,1)} &\le C N^{-(p_{\text{high}}+1)} |u|_{H^{p_{\text{high}}+1}(0,1)} \\ &= C N^{-2p_{\text{low}}} |u|_{H^{p_{\text{high}}+1}(0,1)} \end{aligned} \tag{40}$$

for the high-regularity case, respectively. From (39) and (40) we see that higher regularity interpolation delivers higher order of convergence w.r.t the number of degrees-of-freedom N, in this comparison. Indeed, the convergence rate is almost twice better. At the same time, we should also observe that (40) assumes more regularity of the function u to be interpolated. A more careful study is postponed to Section 3.4.2.

3.4 Analytic functions

In this section we study the behavior of our interpolant in the case of analytic functions. In particular, we compare the case $k = (p + 1)/2$ with the classical case $k = 1$, typical of standard finite elements.

3.4.1 Spectral approximation We first investigate how the interpolation by global polynomials, introduced in Definition 2, behaves when applied to analytic functions on the reference interval $[-1, 1]$. In particular, we are interested in the case of maximum regularity, i.e., $p = 2k - 1$ in this paper. We will show that the convergence with respect to p is in this case poorer than in the $k = 1$ case. In particular, the property that the target function is analytic on $[-1, 1]$ is not sufficient for convergence; we need, instead, the target function to be analytic in a larger region. The sharpness of the theoretical estimates is verified by numerical tests.

In the sequel we will indicate with $\mathcal{E}_R$ the closed ellipse in the complex plane with foci given by the points 1 and -1 on the real line and semiaxes sum equal to $R > 1$. We indicate by r the length of the semiaxis of the ellipse on the real plane, so that $R = r + \sqrt{r^2 - 1}$. Note that any real function which is analytic in $[-1, 1]$ admits an analytic continuation in $\mathcal{E}_R$ for some $R > 1$. We will make use of the following result shown in [11].

Theorem 3 *Let the scalar function $f(z)$ be analytic on $\mathcal{E}_R$, $R > 1$. Then the coefficients α_i, $i \in \mathbb{N} \cup \{0\}$, of the Legendre expansion of $f(z)$ in $[-1, 1]$ satisfy*

$$|\alpha_i| \leq C_{\overline{R}}(2i+1)(\overline{R})^{-i} \max_{z \in \mathcal{E}_{\overline{R}}} |f(z)|$$

for all $1 < \overline{R} < R$, with $C_{\overline{R}} = \frac{\overline{R}}{2(\overline{R}-1)}$.

We will make use also of the following consequence of the well known Cauchy formula.

Theorem 4 *Let $f(z)$ be an analytic function in the closed ball $B(\widehat{z}, \rho)$ of center $\widehat{z}$ and radius ρ. It holds that*

$$|f^{(n)}(\widehat{z})| \leq \frac{n!}{\rho^n} \max_{z \in B(\widehat{z}, \rho)} |f(z)|$$

for all $n \in \mathbb{N} \cup \{0\}$.

We have the following Proposition. Analogous results hold in stronger Sobolev norms.

Proposition 1 *Let the analytic function $u : [-1, 1] \to \mathbb{R}$ have an analytic complex extension on $\mathcal{E}_R$, $R > 1$. Then there exists a constant $C = C(r)$ such that*

$$\|u - u_p\|_{L^2(-1,1)} \leq C\, k^{1/2} \left(\frac{1}{r^2 - 1}\right)^k \max_{z \in \mathcal{E}_R} |u(z)| \ , \qquad (41)$$

where $r = \frac{R^2+1}{2R}$ and $u_p = \widehat{\pi}_{p,k} u$ is given in Definition 2, with $k \geq 1$ and $p = 2k - 1$.

Proof Let $\overline{r}$, with $1 < \overline{r} < r$, represent the length of the real semiaxis of an ellipse $\mathcal{E}_{\overline{R}} \subset \mathcal{E}_R$. It is immediate to verify that it holds $\overline{R} = \overline{r} + \sqrt{\overline{r}^2 - 1}$ and that clearly $1 < \overline{R} < R$. Moreover, it can be checked that

$$\overline{\rho} := \max\{\rho \in \mathbb{R}^+ \mid B(z, \rho) \subset \mathcal{E}_R,\ \forall z \in \mathcal{E}_{\overline{R}}\} = r - \overline{r}\ .$$

Let now α_i, $i \in \mathbb{N}$, represent the coefficients of the Legendre expansion of $u^{(k)}$ on $[-1,1]$. Note that the (extended) function $u^{(k)}$ is analytic in the same domain $\mathcal{E}_R$ as u. First applying Theorem 3 to $u^{(k)}$ and then Theorem 4, by definition of $\overline{\rho}$ we get

$$\begin{aligned} |\alpha_i| &\leq C_{\overline{R}}(2i+1)(\overline{R})^{-i} \max_{z \in \mathcal{E}_{\overline{R}}} |u^{(k)}(z)| \\ &\leq C_{\overline{R}}(2i+1)(\overline{R})^{-i} \frac{k!}{\overline{\rho}^k} \max_{z \in \mathcal{E}_R} |u(z)| \\ &= C_{\overline{R}}(2i+1)k!(\overline{R})^{-i+k}(\overline{R}\overline{\rho})^{-k} M \end{aligned} \tag{42}$$

where $M = \max_{z \in \mathcal{E}_R} |u(z)|$. In order to aim at the best estimate, we now choose $\overline{r}$ that maximizes $\overline{R}\overline{\rho} = (\overline{r} + \sqrt{\overline{r}^2 - 1})(r - \overline{r})$ for $1 < \overline{r} < r$. A basic calculus exercise shows that such maximum is obtained for

$$\overline{r} = \frac{r^2+1}{2r} \quad \Longrightarrow \quad \overline{R}\overline{\rho} = \frac{r^2-1}{2} \ .$$

Taking such choice for $\overline{r}$, using it in (42) and plugging the result into (28) for $m = k$ yields

$$\begin{aligned} \|u - u_p\|^2_{L^2(-1,1)} &\leq \sum_{i=k}^{\infty} |\alpha_i|^2 \frac{2}{2i+1} \frac{(i-k)!}{(i+k)!} \\ &\leq C\,(k!)^2 \left(\frac{2}{r^2-1}\right)^{2k} \sum_{i=k}^{\infty} (2i+1)(\overline{R})^{2(-i+k)} \frac{(i-k)!}{(i+k)!} \end{aligned} \tag{43}$$

where for brevity we write $C = 2(C_{\overline{R}})^2 M^2$. Observing that for all naturals $i \geq k$ it holds $\frac{(i-k)!}{(i+k)!} \leq ((2k)!)^{-1}$, and recalling that $\overline{R} > 1$ and $k \geq 1$,

$$\begin{aligned} \sum_{i=k}^{\infty} (2i+1)(\overline{R})^{2(-i+k)} \frac{(i-k)!}{(i+k)!} &\leq ((2k)!)^{-1} (\overline{R})^{2k} \sum_{i=k}^{\infty} (2i+1)(\overline{R})^{-2i} \\ &\leq C'((2k)!)^{-1}(\overline{R})^{2k} \int_{k-1}^{\infty} (2x+1)\overline{R}^{-2x} \mathrm{d}x \leq C'k(2k!)^{-1} \end{aligned} \tag{44}$$

where the positive constant C' depends on $\overline{R}$. Combining (43) with (44), using the Stirling inequality and recalling the definition of C we get

$$\|u - u_p\|^2_{L^2(-1,1)} \leq C\,C'\,k \frac{(k!)^2}{(2k)!} \left(\frac{2}{r^2-1}\right)^{2k} \leq \overline{C} M^2 k \left(\frac{1}{r^2-1}\right)^{2k} , \tag{45}$$

where the new constant $\overline{C} = 2(C_{\overline{R}})^2 C'$ depends only on R.

From Proposition 1, the following convergence result is immediately obtained.

Corollary 4 *Let* $u : [-1,1] \to \mathbb{R}$ *have an analytic complex extension on* $\mathcal{E}_R$*,* $R > 1$*. Then, as* $k \to \infty$*,* $u_k = \widehat{\pi}_{2k-1,k}u$ *(analytically) converges to* u *in the* $L^2(-1,1)$ *norm if* $r > \sqrt{2}$ *or equivalently* $R > 1+\sqrt{2}$*.*

In fact, when $p = 2k-1$, we have convergence only if the complex extension of u is analytic on a sufficiently large domain, and Corollary 4 expresses a quite sharp condition. This is shown numerically in Figure 2, where the $L^2(-1,1)$-error of the $\widehat{\pi}_{2k-1,k}$ projection is plotted versus the (odd) degree p, for the function $u(x) = (x-r)^{-1}$. This function is analytic in the *interior* of the ellipse $\mathcal{E}_R$, where $R = r + \sqrt{r^2-1}$. It is shown in particular that convergence with respect to p is achieved for $r > \sqrt{2}$, and in fact also for $r = \sqrt{2}$, but not for the values of $r < \sqrt{2}$ considered in the test. Convergence (and divergence) is exponential with respect to $p = 2k-1$, as expected from (41). As a comparison, the error plot for the $L^2(-1,1)$ projection, that is $\widehat{\pi}_{p,0}$, is shown in Figure 3; of course, in this case for any $r > 1$ exponential convergence is observed.

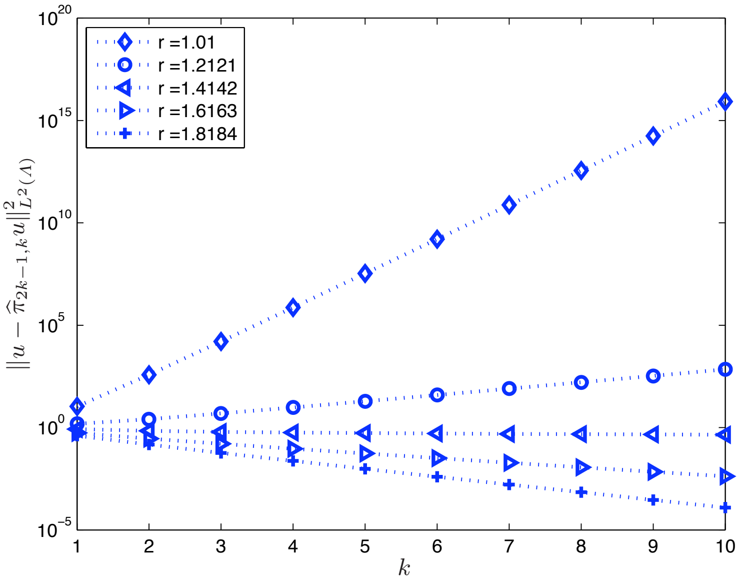

Fig. 2. Error for the $\widehat{\pi}_{2k-1,k}$ projection on $u(x) = (x-r)^{-1}$.

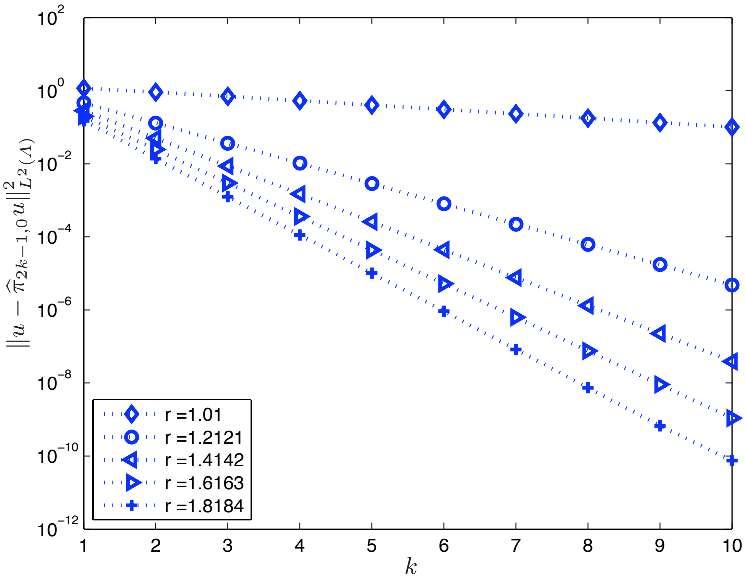

Fig. 3. Approximation error for the $\widehat{\pi}_{2k-1,0}$ projection on $u(x)=(x-r)^{-1}$.

3.4.2 Piecewise polynomial approximation In this section we consider piecewise polynomial approximation and extend the study of Section 3.3.1 to analytic functions. In particular we start recalling estimates (39)–(40):

$$\|u-\pi_{p_{\text{low}},1}u\|_{L^2(0,1)} \le CN^{-(p_{\text{low}}+1)}|u|_{H^{p_{\text{low}}+1}(0,1)}$$

and

$$\begin{aligned}\|u-\pi_{p_{\text{high}},k_{\text{high}}}u\|_{L^2(0,1)} &\le CN^{-(p_{\text{high}}+1)}|u|_{H^{p_{\text{high}}+1}(0,1)}\\ &= CN^{-2p_{\text{low}}}|u|_{H^{2p_{\text{low}}}(0,1)}\end{aligned}$$

which hold, on the same mesh, for the low-regular (p_{low} given, $k_{\text{low}}=1$) and high-regular interpolation ($p_{\text{high}}=2p_{\text{low}}-1$ and $k_{\text{high}}=(p_{\text{high}}+1)/2$), respectively.

The next step is to estimate the Sobolev seminorm $|u|_{H^s(0,1)}$ in terms of s, for a given analytic function u. Similarly to the previous section, we assume u is analytic in a closed ellipse, still denoted by $\mathcal{E}_R$, of the complex plane, having foci 0 and 1, and having semiaxis sum $R > 1/2$. Then a result analogous to Theorem 4 and the Stirling type inequality

$$s! \leq 3\sqrt{s}\left(\frac{s}{e}\right)^{s},$$

give

$$|u|_{H^s(0,1)} \leq C\sqrt{s}\left(\frac{s}{\rho e}\right)^{s}\max_{z\in\mathcal{E}_R}|u(z)|$$

where $\rho > 0$ is such that for any $x \in [0,1]$ there exists a closed ball $B(x,\rho) \subset \mathcal{E}_R$, that is, $\rho \leq \dfrac{(2R-1)^2}{8R}$.

Therefore

$$\|u-\pi_{p_{\text{low}},1}u\|_{L^2(0,1)} \leq CN^{-(p_{\text{low}}+1)}\sqrt{p_{\text{low}}+1}\left(\frac{p_{\text{low}}+1}{\rho e}\right)^{p_{\text{low}}+1}\max_{z\in\mathcal{E}_R}|u(z)| \tag{46}$$

$$\|u-\pi_{p_{\text{high}},k_{\text{high}}}u\|_{L^2(0,1)} \leq CN^{-(2p_{\text{low}})}\sqrt{2p_{\text{low}}}\left(\frac{2p_{\text{low}}}{\rho e}\right)^{2p_{\text{low}}}\max_{z\in\mathcal{E}_R}|u(z)|. \tag{47}$$

In order to evaluate the advantage of high-regular interpolation we compute the ratio of the right-hand-sides of (47) and (46) and bound it from above. We get

$$\begin{aligned}\frac{N^{-(2p_{\text{low}})}\sqrt{2p_{\text{low}}}\left(\frac{2p_{\text{low}}}{\rho e}\right)^{2p_{\text{low}}}}{N^{-(p_{\text{low}}+1)}\sqrt{p_{\text{low}}+1}\left(\frac{p_{\text{low}}+1}{\rho e}\right)^{p_{\text{low}}+1}} &\leq CN^{-p_{\text{low}}+1}\frac{\left(\frac{2p_{\text{low}}}{\rho e}\right)^{2p_{\text{low}}}}{\left(\frac{p_{\text{low}}}{\rho e}\right)^{p_{\text{low}}+1}}\\ &\leq CN^{-p_{\text{low}}+1}\,4^{p_{\text{low}}}\left(\frac{p_{\text{low}}}{\rho e}\right)^{p_{\text{low}}-1}\\ &\leq C\left(\frac{4p_{\text{low}}}{N\rho e}\right)^{p_{\text{low}}-1}\\ &\leq C\left(\frac{4h}{\rho e}\right)^{p_{\text{low}}-1},\end{aligned} \tag{48}$$

where, in the last step, we have also used (38), i.e., $p_{\text{low}}/N = h$.

From (48), we conclude that high-regular interpolation is more accurate than low-regular interpolation for analytic functions, whenever the mesh-size h is kept the same and is small enough compared to the size of the analytic region of the function. Asymptotically, when h goes to zero, the order with respect to h of the interpolation error $u - \pi_{p_{\mathrm{high}},k_{\mathrm{high}}}u$ is (approximately) twice better than the order of $u - \pi_{p_{\mathrm{low}},1}$, on the the same mesh.

4 Basic estimates in two dimensions

In this section we extend the results of the previous part to the two-dimensional case. We will first define the projection operator in the reference square $\hat{Q} = (-1,1) \times (-1,1)$ by a tensor product of the one-dimensional operator defined in the previous section. Therefore, we shall make use of the following spaces: given an integer $s \geq 0$

$$\begin{aligned} V_s^s(\Lambda, L^2(\Lambda)) &= \{u \colon \hat{Q} \to \mathbb{R} \,:\, \iint_{\hat{Q}} (1-x^2)^s \big(\partial_x^s u(x,y)\big)^2 \,dxdy < \infty\}, \\ L^2(\Lambda, V_s^s(\Lambda)) &= \{u \colon \hat{Q} \to \mathbb{R} \,:\, \iint_{\hat{Q}} (1-y^2)^s \big(\partial_y^s u(x,y)\big)^2 \,dxdy < \infty\}. \end{aligned}$$

After obtaining the error bounds in the reference square, we will extend them to a generic square by a scaling argument, in a similar fashion to what has been done in the one-dimensional case. Given $Q = (a,b) \times (c,d)$ and two non-negative integers k_1, k_2, we will write $H^{k_1,k_2}(Q) = H^{k_1}((a,b), H^{k_2}(c,d))$. Besides, for $p_1 \geq 0$, $p_2 \geq 0$ integers, we will denote by $\mathcal{S}^{p_1,p_2}(Q)$ the space of polynomials of degree p_1 in the first variable and p_2 in the second, that is,

$$\mathcal{S}^{p_1,p_2}(Q) = \{u \colon Q \to \mathbb{R} \,:\, u(\cdot,y) \in \mathcal{S}^{p_1}(a,b),\; u(x,\cdot) \in \mathcal{S}^{p_2}(c,d)\}.$$

We set $p = (p_1, p_2)$ or, by abuse of notation, when $p_1 = p_2$, $p = p_1 = p_2$. In the latter case, we write $\mathcal{S}^p(Q) = \mathcal{S}^{p_1,p_2}(Q)$.

4.1 Polynomial approximation in the reference element $\hat{Q}$

As mentioned above, the two-dimensional projection operator we will consider is defined as the tensor product of the one-dimensional operator $\pi_{p,k}$. However, we will allow different polynomial degree as well as different regularity order on each variable.

Definition 4 *Let $p = (p_1, p_2)$, $k = (k_1, k_2)$ where p_1, p_2, k_1 and k_2 are non-negative integers. We define $\widehat{\Pi}_{p,k} \colon H^{k_1,k_2}(\hat{Q}) \to \mathcal{S}^{p_1,p_2}(\hat{Q})$ as*

$$\widehat{\Pi}_{p,k} = \widehat{\pi}_{p_1,k_1} \otimes \widehat{\pi}_{p_2,k_2}.$$

This operator can also be interpreted as a composition of two-dimensional operators. Let p_1, p_2, k_1 and k_2 be non-negative integers and $u \in H^{k_1,k_2}(\hat{Q})$. Given $y \in \Lambda$, let $\varphi_y \colon \Lambda \to \mathbb{R}$, $\varphi_y(x) = u(x,y)$. We define

$$\widehat{\pi}^{(x)}_{p_1,k_1} u(x,y) = \widehat{\pi}_{p_1,k_1}\varphi_y(x).$$

Similarly, for $x \in \Lambda$, let $\phi_x \colon \Lambda \to \mathbb{R}$, $\phi_x(y) = u(x,y)$. We define

$$\widehat{\pi}^{(y)}_{p_2,k_2} u(x,y) = \widehat{\pi}_{p_2,k_2}\phi_x(y).$$

Since each of these operators acts on one variable, due to their linearity and continuity, $\widehat{\pi}^{(x)}_{p_1,k_1} \circ \widehat{\pi}^{(y)}_{p_2,k_2} = \widehat{\pi}^{(y)}_{p_2,k_2} \circ \widehat{\pi}^{(x)}_{p_1,k_1}$ and

$$\widehat{\Pi}_{p,k} = \widehat{\pi}^{(x)}_{p_1,k_1} \circ \widehat{\pi}^{(y)}_{p_2,k_2} = \widehat{\pi}^{(y)}_{p_2,k_2} \circ \widehat{\pi}^{(x)}_{p_1,k_1}.$$

Lemma 6 *Let $p = (p_1, p_2)$, $k = (k_1, k_2)$ where p_1, p_2, k_1 and k_2 are non-negative integers, $\widehat{\Pi}_{p,k}$ as in Definition 4. Let moreover $\hat{v}_n$ and $\hat{\gamma}_n$, $n = 1,2,3,4$ be respectively the vertexes and edges of $\hat{Q}$. We order the edges in such a way that $\hat{\gamma}_1$ represents the rightmost vertical edge and the remaining ones are labeled in anti-clockwise order. Then if $u \in H^{k_1,k_2}(\hat{Q})$, for $i = 0,\dots,k_1-1$, $j = 0,\dots,k_2-1$, one has*

$$(\partial^i_x \partial^j_y \widehat{\Pi}_{p,k} u)|_{\hat{\gamma}_n} = \big[\partial^i_x \widehat{\pi}^{(x)}_{p_1,k_1}(\partial^j_y u)\big]|_{\hat{\gamma}_n}, \quad n = 2,4, \tag{49}$$

$$(\partial^i_x \partial^j_y \widehat{\Pi}_{p,k} u)|_{\hat{\gamma}_n} = \big[\partial^j_y \widehat{\pi}^{(y)}_{p_2,k_2}(\partial^i_x u)\big]|_{\hat{\gamma}_n}, \quad n = 1,3, \tag{50}$$

$$\partial^i_x \partial^j_y \widehat{\Pi}_{p,k} u(\hat{v}_n) = \partial^i_x \partial^j_y u(\hat{v}_n), \quad n = 1,2,3,4. \tag{51}$$

Proof It is easily checked that $\partial_x \widehat{\pi}^{(y)}_{p_2,k_2}\cdot = \widehat{\pi}^{(y)}_{p_2,k_2}\partial_x\cdot$ and $\partial_y \widehat{\pi}^{(x)}_{p_1,k_1}\cdot = \widehat{\pi}^{(x)}_{p_1,k_1}\partial_y\cdot$, therefore

$$\begin{aligned}\partial^i_x \partial^j_y \widehat{\Pi}_{p,k} u(x,y) =& \partial^j_y\, \widehat{\pi}^{(y)}_{p_2,k_2}\big(\partial^i_x\, \widehat{\pi}^{(x)}_{p_1,k_1} u(x,y)\big)\\ =& \partial^i_x\, \widehat{\pi}^{(x)}_{p_1,k_1}\big(\partial^j_y\, \widehat{\pi}^{(y)}_{p_2,k_2} u(x,y)\big).\end{aligned}$$

Now, taking into account that the values of the derivatives up to order k_1-1 or k_2-1 of the one-dimensional projection coincide with those of the derivatives of the function at the endpoints of the interval, using the first line above

$$\partial^i_x\, \partial^j_y\, \widehat{\Pi}_{p,k} u(x,\pm 1) = \partial^j_y\, \big(\partial^i_x\, \widehat{\pi}^{(x)}_{p_1,k_1} u(x,\pm 1)\big) = \partial^i_x \widehat{\pi}^{(x)}_{p_1,k_1}\big(\partial^j_y u(x,\pm 1)\big),$$

for $j = 0,\dots,k_2-1$, which is exactly (49). Similarly, (50) is obtained by the same one-dimensional argument applied to the second line above, with $x = \pm 1$:

$$\partial^i_x\, \partial^j_y \widehat{\Pi}_{p,k} u(\pm 1, y) = \partial^i_x\big(\partial^j_y\, \widehat{\pi}^{(y)}_{p_2,k_2} u(\pm 1, y)\big) = \partial^j_y \widehat{\pi}^{(y)}_{p_2,k_2}\big(\partial^i_x u(\pm 1, y)\big),$$

for $i = 0, \dots, k_1 - 1$. Finally, applying the same argument once again,

$$\partial_x^i \, \partial_y^j \, \widehat{\Pi}_{p,k} u(\pm 1, \pm 1) = \partial_x^i \, \partial_y^j \, u(\pm 1, \pm 1), \quad i = 0, \dots, k_1 - 1, \; j = 0, \dots, k_2 - 1,$$

which is (51).

In order to bound the error of the projector $\widehat{\Pi}_{p,k}$, we need to deduce two-dimensional estimates for $\widehat{\pi}^{(x)}_{p_1,k_1}$ and $\widehat{\pi}^{(y)}_{p_2,k_2}$ from the one-dimensional results obtained in Section 3.2.

Lemma 7 *For $d = 1, 2$, let p_d, k_d be non-negative integers, $p_d \geq 2k_d - 1$, and denote $\kappa_d = p_d - k_d + 1$.*

Let $u \colon \hat{Q} \to \mathbb{R}$ be such that $\partial_x^{k_1} u \in V^{s_1}_{s_1}(\Lambda, L^2(\Lambda))$ for some $0 \leq s_1 \leq \kappa_1$. Then, for any $\ell = 0, \dots, k_1$,

$$\left\| \partial_x^\ell (u - \widehat{\pi}^{(x)}_{p_1,k_1} u) \right\|^2_{L^2(\hat{Q})} \leq \frac{(\kappa_1 - s_1)!}{(\kappa_1 + s_1)!} \frac{(\kappa_1 - (k_1 - \ell))!}{(\kappa_1 + (k_1 - \ell))!} \iint_{\hat{Q}} (1 - x^2)^{s_1} (\partial_x^{s_1 + k_1} u(x, y))^2 \, dx dy. \quad (52)$$

Analogously, if $\partial_y^{k_2} u \in L^2(\Lambda, V^{s_2}_{s_2}(\Lambda))$ for some $0 \leq s_2 \leq \kappa_2$, then, for any $\ell = 0, \dots, k_2$,

$$\left\| \partial_y^\ell (u - \widehat{\pi}^{(y)}_{p_2,k_2} u) \right\|^2_{L^2(\hat{Q})} \leq \frac{(\kappa_2 - s_2)!}{(\kappa_2 + s_2)!} \frac{(\kappa_2 - (k_2 - \ell))!}{(\kappa_2 + (k_2 - \ell))!} \iint_{\hat{Q}} (1 - y^2)^{s_2} (\partial_y^{s_2 + k_2} u(x, y))^2 \, dx dy. \quad (53)$$

Proof Since

$$\left\| \partial_x^\ell (u - \widehat{\pi}^{(x)}_{p_1,k_1} u) \right\|^2_{L^2(\hat{Q})} = \int_{-1}^{1} \left\| \partial_x^\ell (u - \widehat{\pi}^{(x)}_{p_1,k_1} u)(\cdot, y) \right\|^2_{L^2(\Lambda)} dy,$$

we can apply (27) to the integrand with the function $\varphi_y(x) = u(x, y)$ to obtain

$$\left\| \partial_x^\ell (u - \widehat{\pi}^{(x)}_{p_1,k_1} u) \right\|^2_{L^2(\hat{Q})} \leq \frac{(\kappa_1 - s_1)!}{(\kappa_1 + s_1)!} \frac{(\kappa_1 - (k_1 - \ell))!}{(\kappa_1 + (k_1 - \ell))!} \int_{-1}^{1} \int_{-1}^{1} (1 - x^2)^{s_1} (\partial_x^{s_1 + k_1} u(x, y))^2 \, dx \, dy,$$

which is the desired estimate. The second inequality is proved in the same fashion, taking into account that

$$\left\| \partial_y^\ell (u - \widehat{\pi}^{(y)}_{p_2,k_2} u) \right\|^2_{L^2(\hat{Q})} = \int_{-1}^{1} \left\| \partial_y^\ell (u - \widehat{\pi}^{(y)}_{p_2,k_2} u)(x, \cdot) \right\|^2_{L^2(\Lambda)} dx.$$

Corollary 5 *Let* p_1, k_1 *be non-negative integers,* $p_1 \geq 2k_1 - 1$, $\kappa_1 = p_1 - k_1 + 1$ *and let* $u\colon \hat{Q} \to \mathbb{R}$ *be such that* $\partial_x^{k_1} u \in L^2(\hat{Q})$. *Then, for any* $\ell = 0, \ldots, k_1$,

$$\left\|\partial_x^\ell(\widehat{\pi}_{p_1,k_1}^{(x)} u)\right\|_{L^2(\hat{Q})}^2 \leq 2\left\|\partial_x^\ell u\right\|_{L^2(\hat{Q})}^2 + 2\frac{(\kappa_1-(k_1-\ell))!}{(\kappa_1+(k_1-\ell))!}\left\|\partial_x^{k_1} u\right\|_{L^2(\hat{Q})}^2. \tag{54}$$

Analogously, if p_2, k_2 *are non-negative integers,* $p_2 \geq 2k_2 - 1$, $\kappa_2 = p_2 - k_2 + 1$ *and* $\partial_y^{k_2} u \in L^2(\hat{Q})$, *then, for any* $\ell = 0, \ldots, k_2$,

$$\left\|\partial_y^\ell(\widehat{\pi}_{p_2,k_2}^{(y)} u)\right\|_{L^2(\hat{Q})}^2 \leq 2\left\|\partial_y^\ell u\right\|_{L^2(\hat{Q})}^2 + 2\frac{(\kappa_2-(k_2-\ell))!}{(\kappa_2+(k_2-\ell))!}\left\|\partial_y^{k_2} u\right\|_{L^2(\hat{Q})}^2. \tag{55}$$

Proof We will only prove (54) since (55) is completely analogous. From the triangular inequality,

$$\left\|\partial_x^\ell(\widehat{\pi}_{p_1,k_1}^{(x)} u)\right\|_{L^2(\hat{Q})}^2 \leq 2\left\|\partial_x^\ell u\right\|_{L^2(\hat{Q})}^2 + 2\left\|\partial_x^\ell(u - \widehat{\pi}_{p_1,k_1}^{(x)} u)\right\|_{L^2(\hat{Q})}^2.$$

We use now (52) with $s_1 = 0$ to bound the second term above, which gives (54).

Theorem 5 *Let* p_d, k_d *be non-negative integers,* $p_d \geq 2k_d - 1$, $\kappa_d = p_d - k_d + 1$, *for* $d = 1, 2$, *and* $\widehat{\Pi}_{p,k}$ *the projection operator described in Definition 4. Let* $u\colon \hat{Q} \to \mathbb{R}$, *let* $0 \leq i \leq k_1$, $0 \leq j \leq k_2$, *and* $\partial_x^{k_1} \partial_y^j u \in V_{s_1}^{s_1}(\Lambda, L^2(\Lambda))$, $\partial_x^i \partial_y^{k_2} u \in L^2(\Lambda, V_{s_2}^{s_2}(\Lambda))$ *and* $\partial_x^{k_1} \partial_y^{k_2} u \in L^2(\Lambda, V_{s_3}^{s_3}(\Lambda))$ *for certain* $0 \leq s_1 \leq \kappa_1$, $0 \leq s_2, s_3 \leq \kappa_2$. *Then, we have*

$$\begin{aligned}
&\left\|\partial_x^i \partial_y^j (u - \widehat{\Pi}_{p,k} u)\right\|_{L^2(\hat{Q})}^2 \leq \\
&2\frac{(\kappa_1-s_1)!}{(\kappa_1+s_1)!}\frac{(\kappa_1-(k_1-i))!}{(\kappa_1+(k_1-i))!}\iint_{\hat{Q}} (1-x^2)^{s_1} (\partial_x^{s_1+k_1} \partial_y^j u(x,y))^2 \, dxdy \\
&+ 4\frac{(\kappa_2-s_2)!}{(\kappa_2+s_2)!}\frac{(\kappa_2-(k_2-j))!}{(\kappa_2+(k_2-j))!}\iint_{\hat{Q}} (1-y^2)^{s_2} (\partial_x^i \partial_y^{s_2+k_2} u(x,y))^2 \, dxdy \\
&\qquad + 4\frac{(\kappa_1-(k_1-i))!}{(\kappa_1+(k_1-i))!}\frac{(\kappa_2-s_3)!}{(\kappa_2+s_3)!}\frac{(\kappa_2-(k_2-j))!}{(\kappa_2+(k_2-j))!}\times \\
&\qquad\qquad \iint_{\hat{Q}} (1-y^2)^{s_3} (\partial_x^{k_1} \partial_y^{s_3+k_2} u(x,y))^2 \, dxdy.
\end{aligned} \tag{56}$$

Proof We use first the triangular inequality and the commutativity of derivatives with respect to one variable and projection with respect to the other variable, to get

$$
\begin{aligned}
&\big\|\partial_x^i\partial_y^j(u-\widehat{\Pi}_{p,k}u)\big\|^2_{L^2(\hat{Q})}\\
&=\big\|\partial_x^i\partial_y^j u-\partial_x^i\partial_y^j(\widehat{\pi}^{(x)}_{p_1,k_1}u)+\partial_x^i\partial_y^j(\widehat{\pi}^{(x)}_{p_1,k_1}u)-\partial_x^i\partial_y^j\big(\widehat{\pi}^{(x)}_{p_1,k_1}\circ\widehat{\pi}^{(y)}_{p_2,k_2}u\big)\big\|^2_{L^2(\hat{Q})}\\
&\le 2\big\|\partial_x^i\big(\partial_y^j u-\partial_y^j(\widehat{\pi}^{(x)}_{p_1,k_1}u)\big)\big\|^2_{L^2(\hat{Q})}+2\big\|\partial_x^i\partial_y^j\widehat{\pi}^{(x)}_{p_1,k_1}\big(u-\widehat{\pi}^{(y)}_{p_2,k_2}u\big)\big\|^2_{L^2(\hat{Q})}\\
&\le 2\big\|\partial_x^i\big(\partial_y^j u-\widehat{\pi}^{(x)}_{p_1,k_1}\partial_y^j u\big)\big\|^2_{L^2(\hat{Q})}+2\big\|\partial_x^i\widehat{\pi}^{(x)}_{p_1,k_1}\big(\partial_y^j u-\partial_y^j\widehat{\pi}^{(y)}_{p_2,k_2}u\big)\big\|^2_{L^2(\hat{Q})}.
\end{aligned}
$$

We apply now (54), with $m=i$, to bound the second term and get

$$
\begin{aligned}
&\big\|\partial_x^i\partial_y^j(u-\widehat{\Pi}_{p,k}u)\big\|^2_{L^2(\hat{Q})}\\
&\quad\le 2\big\|\partial_x^i\big(\partial_y^j u-\widehat{\pi}^{(x)}_{p_1,k_1}\partial_y^j u\big)\big\|^2_{L^2(\hat{Q})}+4\big\|\partial_x^i\big(\partial_y^j u-\partial_y^j\widehat{\pi}^{(y)}_{p_2,k_2}u\big)\big\|^2_{L^2(\hat{Q})}\\
&\qquad\quad+4\frac{(\kappa_1-(k_1-i))!}{(\kappa_1+(k_1-i))!}\big\|\partial_x^{k_1}\big(\partial_y^j u-\partial_y^j\widehat{\pi}^{(y)}_{p_2,k_2}u\big)\big\|^2_{L^2(\hat{Q})}\\
&\quad\le 2\big\|\partial_x^i\big(\partial_y^j u-\widehat{\pi}^{(x)}_{p_1,k_1}\partial_y^j u\big)\big\|^2_{L^2(\hat{Q})}+4\big\|\partial_y^j\big(\partial_x^i u-\widehat{\pi}^{(y)}_{p_2,k_2}\partial_x^i u\big)\big\|^2_{L^2(\hat{Q})}\\
&\qquad\qquad\quad+4\frac{(\kappa_1-(k_1-i))!}{(\kappa_1+(k_1-i))!}\big\|\partial_y^j\big(\partial_x^{k_1}u-\widehat{\pi}^{(y)}_{p_2,k_2}\partial_x^{k_1}u\big)\big\|^2_{L^2(\hat{Q})}.
\end{aligned}
$$

The argument finishes by applying (52) with $s=s_1$ and $\partial_y^j u$ instead of u to bound the first summand, (53) with $s=s_2$ and $\partial_x^i u$ instead of u to bound the second term, and again (53) with $s=s_3$ and $\partial_x^{k_1}u$ instead of u for the last term.

Corollary 6 *Let p_d, k_d, κ_d be non-negative integers, $p_d\ge 2k_d-1$, $\kappa_d=p_d-k_d+1$, for $d=1,2$, $\widehat{\Pi}_{p,k}$ the projection operator given in Definition 4 and $u\in H^\sigma(\hat{Q})$ for some $k_1+k_2\le\sigma\le\min\{p_1,p_2\}+1$. Then, for $i=0,\dots,k_1$, $j=0,\dots,k_2$ there exists a positive constant, C, independent of σ, i, j, p_1, p_2, k_1 and k_2, such that*

$$
\begin{aligned}
&\big\|\partial_x^i\partial_y^j(u-\widehat{\Pi}_{p,k}u)\big\|^2_{L^2(\hat{Q})}\le\\
&\qquad C\Big(\frac{e}{2}\Big)^{2(\sigma-(i+j))}\Big\{\kappa_1^{-2(\sigma-(i+j))}\big\|\partial_x^{\sigma-j}\partial_y^j u\big\|^2_{L^2(\hat{Q})}\\
&\qquad\qquad+\kappa_2^{-2(\sigma-(i+j))}\big\|\partial_x^i\partial_y^{\sigma-i}u\big\|^2_{L^2(\hat{Q})}\\
&\qquad\qquad+\kappa_1^{-2(k_1-i)}\kappa_2^{-2(\sigma-k_1-j)}\big\|\partial_x^{k_1}\partial_y^{\sigma-k_1}u\big\|^2_{L^2(\hat{Q})}\Big\}. \qquad (57)
\end{aligned}
$$

In particular, if $p_1=p_2=p$ and $k^*=\max\{k_1,k_2\}$, then

$$
\big\|\partial_x^i\partial_y^j(u-\widehat{\Pi}_{p,k}u)\big\|_{L^2(\hat{Q})}\le C\Big(\frac{e}{2}\Big)^{\sigma-(i+j)}(p-k^*+1)^{-(\sigma-(i+j))}|u|_{H^\sigma(\hat{Q})}. \qquad (58)
$$

Proof We make use of bound (22) for quotients of factorials, as well as the fact that $(1-\xi^2)^s \leq 1$ for $\xi \in \Lambda$, to reduce (56) as follows,

$$\begin{aligned}
&\left\|\partial_x^i \partial_y^j (u - \widehat{\Pi}_{p,k} u)\right\|^2_{L^2(\hat{Q})} \\
&\qquad \leq C\Big(\frac{e}{2}\Big)^{2(s_1+k_1-i)} \kappa_1^{-2(s_1+k_1-i)} \left\|\partial_x^{s_1+k_1}\partial_y^j u\right\|^2_{L^2(\hat{Q})} \\
&\qquad + C\Big(\frac{e}{2}\Big)^{2(s_2+k_2-j)} \kappa_2^{-2(s_2+k_2-j)} \left\|\partial_x^{i}\partial_y^{s_2+k_2} u\right\|^2_{L^2(\hat{Q})} \\
&+ C\Big(\frac{e}{2}\Big)^{2(s_3+k_1+k_2-i-j)} \kappa_1^{-2(k_1-i)} \kappa_2^{-2(s_3+k_2-j)} \left\|\partial_x^{k_1}\partial_y^{s_3+k_2} u\right\|^2_{L^2(\hat{Q})}.
\end{aligned}$$

We now take $s_1 = \sigma - k_1 - j$, $s_2 = \sigma - k_2 - i$ and $s_3 = \sigma - k_1 - k_2$ to yield

$$\begin{aligned}
&\left\|\partial_x^i \partial_y^j (u - \widehat{\Pi}_{p,k} u)\right\|^2_{L^2(\hat{Q})} \\
&\qquad \leq C\Big(\frac{e}{2}\Big)^{2(\sigma-(i+j))} \kappa_1^{-2(\sigma-(i+j))} \left\|\partial_x^{\sigma-j}\partial_y^j u\right\|^2_{L^2(\hat{Q})} \\
&\qquad + C\Big(\frac{e}{2}\Big)^{2(\sigma-(i+j))} \kappa_2^{-2(\sigma-(i+j))} \left\|\partial_x^{i}\partial_y^{\sigma-i} u\right\|^2_{L^2(\hat{Q})} \\
&\quad + C\Big(\frac{e}{2}\Big)^{2(\sigma-(i+j))} \kappa_1^{-2(k_1-i)} \kappa_2^{-2(\sigma-k_1-j)} \left\|\partial_x^{k_1}\partial_y^{\sigma-k_1} u\right\|^2_{L^2(\hat{Q})},
\end{aligned}$$

which is the desired result (57). Finally, (58) is a direct consequence of (57) and the definition of p and k^*.

4.2 Polynomial approximation in a generic rectangle

A simple scaling argument yields the following Lemma.

Lemma 8 *Let* $Q = (\zeta_{i_1,1}, \zeta_{i_1+1,1}) \times (\zeta_{i_2,2}, \zeta_{i_2+1,2})$, $h_1 = \zeta_{i_1+1,1} - \zeta_{i_1,1}$, $h_2 = \zeta_{i_2+1,2} - \zeta_{i_2,2}$, $T_Q \colon \hat{Q} \to Q$,

$$T_Q(\xi,\eta) = \Big(\frac{\zeta_{i_1+1,1}-\zeta_{i_1,1}}{2}\xi + \frac{\zeta_{i_1+1,1}+\zeta_{i_1,1}}{2}, \frac{\zeta_{i_2+1,2}-\zeta_{i_2,2}}{2}\eta + \frac{\zeta_{i_2+1,2}+\zeta_{i_2,2}}{2}\Big),$$

$v \colon Q \to \mathbb{R}$ *and* $u = v \circ T_Q$. *Then*

$$\left\|\partial_x^i \partial_y^j v\right\|^2_{L^2(Q)} = \Big(\frac{h_1}{2}\Big)^{-2i+1} \Big(\frac{h_2}{2}\Big)^{-2j+1} \left\|\partial_\xi^i \partial_\eta^j u\right\|^2_{L^2(\hat{Q})}.$$

Given $p = (p_1, p_2)$, and $k = (k_1, k_2)$ with $p_d \geq 2k_d - 1$ $d = 1, 2$, we can define the projection operator for functions defined in the rectangle Q as follows: $\Pi^Q_{p,k} : H^\sigma(Q) \to \mathcal{S}^p(Q)$, $\sigma \geq k_1 + k_2$

$$(\Pi^Q_{p,k} v) \circ T_Q = \widehat{\Pi}_{p,k}(v \circ T_Q). \tag{59}$$

We have then the following immediate consequences of Theorem 5.

Corollary 7 *Let $p_1 = p_2 = p$ and k_d be non-negative integers, $p \geq 2k_d - 1$ for $d = 1, 2$, and $\Pi^Q_{p,k}$ the projection operator given in (59). Then, there exists a positive constant C, independent of σ, i, j, p and k_1, k_2, such that, for all $v \in H^\sigma(Q)$ with $k_1 + k_2 \leq \sigma \leq p + 1$ and for all $i = 0, \dots, k_1$, $j = 0, \dots, k_2$,*

$$\left\| \partial^i_x \partial^j_y (v - \Pi^Q_{p,k} v) \right\|_{L^2(Q)} \leq C (p - k^* + 1)^{-(\sigma - (i+j))} h^{(\sigma - (i+j))} |v|_{H^\sigma(Q)}. \tag{60}$$

where $k^* = \max\{k_1, k_2\}$ and $h = \max\{h_1, h_2\}$.

Proof If we denote $u = v \circ T_Q$, by Lemma 8, we have

$$\left\| \partial^i_x \partial^j_y (v - \Pi^Q_{p,k} v) \right\|^2_{L^2(Q)} = \left(\frac{h_1}{2}\right)^{-2i+1} \left(\frac{h_2}{2}\right)^{-2j+1} \left\| \partial^i_x \partial^j_y (u - \widehat{\Pi}_{p,k}) u \right\|^2_{L^2(\hat{Q})}. \tag{61}$$

Note that estimate (57) still holds if we substitute k_1, k_2 with k^*. Applying such bound to identity (61) it follows

$$\begin{aligned} \left\| \partial^i_x \partial^j_y (v - \Pi^Q_{p,k} v) \right\|^2_{L^2(Q)} \leq C \left(\frac{e}{2}\right)^{\sigma - (i+j)} \left(\frac{h_1}{2}\right)^{-2i+1} \left(\frac{h_2}{2}\right)^{-2j+1} \\ \cdot (p - k^* + 1)^{-2(\sigma - (i+j))} \\ \cdot \left\{ \left\| \partial^{\sigma - j}_x \partial^j_y u \right\|^2_{L^2(\hat{Q})} + \left\| \partial^i_x \partial^{\sigma - i}_y u \right\|^2_{L^2(\hat{Q})} + \left\| \partial^{k_1}_x \partial^{\sigma - k_1}_y v \right\|^2_{L^2(\hat{Q})} \right\}. \end{aligned}$$

Applying again the scaling argument of Lemma 8, on the reverse direction, and bounding all the terms h_1, h_2 with h, we get the desired bound.

The following result follows immediately from Corollary 7.

Corollary 8 *Under the same hypotheses and notation of Corollary 7, for all integers $0 \leq \ell \leq k_* := \min\{k_1, k_2\}$ it holds*

$$|v - \Pi^Q_{p,k} v|^2_{H^\ell(Q)} \leq C (p - k^* + 1)^{-2(\sigma - \ell)} h^{2(\sigma - \ell)} |v|^2_{H^\sigma(Q)}. \tag{62}$$

Remark 1 *Extending this results to the three dimensional case it is not straightforward, unless accepting more severe requirements on the discrete parameters k, p and on the target function regularity. In particular, following the above steps would give in Corollary 7 the condition $k_1+k_2+k_3 \leq \sigma \leq p+1$, where k_i, $i=1,2,3$, represents as usual the continuity constant for the i-th variable. Considering for instance the case $k_1=k_2=k_3=k$, this would enforce the relation $p \geq 3k-1$, which is definitely more severe than the one required to define the interpolation operator ($p \geq 2k-1$).*

5 Approximation by NURBS in the 2D physical domain

In this section we estimate the approximation error of our h, k, p-interpolant in the framework of Isogeometric Analysis, in the case $d = 2$. The simpler one-dimensional case is in fact trivially obtained from the present study, while treating the $d=3$ case is not straightforward, see also Remark 1.

We recall the notation of Section 2: a scalar spline space $\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = S^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}(\mathcal{Q}_h)$ on the parametric domain $\widehat{\Omega}$ is given, and the corresponding NURBS space on the physical domain Ω, associated with the functions w, $\mathbf{F}$, is denoted $\mathcal{V}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} = \mathcal{V}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}(\mathcal{Q}_h, w, \mathbf{F})$. For the sake of simplicity, we assume $p_1 = p_2 = p$. We associate to each internal edge e of the mesh in the parametric domain a number $k_e = k_e(\boldsymbol{k}_1, \boldsymbol{k}_2)$, representing the regularity of $\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ across the edge e, that is, k_e is the largest natural such that all functions v in $\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ satisfy

$$\partial^l_{\mathbf{n}} v^+|_e = \partial^l_{\mathbf{n}} v^-|_e \qquad l = 0, 1, .., k_e - 1, \tag{63}$$

where $\partial_{\mathbf{n}}$ represents the normal derivative to the edge e and $v^{\pm} = v|_{Q^{\pm}}$ with Q^+, Q^- the two elements separated by e. Furthermore, from Section 2 it follows that $k_e \leq k_{max}$.

In the following, C will indicate a generic constant independent of the mesh-size h, the degree $p = p_1 = p_2$ and the regularity $\boldsymbol{k}_1, \boldsymbol{k}_2$, which may change at each occurrence. Notice that, since in Isogeometric Analysis the map $\mathbf{F}$ and weight w are fixed at the coarse mesh level, in the sequel the dependence of the estimates on $\mathbf{F}, w$ is left implicit and included in the constants. The Lemma below is a simplified version of Lemma 3.5 in [3].

Lemma 9 *Let ℓ be a non-negative integer, $Q = (a,b) \times (c,d)$ and $\mathbf{F} : Q \to K$ a smooth bijective mapping with smooth inverse. For all functions $v \in H^\ell(K)$ and $u \in H^\ell(Q)$ it holds*

$$|v \circ \mathbf{F}|_{H^\ell(Q)} \leq C\, \|v\|_{H^\ell(K)}\ , \quad |u \circ \mathbf{F}^{-1}|_{H^\ell(K)} \leq C\, \|u\|_{H^\ell(Q)}$$

with $C = C(\mathbf{F})$.

Let $Q = (\zeta_{i_1,1}\,,\,\zeta_{i_1+1,1}) \times (\zeta_{i_2,2}\,,\,\zeta_{i_2+1,2})$, with $1 \le i_1 \le m_1 - 1$ and $1 \le i_2 \le m_2 - 1$, be a general element in the parametric domain. Let the integers

$$\begin{aligned} &k_1 = \max\{k_{i_1,1}, k_{i_1+1,1}\}\ , && k_2 = \max\{k_{i_2,2}, k_{i_2+1,2}\}\ , \\ &k^* = \max\{k_1, k_2\}\ , && k_* = \min\{k_1, k_2\}\ . \end{aligned} \tag{64}$$

where the dependence of k_1, k_2, k^*, k_* on Q is left implicit for ease of notation.

On each element Q we have then the projector $\Pi^Q_{p,k}$, with $p = p_1 = p_2$ and $k = (k_1, k_2)$, introduced in (59). Observe that for each element $Q \in \mathcal{Q}_h$ a different $k = (k_1, k_2)$ is selected, according with (64). We introduce the local NURBS interpolant on the element $K = \mathbf{F}(Q)$ in the physical domain

$$\Pi^K_{\mathcal{V}}(v) = \frac{\Pi^Q_{p,k}(w\,(v \circ \mathbf{F}))}{w} \circ \mathbf{F}^{-1}\ , \qquad v \in H^\sigma(K)\ ,$$

with $\sigma \ge k_1 + k_2$.

We then have the following result.

Theorem 6 *Let* $2k_i - 1 \le p$, $i = 1, 2$, $v \in H^\sigma(K)$ *with* $k_1 + k_2 \le \sigma \le p + 1$ *and the non-negative integer* $\ell \le k_*$. *Then it holds*

$$|v - \Pi^K_{\mathcal{V}}(v)|_{H^\ell(K)} \le C(p - k^* + 1)^{-(\sigma-\ell)} h_K^{\sigma-\ell} \|v\|_{H^\sigma(K)}, \tag{65}$$

where h_K *indicates the diameter of* K.

Proof We start recalling that both the mapping $\mathbf{F}$ and the weight w are smooth when restricted to the element $Q = \mathbf{F}^{-1}(K)$. The same holds for the inverse of $\mathbf{F}$, mapping K into Q. Moreover, by definition, w is bounded away from 0, which implies that also $1/w$ is smooth in Q. By definition of $\Pi^K_{\mathcal{V}}$, first using Lemma 9 and then a standard bound, it follows

$$\begin{aligned} |v - \Pi^K_{\mathcal{V}}(v)|_{H^\ell(K)} &= \Big|\frac{(w\,v \circ \mathbf{F}) - \Pi^Q_{p,k}(w\,v \circ \mathbf{F})}{w} \circ \mathbf{F}^{-1}\Big|_{H^\ell(K)} \\ &\le C\Big\|\frac{(w\,v \circ \mathbf{F}) - \Pi^Q_{p,k}(w\,v \circ \mathbf{F})}{w}\Big\|_{H^\ell(Q)} \\ &\le C\|(w\,v \circ \mathbf{F}) - \Pi^Q_{p,k}(w\,v \circ \mathbf{F})\|_{H^\ell(Q)}\ , \end{aligned} \tag{66}$$

where $C = C(\mathbf{F}, w)$.

Recalling the definition of $\Pi_{p,k}^{Q}$ and applying Corollary 8 in (66) yields

$$|v-\Pi_{\mathcal{V}}^{K}(v)|_{H^{\ell}(K)} \leq C(p-k^{*}+1)^{-(\sigma-\ell)} h_{Q}^{\sigma-\ell} |w\, v \circ \mathbf{F}\,|_{H^{\sigma}(Q)} \ , \tag{67}$$

with $C=C(\mathbf{F}\,,w)$ and $h_Q=\max\{\zeta_{i_1+1,1}-\zeta_{i_1,1}, \zeta_{i_2+1,2}-\zeta_{i_2,2}\}$.

From (67), using a standard bound and Lemma 9 we get

$$\begin{aligned}|v-\Pi_{\mathcal{V}}^{K}(v)|_{H^{\ell}(K)} &\leq C(p-k^{*}+1)^{-(\sigma-\ell)} h_{Q}^{\sigma-\ell} \|v \circ \mathbf{F}\,\|_{H^{\sigma}(Q)} \\ &\leq C(p-k^{*}+1)^{-(\sigma-\ell)} h_{Q}^{\sigma-\ell} \|v\|_{H^{\sigma}(K)}\end{aligned} \tag{68}$$

with $C=C(\mathbf{F}\,,w)$. Finally, the result follows using that, since $Q=\mathbf{F}^{\,-1}(K)$,

$$h_Q \leq C' h_K$$

with $C'=C'(\mathbf{F}\,)$.

Let now v be a function in $H^{\sigma}(\Omega)$, $\sigma \geq k_{\max,1}+k_{\max,2}$, where we recall that $k_{\max,d} := \max\{k_{i,d} : i=2,\ldots,m_d-1\}$. Then, we can define the global NURBS interpolant $\Pi_{\mathcal{V}}$ by

$$\Pi_{\mathcal{V}}(v)|_K = \Pi_{\mathcal{V}}^{K}(v|_K) \qquad \forall K \in \mathcal{K}_h \ ,$$

where $\mathcal{K}_h$ indicates the set of mesh elements in the physical domain. Then the following lemma holds.

Lemma 10 *Let $v \in H^{\sigma}(\Omega)$ with $\sigma \geq k_{\max,1}+k_{\max,2}$. If $2k_{\max,i}-1 \leq p$ for $i=1,2$, then $\Pi_{\mathcal{V}}(v) \in \mathcal{V}_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{p_1,p_2}$.*

Proof By definition, for all $K \in \mathcal{K}_h$,

$$\Pi_{\mathcal{V}}(v)|_K = \frac{\Pi_{p,k}^{Q}((w\, v \circ \mathbf{F}\,)|_Q)}{w} \circ \mathbf{F}^{\,-1}|_K \ ,$$

where $Q=\mathbf{F}^{\,-1}(K)$. Then, in order to prove that $\Pi_{\mathcal{V}}(v) \in \mathcal{V}_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{p_1,p_2}$ we have to show that

$$\Pi_{p,k}(w\, v \circ \mathbf{F}\,) \in \mathcal{S}_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{p_1,p_2},$$

where the global spline interpolant on the parametric space is

$$\Pi_{p,k}(u)|_Q = \Pi_{p,k}^{Q}(u|_Q) \qquad \forall Q \in \mathcal{Q}_h \ ,$$

with u piecewise smooth. We therefore have to show that $\Pi_{p,k}(wv \circ \mathbf{F}\,)$ satisfies (63) for all internal edges e of the parametric mesh. Note that both the weight w and the map $\mathbf{F}$ are piecewise C^{∞} and at least of regularity C^{k_e-1} across each edge e of the parametric mesh, i.e. satisfy condition (63). As a consequence, since $v \in H^{\sigma}(\Omega)$ with $\sigma \geq k_{\max,1} + k_{\max,2} \geq k_e$, it is easy to check that $w\, v \circ \mathbf{F}$ satisfies (63) for all internal edges of $\mathcal{Q}_h$.

Let e be an internal vertical edge, i.e.

$$e = \{(x,y) \in \widehat{\Omega} \mid x = \zeta_{i_1,1},\ \zeta_{i_2,2} \leq y \leq \zeta_{i_2+1,2}\}$$

for some $2 \leq i_1 \leq m_1 - 1$ and $1 \leq i_2 \leq m_2 - 1$. We call Q^+ and Q^- the two (respectively left and right) elements sharing e, namely

$$Q^+ = (\zeta_{i_1-1,1}, \zeta_{i_1,1}) \times (\zeta_{i_2,2}, \zeta_{i_2+1,2})\ , \quad Q^- = (\zeta_{i_1,1}, \zeta_{i_1+1,1}) \times (\zeta_{i_2,2}, \zeta_{i_2+1,2})\ ,$$

and $T_{\pm}$ the respective linear maps introduced in Lemma 8. We call $(\Delta x)_+ = \zeta_{i_1,1} - \zeta_{i_1-1,1}$ the width of Q^+ and analogously for $(\Delta x)_-$. Moreover, with an obvious notation, we define the restrictions $\varphi^{\pm} = (w\ v \circ \mathbf{F}\,)|_{Q^{\pm}}$ and indicate with $k_1^{\pm}, k_2^{\pm}$ the regularity indexes introduced in (64), respectively for $Q^{\pm}$. Note that, thanks to the tensor product structure, it clearly holds

$$k_2^+ = \max\{k_{i_2,2}, k_{i_2+1,2}\} = k_2^-\ . \tag{69}$$

By definition of $k_e = k_{i_1,1}$, and by definition of normal derivative, condition (63) for $\Pi_{p,k}(w\ v \circ \mathbf{F}\,)$ across the vertical edge e can be written as

$$\partial_x^i \Pi_{p,k}(\varphi^+)|_e = \partial_x^i \Pi_{p,k}(\varphi^-)|_e \qquad i = 0, 1, .., k_{i_1,1} - 1\ . \tag{70}$$

Applying a change of variable, (59), (50) with $j = 0$ and finally the aforementioned regularity of $\varphi = wv \circ \mathbf{F}$, we get, for $i = 0, 1, .., k_1^+ - 1$,

$$\begin{aligned}
\partial_x^i \Pi_{p,k}(\varphi^+)|_e &= \partial_x^i \big(\widehat{\Pi}_{p,k}(\varphi^+ \circ T_+) \circ T_+^{-1}\big)|_e \\
&= \left(\frac{\Delta x}{2}\right)_+^{-i} \partial_{\hat{x}}^i \widehat{\Pi}_{p,k}(\varphi^+ \circ T_+)|_{\hat{\gamma}_1} \\
&= \left(\frac{\Delta x}{2}\right)_+^{-i} \widehat{\pi}^{(y)}_{p,k_2^+}\big(\partial_{\hat{x}}^i(\varphi^+ \circ T_+)\big)|_{\hat{\gamma}_1} \\
&= \left(\frac{\Delta x}{2}\right)_+^{-i} \widehat{\pi}^{(y)}_{p,k_2^+}\left(\left(\frac{\Delta x}{2}\right)_+^{i} \partial_x^i(\varphi^+) \circ T_+\right)\Bigg|_{\hat{\gamma}_1} \\
&= \widehat{\pi}^{(y)}_{p,k_2^+}\big(\partial_x^i(\varphi^+) \circ T_+\big)|_{\hat{\gamma}_1} = \widehat{\pi}^{(y)}_{p,k_2^+}\big(\partial_x^i(\varphi^-) \circ T_-\big)|_{\hat{\gamma}_3}\ .
\end{aligned} \tag{71}$$

Recalling (69), the above identity grants

$$\partial_x^i \Pi_{p,k}(\varphi^+)|_e = \widehat{\pi}^{(y)}_{p,k_2^-}\big(\partial_x^i(\varphi^-) \circ T_-\big)|_{\hat{\gamma}_3}\ , \qquad i = 0, 1, ..., k_1^+ - 1\ . \tag{72}$$

Similarly, applying (59) and (50) for the element Q^- gives

$$\partial_x^i \Pi_{p,k}(\varphi^-)|_e = \widehat{\pi}^{(y)}_{p,k_2^-}\big(\partial_x^i(\varphi^-)\circ T_-\big)|_{\hat{\gamma}_3} \,, \qquad i=0,1,...,k_1^- -1 \,. \tag{73}$$

Since, by definition, $k_1^+ \geq k_{i_1,1}$ and $k_1^- \geq k_{i_1,1}$, equations (72) and (73) imply (70). Horizontal edges are handled identically using (49).

Lemma 10 shows that the operator

$$\Pi_{\mathcal{V}} \colon H^\sigma(\Omega) \longrightarrow \mathcal{V}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} \,, \qquad \sigma \geq k_{\max,1} + k_{\max,2} \,,$$

is well defined, while Theorem 6 shows its (local) approximation properties. This concludes our analysis, and, in particular, implies the following global error estimate.

Theorem 7 *Let* $2k_{\max,i}-1 \leq p$ *for* $i=1,2$, $v \in H^\sigma(\Omega)$ *with* $k_{\max,1}+k_{\max,2} \leq \sigma \leq p+1$ *and the non-negative integer* $\ell \leq \min\{k_{i,d} : i = 2,\dots,m_d-1,\ d=1,2\}$. *Then it holds*

$$|v - \Pi_{\mathcal{V}}(v)|_{H^\ell(\Omega)} \leq C(p-k^*_{\max}+1)^{-(\sigma-\ell)} h^{\sigma-\ell} \|v\|_{H^\sigma(\Omega)}, \tag{74}$$

where $k^*_{\max} = \max\{k_{\max,1}, k_{\max,2}\}$ *and* h *indicates the maximum diameter of the elements* $K \in \mathcal{K}_h$.

Remark 2 *Reasoning as in Sections 3.3.1 and 3.4.2, from the estimate (74) it can be shown that given a mesh and the total number of degrees-of-freedom, high-regular interpolation is advantageous.*

6 Conclusions

We have presented here a first result on error estimates for NURBS approximation of smooth functions, explicit in the mesh-size h, degrees p, space regularities $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, that determine the approximation. However, a restriction on the regularity of this approximant must be imposed, namely, $k_{\max,1}+k_{\max,2}-1 \leq p$. This condition is directly related to the way in which we have defined the one-dimensional projection operator, in which the support of each basis function is restricted to a unique mesh interval and $2k$ conditions are imposed on each of these intervals. This fact indicates that a different approach must be applied when more regularity, up to C^{p-1} continuity, in the approximation is desired.

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