

# A Lagrange multiplier method for the finite element solution of elliptic interface problems using non-matching meshes

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**Summary** In this paper we propose a Lagrange multiplier method for the finite element solution of multi-domain elliptic partial differential equations using non-matching meshes. The interface Lagrange multiplier is chosen with the purpose of avoiding the cumbersome integration of products of functions on unrelated meshes (e.g. we will consider global polynomials as multiplier). The ideas are illustrated using Poisson's equation as a model, and the proposed method is shown to be stable and optimally convergent. Numerical experiments demonstrating the theoretical results are also presented.

## 1 Introduction

When considering multi-domain problems with non-matching meshes using Lagrange multiplier techniques, two basic problems occur. First and foremost, the discrete spaces for the discretization of the primal variable and the multipliers have to fulfill the *inf-sup* condition [8] in order the resulting numerical scheme to be stable; it then turns out that many natural choices of approximations do not. Fortunately, this problem can be alleviated by use of stabilized multiplier methods [16, 19, 1, 4], or by using mesh-dependent penalty

methods [2,3]. The second problem is that products of traces of the primal variable and the multipliers have to be integrated on the interfaces. For methods known to fulfill the *inf-sup* condition, such as the mortar element method (see, e.g., [5,6] and [20]), as well as most stabilized methods, this will mean integrating products of piecewise polynomials on unrelated meshes. This is not easily done in practice for problems in  $\mathbb{R}^3$  (see, however, [17]). To mitigate these two problems, we suggest a stabilization method which avoids the cumbersome integration of products of unrelated mesh functions. For example, as already done in [14], global polynomial multipliers can be used on the interfaces: only products of global polynomials and local polynomials have to be considered and this makes the integration problem much simpler in many cases. The obtained method is stable and optimally convergent. We remark that stability is achieved under the very mild assumption that the approximation space for the interface multiplier contains the constants. The method is presented and analyzed using a two-dimensional Poisson equation as a model with two subdomains. From a geometric point of view, two situations can occur: the case where the intersections of the boundaries of both subdomains with the outer boundary have non-zero 1D measure, and the case where the interface is a closed curve (see Fig. 1 below). We point out that the stability analysis of the former case, for which similar arguments as in [1] could be applied, is simpler than that of the latter case. In this paper we present an analysis that covers both cases.

As a basic motivation for this work, we have in mind the contact problem in elasticity. In standard commercial codes for computing contact between two elastic bodies, the contact condition is only checked at the nodes either on one or on both of the bodies. This corresponds to choosing discrete Lagrange multipliers which is not natural from the perspective of the variational formulation of the problem. The stability and convergence properties of these approaches are in general not known, and the results have to be carefully interpreted, which requires some experience. Furthermore, in our experience, plenty of choices have to be made in discretizing the interface, choosing the master surface,

etc. An obvious reason for choosing discrete multipliers is that it makes the integration problem particularly easy; as mentioned above this is also the aim of the method to be presented.

One could interpret our approach as covering the contact surface with, for example, a polynomial layer which acts as an intermediate between the two surfaces (which do not have to be known in advance). Even though this results in a global coupling of all the variables on the contact surface, the typical contact application has a small zone of contact and the global coupling will not cause the problem to grow excessively in size.

The outline of the paper is as follows. In Section 2 the interface Lagrange multiplier method with a generic discretization of the multiplier is presented, after introducing the model problem together with some notation, and discussing the motivation of the present work. The stability and error analysis of the new method is carried out in Section 3, and numerical experiments (with a global polynomial multiplier) demonstrating the theoretical results are presented in Section 4. The paper ends with some conclusions in Section 5.

## 2 Formulation of the method

In this section we introduce an interface Lagrange multiplier method for the finite element discretization of elliptic problems on non-matching grids. Before doing that, we make precise the model problem we will be working on, together with some notation and motivation of the present work.

### 2.1 Model problem

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , with boundary  $\partial\Omega$ . (The extension to  $\mathbb{R}^3$  is straightforward.) As a model problem, we consider a stationary heat conduction problem in the case where there is a piecewise straight internal boundary  $\Gamma$  dividing  $\Omega$  into two subdomains

$\Omega_1$  and  $\Omega_2$ . Thus, we want to solve for  $u$  the problem

$$\begin{aligned} -\nabla \cdot (\kappa_i \nabla u_i) &= f \quad \text{in } \Omega_i, \\ u_i &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega, \\ u_1 - u_2 &= 0 \quad \text{on } \Gamma, \\ \mathbf{n}_1 \cdot \kappa_1 \nabla u_1 + \mathbf{n}_2 \cdot \kappa_2 \nabla u_2 &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

for  $i = 1, 2$ , where we have denoted by  $u_i$  the restriction of  $u$  to  $\Omega_i$ . Here  $f$  is a given function,  $\kappa_i$ , which is assumed to be positive and smooth in  $\Omega_i$ , is the conductivity, and  $\mathbf{n}_i$  is the outward pointing normal to  $\Omega_i$  at  $\Gamma$ ,  $i = 1, 2$ . This formulation of the standard Poisson problem is common in the context of domain decomposition problems, cf. [13].

Define

$$V = \{v : v_i \in H^1(\Omega_i), v_i = 0 \text{ on } \partial\Omega_i \setminus \Gamma, i = 1, 2\}$$

and

$$\Lambda = (H_{00}^{1/2}(\Gamma))',$$

the dual space of  $H_{00}^{1/2}(\Gamma)$  (see, e.g., [15]). Notice that, whenever  $\Gamma \cap \partial\Omega = \emptyset$  (see Fig. 1, right),  $H_{00}^{1/2}(\Gamma)$  coincides with  $H^{1/2}(\Gamma)$  and  $\Lambda = H^{-1/2}(\Gamma)$ .

A weak form of (1) using the Lagrange multiplier approach is as follows:

Find  $(u, \lambda) \in V \times \Lambda$  such that

$$\begin{aligned} \sum_i \int_{\Omega_i} \kappa_i \nabla u_i \cdot \nabla v_i \, dx + \int_{\Gamma} \lambda [v] \, ds &= \sum_i \int_{\Omega_i} f v_i \, dx \quad \forall v \in V, \\ \int_{\Gamma} [u] \mu \, ds &= 0 \quad \forall \mu \in \Lambda, \end{aligned} \tag{2}$$

where  $[v] := (v_1 - v_2)|_{\Gamma}$  is the jump of  $v$  across  $\Gamma$ . Notice that

$$\lambda = -\kappa_1 \nabla u_1 \cdot \mathbf{n}_1 = \kappa_2 \nabla u_2 \cdot \mathbf{n}_2 \quad \text{on } \Gamma.$$

## 2.2 Notation

We introduce the necessary notation for the definition of the method we are going to present and its subsequent analysis, focusing, for simplicity, on the case of triangular elements. Therefore, we assume that we are given a triangular mesh  $\mathcal{T}_i^h$  of the domain  $\Omega_i$ ,

$i = 1, 2$ . We denote by  $h_i$  the mesh-size of  $\mathcal{T}_i^h$ . Obviously,  $\mathcal{T}^h = \mathcal{T}_1^h \cup \mathcal{T}_2^h$  provides a mesh for  $\Omega$ , whose mesh-size is  $h = \max\{h_1, h_2\}$ . We introduce the (family of) finite element space

$$V^h = \{v \in V : v|_K \in P^k(K), \forall K \in \mathcal{T}^h\},$$

where  $P^k(K)$  denotes the space of polynomials of degree at most  $k$  on  $K$ , with  $k \geq 1$ .

On  $\Gamma$  we introduce a family of spaces  $\Lambda^p$  of discrete multipliers such that

$$P^0(\Gamma) \subset \Lambda^p, \quad (3)$$

$P^0(\Gamma)$  being the space of constants on the whole interface  $\Gamma$ . As a particular case, we will consider the space  $\Lambda^p$  of global polynomials defined as follows: the interface  $\Gamma$  is decomposed as the union  $\Gamma = \bigcup \Gamma_j$  of  $n_\Gamma$  straight lines  $\Gamma_j$  of length  $\ell_j$  (see Fig. 1); we associate with each  $\Gamma_j$  the non-negative integer  $p_j$  and define  $\underline{p} := [p_1, \dots, p_{n_\Gamma}]$ ; then our particular choice is

$$\Lambda^p = \{\mu \in \Lambda : \mu|_{\Gamma_j} \in P^{p_j}(\Gamma_j), j = 1, \dots, n_\Gamma\}, \quad (4)$$

with  $P^{p_j}(\Gamma_j)$  denoting the space of polynomials of degree at most  $p_j$  on  $\Gamma_j$  with respect to a local coordinate. In this particular case the elements of  $\Lambda^p$  can be discontinuous at the endpoints of the  $\Gamma_j$ 's.

We also remark that two different situations can occur from a geometric point of view:

1. both  $\partial\Omega_1 \cap \partial\Omega$  and  $\partial\Omega_2 \cap \partial\Omega$  have nonzero 1-D measure (see Fig. 1, left);
2. either  $\partial\Omega_1 \cap \partial\Omega$  or  $\partial\Omega_2 \cap \partial\Omega$  has zero 1-D measure (see Fig. 1, right).

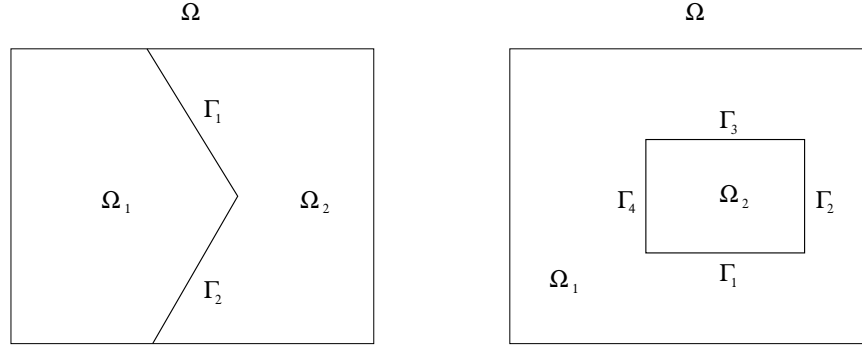
As we will see in the next section, the stability analysis is more difficult for the second case.

### 2.3 Background

A standard penalty method for domain decomposition problems is the following (cf. [2]):

Find  $u^h \in V^h$  such that

$$\sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \frac{1}{\gamma} [u^h][v] \, ds = \sum_i \int_{\Omega_i} f v_i \, dx \quad \forall v \in V^h. \quad (5)$$



**Fig. 1.** Two different geometric situations.

There is a consistency error present in (5), but by letting the penalty function  $\gamma$  depend on the mesh-size, i.e.,  $\gamma$  of the order of  $h^r$ , for suitable values of  $r > 0$ , this consistency error will not dominate the discretization error in energy-like norms (see [3] for an extensive investigation).

The main problem of implementation of (5) is how to evaluate integrals of the type

$$\int_{\Gamma} u_i^h v_j ds, \quad i \neq j,$$

especially in three dimensions. If quadrature is used, we have an expensive search problem in locating elements in the mesh on  $\Omega_i$  containing quadrature points in the elements of the mesh on  $\Omega_j$ . Exact integration is within reach; Priestley [17] has made an implementation of exact quadrature in the case of triangular surface meshes with common boundary. Here we take an alternative route which greatly simplifies the implementation, by introducing a suitable multiplier on the interface  $\Gamma$ .

Therefore, starting from (5), one could consider the following *inconsistent* perturbed Lagrange multiplier method:

Find  $(u^h, \lambda^p) \in V^h \times \Lambda^p$  such that

$$\begin{aligned} \sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i dx + \int_{\Gamma} \lambda^p [v] ds &= \sum_i \int_{\Omega_i} f v_i dx \quad \forall v \in V^h, \\ \int_{\Gamma} [u^h] \mu ds - \int_{\Gamma} \gamma \lambda^p \mu ds &= 0 \quad \forall \mu \in \Lambda^p. \end{aligned} \tag{6}$$

Now, as  $\gamma \rightarrow 0$ , the problem will be a standard saddle-point problem which requires compatibility between the discrete spaces for  $\lambda^p$  and  $u^h$  in order to fulfill a Babuška-Brezzi condition [8]. Again, we can let  $\gamma$  to be of the order of  $h^r$ , so that the problem of balancing  $\lambda^p$  and  $V^h$  is alleviated. In case of (4) the product of basis functions and global polynomials can easily be integrated exactly, at least for simplicial elements.

On the other hand, due to inconsistency, the formulation (6) exhibits a loss of accuracy, as our numerical results in Section 4 indicate. It also requires coupling the parameter  $\alpha$  to the polynomial degree of approximation, which is detrimental to the conditioning of the system. In order to overcome these drawbacks, we are going to present a *consistent* version of (6).

#### 2.4 The consistent method: A Nitsche-type interface condition

The classical method of Nitsche [16] for handling Dirichlet boundary conditions weakly was extended to the case of domain decomposition with non-matching meshes by Becker, Hansbo and Stenberg [4]. The problem of having to integrate products of functions on one side of the interface with functions on the other side arises also in their method. However, Nitsche-type methods are consistent and thus optimally convergent with a fixed “penalty” parameter of  $O(h^{-1})$ , which does not destroy the conditioning of the resulting system. Thus, there could be some advantages to formulating a Nitsche-type interface formulation using the interjacent multiplier space. To this end we define  $\mathbf{n} := \mathbf{n}_1 = -\mathbf{n}_2$  on  $\Gamma$  and

$$\{\mathbf{n} \cdot \mathbf{w}\} := \alpha \mathbf{n} \cdot \mathbf{w}_1 + (1 - \alpha) \mathbf{n} \cdot \mathbf{w}_2,$$

where  $0 \leq \alpha \leq 1$ . We propose an unsymmetric method and its symmetric variant, the latter one only for the choices  $\alpha = 0$  and  $\alpha = 1$ .

**2.4.1 The unsymmetric method** For any choice of  $0 \leq \alpha \leq 1$ , the unsymmetric method reads as follows:

Find  $(u^h, \lambda^p) \in V^h \times \Lambda^p$  such that

$$\begin{aligned} \sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_{\Gamma} \lambda^p [v] \, ds &= \sum_i \int_{\Omega_i} f v_i \, dx \quad \forall v \in V^h, \\ \int_{\Gamma} [u^h] \mu \, ds - \int_{\Gamma} \gamma \{ \mathbf{n} \cdot \kappa \nabla u^h \} \mu \, ds - \int_{\Gamma} \gamma \lambda^p \mu \, ds &= 0 \quad \forall \mu \in \Lambda^p, \end{aligned} \quad (7)$$

where  $\gamma$  is the function of  $L^\infty(\Gamma)$  defined as follows. Denote by  $\mathcal{N}_\Gamma$  the set of nodes of  $\mathcal{T}_1^h$  and  $\mathcal{T}_2^h$  lying on  $\Gamma$ . Fix a point  $\mathbf{x}$  on  $\Gamma \setminus \mathcal{N}_\Gamma$  and let  $K_1$  and  $K_2$  be the two elements of  $\mathcal{T}_1^h$  and  $\mathcal{T}_2^h$ , respectively, such that the interior of  $\partial K_1 \cap \partial K_2 \cap \Gamma$  is non empty and  $\mathbf{x} \in \partial K_1 \cap \partial K_2 \cap \Gamma$ . Denote by  $h_{K_1}$  and  $h_{K_2}$  the diameters of  $K_1$  and  $K_2$ , respectively.

For  $\mathbf{x} \in \Gamma \setminus \mathcal{N}_\Gamma$ , we define

$$\gamma(\mathbf{x}) = \gamma_0 \min \left\{ \alpha \frac{h_{K_1}}{\kappa_1(\mathbf{x})}, (1 - \alpha) \frac{h_{K_2}}{\kappa_2(\mathbf{x})} \right\}, \quad (8)$$

with  $\gamma_0$  a constant independent on the mesh-size and the material properties. Restrictions on  $\gamma_0$  will be made precise later on. Notice that  $\gamma$  is defined almost everywhere on  $\Gamma$ . We remark that the function  $\gamma$  here plays the same role as the  $\gamma$  in the formulations (5) and (6).

*2.4.2 The symmetric variant* In the cases of  $\alpha = 0$  and  $\alpha = 1$ , the method (7) can be symmetrized without introducing integration of cross terms across the interface; this would not be possible for  $0 < \alpha < 1$ . The symmetric formulation reads as follows:

Find  $(u^h, \lambda^p) \in V^h \times \Lambda^p$  such that

$$\begin{aligned} \sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_{\Gamma} \lambda^p [v] \, ds - \int_{\Gamma} \gamma \lambda^p (\mathbf{n} \cdot \kappa_j \nabla v_j) \, ds \\ - \int_{\Gamma} \gamma (\mathbf{n} \cdot \kappa_j \nabla u_j^h) (\mathbf{n} \cdot \kappa_j \nabla v_j) \, ds &= \sum_i \int_{\Omega_i} f v_i \, dx \quad \forall v \in V^h, \\ \int_{\Gamma} [u^h] \mu \, ds - \int_{\Gamma} \gamma (\mathbf{n} \cdot \kappa_j \nabla u_j^h) \mu \, ds - \int_{\Gamma} \gamma \lambda^p \mu \, ds &= 0 \quad \forall \mu \in \Lambda^p, \end{aligned} \quad (9)$$

with subscript  $j = 2$  if  $\alpha = 0$ , and  $j = 1$  if  $\alpha = 1$ . Again, the function  $\gamma$  is as in (8). We remark that, as pointed out in the discussion after Theorem 2 below,  $\alpha = 0$  and  $\alpha = 1$  are reasonable choices and in particular the sole choices which give the best convergence results whenever the characteristic mesh-sizes of the two domains are significantly different

from each other. One reason of using the symmetric formulation is its adjoint consistency, that is useful whenever duality arguments need to be applied.

Both the formulations (7) and (9) are consistent (see Lemma 1 below).

### 3 Analysis of the method

In this section we develop stability analysis and derive error estimates of the methods (7) and (9). The analysis will show that, in the particular case of global polynomial multipliers (i.e., (4)) and quasi-uniform meshes  $\mathcal{T}^h$  of size  $h$ , for analytical solutions in  $H^2(\Omega_i)$  the choice of the polynomial approximation orders for  $\lambda$  that give the best error estimate is  $p_j \approx \ell_j/h$ ,  $j = 1, \dots, n_\Gamma$ . In correspondence to that, and for  $\gamma$  of the order of  $h^{-1}$ , we obtain the optimal error estimate

$$\| \| (u - u^h, \lambda - \lambda^p) \| \| \leq Ch \sum_i |u_i|_{H^2(\Omega_i)},$$

where  $\| \| \cdot \| \|$  is an energy-like norm, and  $C$  is a positive constant independent of  $h$ . More in general, the estimates we will derive take into account the *local* mesh-size and material properties.

Before proceeding with the analysis, we rewrite the formulations (7) and (9) in a unified way:

Find  $(u^h, \lambda^p) \in V^h \times \Lambda^p$  such that

$$\begin{aligned} \sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \lambda^p [v] \, ds - S \int_\Gamma \gamma \lambda^p \{ \mathbf{n} \cdot \kappa \nabla v \} \, ds \\ - S \int_\Gamma \gamma \{ \mathbf{n} \cdot \kappa \nabla u^h \} \{ \mathbf{n} \cdot \kappa \nabla v \} \, ds = \sum_i \int_{\Omega_i} f v_i \, dx \quad \forall v \in V^h, \quad (10) \\ \int_\Gamma [u^h] \mu \, ds - \int_\Gamma \gamma \{ \mathbf{n} \cdot \kappa \nabla u^h \} \mu \, ds - \int_\Gamma \gamma \lambda^p \mu \, ds = 0 \quad \forall \mu \in \Lambda^p, \end{aligned}$$

either with  $S = 0$  and any  $0 \leq \alpha \leq 1$  (to get the unsymmetric formulation (7)), or with  $S = 1$  and  $\alpha = 0$  or  $\alpha = 1$  (to get the symmetric formulation (9)).

Introducing

$$\begin{aligned} \mathcal{B}^h(w, \nu; v, \mu) &:= \sum_i \int_{\Omega_i} \kappa_i \nabla w_i \cdot \nabla v_i \, dx + \int_{\Gamma} \nu [v] \, ds \\ &\quad - S \int_{\Gamma} \gamma \nu \{ \mathbf{n} \cdot \kappa \nabla v \} \, ds - S \int_{\Gamma} \gamma \{ \mathbf{n} \cdot \kappa \nabla w \} \{ \mathbf{n} \cdot \kappa \nabla v \} \, ds \\ &\quad - \int_{\Gamma} [w] \mu \, ds + \int_{\Gamma} \gamma \{ \mathbf{n} \cdot \kappa \nabla w \} \mu \, ds + \int_{\Gamma} \gamma \nu \mu \, ds, \end{aligned}$$

and

$$\mathcal{F}^h(v, \mu) = \sum_i \int_{\Omega_i} f v_i \, dx,$$

the formulation (10) can be written as follows:

Find  $(u^h, \lambda^p) \in V^h \times \Lambda^p$  such that

$$\mathcal{B}^h(u^h, \lambda^p; v, \mu) = \mathcal{F}^h(v, \mu) \quad \forall (v, \mu) \in V^h \times \Lambda^p. \quad (11)$$

**Lemma 1** *The method (11) is consistent in the sense that, if  $(u, \lambda)$  is the solution of (2), then*

$$\mathcal{B}^h(u - u^h, \lambda - \lambda^p; v, \mu) = 0 \quad \forall (v, \mu) \in V^h \times \Lambda^p.$$

*Proof* Inserting  $(u, \lambda)$  solution of (2) in the place of  $(u^h, \lambda^p)$  in (11), recalling that  $\lambda = -\kappa_1 \nabla u_1 \cdot \mathbf{n} = -\kappa_2 \nabla u_2 \cdot \mathbf{n}$  on  $\Gamma$ , we find that

$$\begin{aligned} \sum_i \int_{\Omega_i} \kappa_i \nabla (u_i - u_i^h) \cdot \nabla v_i \, dx + \int_{\Gamma} (\lambda - \lambda^p) [v] \, ds &= 0, \\ \int_{\Gamma} \gamma (\lambda - \lambda^p) \{ \mathbf{n} \cdot \kappa \nabla v \} \, ds + \int_{\Gamma} \gamma \{ \mathbf{n} \cdot \kappa \nabla (u - u^h) \} \{ \mathbf{n} \cdot \kappa \nabla v \} \, ds &= 0, \\ \int_{\Gamma} [u - u^h] \mu \, ds - \int_{\Gamma} \gamma \{ \mathbf{n} \cdot \kappa \nabla (u - u^h) \} \mu \, ds - \int_{\Gamma} \gamma (\lambda - \lambda^p) \mu \, ds &= 0, \end{aligned}$$

for all  $v \in V^h$  and  $\mu \in \Lambda^p$ . The assertion immediately follows.

### 3.1 Stability

Defining the (weighted) broken  $H^1$ -norm

$$\|w\|_V = \left( \sum_i \|\kappa_i^{1/2} \nabla w_i\|_{L_2(\Omega_i)}^2 + \|aw\|_{L_2(\Omega)}^2 \right)^{1/2},$$

with  $a = \kappa^{1/2}/\text{diam}(\Omega)$ , we introduce the norm

$$\| \! \| (w, \nu) \! \| := \left( \|w\|_V^2 + \|\gamma^{1/2} \nu\|_{L_2(\Gamma)}^2 \right)^{1/2}.$$

Define the constant  $\mathbf{k}$  as the mean value of the function  $\kappa$  on  $\Omega$  and  $b := (\mathbf{k}/\text{diam}(\Omega))^{1/2}$ .

On the interface  $\Gamma$ , we will use the norm

$$\|\varphi\|_{1/2,\Gamma} := \left( \|b\varphi\|_{L_2(\Gamma)}^2 + |\mathbf{k}^{1/2}\varphi|_{H_{00}^{1/2}(\Gamma)}^2 \right)^{1/2},$$

together with its dual denoted by  $\|\cdot\|_{-1/2,\Gamma}$ . We recall that, whenever  $\Gamma \cap \partial\Omega = \emptyset$ ,

$|\cdot|_{H_{00}^{1/2}(\Gamma)}$  coincides with  $|\cdot|_{H^{1/2}(\Gamma)}$  (see [15]).

*Remark 1* The norm  $\|\cdot\|_{1/2,\Gamma}$  is the natural norm for the traces on  $\Gamma$  of functions belonging to  $V$ , when  $V$  is endowed with the  $\|\cdot\|_V$ -norm.

We proceed by proving continuity and inf-sup properties of the form  $\mathcal{B}^h$ . In the sequel  $C, C_1, \dots$  denote generic *strictly positive* constants possibly depending on the shape of the domain, on the shape regularity of the meshes, on the quantity  $\max \kappa / \min \kappa$ , and on the polynomial approximation degrees of  $V^h$ , but independent of the mesh-size and of  $\Lambda^p$ .

**Proposition 1** *For all  $w, v \in V, \nu, \mu \in \Lambda$  we have*

$$\mathcal{B}^h(w, \nu; v, \mu) \leq C \left( \|(w, \nu)\| + \|\gamma^{-1/2}[w]\|_{L_2(\Gamma)} + \|\nu\|_{-1/2,\Gamma} \right) \|(v, \mu)\|. \quad (12)$$

*Proof* The bound follows from the Cauchy-Schwarz inequality, from the duality inequality with respect to the norms  $\|\cdot\|_{1/2,\Gamma}$  and  $\|\cdot\|_{-1/2,\Gamma}$ , the inverse inequalities

$$\|\kappa_i^{-1/2} h_i^{1/2} \mathbf{n}_i \cdot \kappa_i \nabla w_i\|_{L_2(\Gamma)}^2 \leq C_I \|\kappa_i^{1/2} \nabla w_i\|_{L_2(\Omega_i)}^2 \quad \forall w \in V_i^h, \quad i = 1, 2, \quad (13)$$

and from

$$\|[w]\|_{1/2,\Gamma} \leq \|w_1\|_{1/2,\Gamma} + \|w_2\|_{1/2,\Gamma} \leq C \|w\|_V$$

(see Remark 1).

**Proposition 2** *Provided that  $\gamma_0$  is small enough (see Remark 2 below), for all  $(w, \nu) \in V^h \times \Lambda^p$  there is  $(v, \mu) \in V^h \times \Lambda^p$  such that*

$$\|(v, \mu)\| \leq C_1 \|(w, \nu)\|,$$

$$\mathcal{B}^h(w, \nu; v, \mu) \geq C_2 \|(w, \nu)\|^2,$$

*Proof* Let  $(w, \nu) \in V^h \times \Lambda^p$  and take  $(v, \mu) \in V^h \times \Lambda^p$  as  $v = w$  and  $\mu = \mu_1 + \delta\mu_2$ , with  $\mu_1 = \nu$  and  $\mu_2 = -b^2\Pi_0[w]$ , where  $\Pi_0$  is the  $L_2(\Gamma)$ -projection operator onto  $P^0(\Gamma)$ , and  $\delta$  is a positive parameter still at our disposal. From the definition of  $\mu_2$ , the bound  $b\gamma^{1/2} \leq C\gamma_0^{1/2}$ , and a trace inequality, we have

$$\|\gamma^{1/2}\mu_2\|_{L_2(\Gamma)} = b\|(b\gamma^{1/2})\Pi_0[w]\|_{L_2(\Gamma)} \leq C\gamma_0^{1/2}b\|[w]\|_{L_2(\Gamma)} \leq C\gamma_0^{1/2}\|w\|_V.$$

The continuity estimate  $|(v, \mu)| \leq C_1|(w, \nu)|$  immediately follows.

For the coercivity, we proceed in two steps. First, from the definition of  $v$  and  $\mu_1$  we have

$$\begin{aligned} \mathcal{B}^h(w, \nu; v, \mu_1) &= \sum_i \|\kappa_i^{1/2}\nabla w_i\|_{L_2(\Omega_i)}^2 + (1-S) \int_{\Gamma} \gamma \{\mathbf{n} \cdot \kappa \nabla w\} \nu \, ds \\ &\quad - S \|\gamma^{1/2} \{\mathbf{n} \cdot \kappa \nabla w\}\|_{L_2(\Gamma)}^2 + \|\gamma^{1/2}\nu\|_{L_2(\Gamma)}^2. \end{aligned}$$

By combining a weighted Cauchy-Schwarz inequality with the inverse inequality (13), provided that  $\gamma_0 < 1/C_I$ , making use of the Young inequality, we obtain

$$\mathcal{B}^h(w, \nu; v, \mu_1) \geq C \left( \sum_i \|\kappa_i^{1/2}\nabla w_i\|_{L_2(\Omega_i)}^2 + \|\gamma^{1/2}\nu\|_{L_2(\Gamma)}^2 \right), \quad (14)$$

where  $C$  depends on  $C_I$  and  $\gamma_0$ , but is independent of the mesh-size.

For the second step, we can write

$$\begin{aligned} \mathcal{B}^h(w, \nu; 0, \mu_2) &= \int_{\Gamma} b[w] b \Pi_0[w] \, ds - \int_{\Gamma} b^2\gamma \{\mathbf{n} \cdot \kappa \nabla w\} \Pi_0[w] \, ds - \int_{\Gamma} b^2\gamma \nu \Pi_0[w] \, ds \\ &= \|b \Pi_0[w]\|_{L_2(\Gamma)}^2 - \int_{\Gamma} b^2\gamma \{\mathbf{n} \cdot \kappa \nabla w\} \Pi_0[w] \, ds - \int_{\Gamma} b^2\gamma \nu \Pi_0[w] \, ds. \end{aligned}$$

Using suitably weighted Cauchy-Schwarz inequalities for the last two integrals and the inverse inequality (13), we obtain

$$\mathcal{B}^h(w, \nu; 0, \mu_2) \geq \frac{1}{2} \|b \Pi_0[w]\|_{L_2(\Gamma)}^2 - CC_I\gamma_0^2 \sum_i \|\kappa_i^{1/2}\nabla w_i\|_{L_2(\Omega_i)}^2 - C\gamma_0 \|\gamma^{1/2}\nu\|_{L_2(\Gamma)}^2. \quad (15)$$

By adding together (14) and (15) multiplied by  $\delta$ , and taking  $\delta$  small enough (depending on the constants  $C$ ,  $C_I$  and  $\gamma_0$ ), we obtain

$$\mathcal{B}^h(w, \nu; v, \mu) \geq C_3 \left( \sum_i \|\kappa_i^{1/2}\nabla w_i\|_{L_2(\Omega_i)}^2 + \|b \Pi_0[w]\|_{L_2(\Gamma)}^2 + \|\gamma^{1/2}\nu\|_{L_2(\Gamma)}^2 \right), \quad (16)$$

with a positive constant  $C_3$  only depending on  $C_I$  and  $\gamma_0$ , therefore independent of the mesh-size.

In order to complete the proof of the proposition, we need to show that

$$\|aw\|_{L_2(\Omega)}^2 \leq C \left( \sum_i \|\kappa_i^{1/2} \nabla w_i\|_{L_2(\Omega_i)}^2 + \|b \Pi_0[w]\|_{L_2(\Gamma)}^2 \right), \quad (17)$$

with a constant  $C$  independent of the mesh-size. The coercivity estimate  $\mathcal{B}^h(w, \nu; v, \mu) \geq C_2 \|((w, \nu))\|^2$  will follow easily. In the case where both  $\partial\Omega_1$  and  $\partial\Omega_2$  contain a part of the Dirichlet boundary with nonzero 1-dimensional measure (see Fig. 1, left), the estimate (17) directly follows from the standard Poincaré's inequality. On the other hand, if either of  $\partial\Omega_1$  or  $\partial\Omega_2$  does not contain a part of the Dirichlet boundary with nonzero 1-dimensional measure (see Fig. 1, right), we still make use of the Poincaré inequality, but the proof is not straightforward. A similar result has been proven in [7] for piecewise  $H^1$  functions. We develop the latter case in detail.

Assume, to fix the ideas, that  $\partial\Omega_2$  does not intersect  $\partial\Omega$ . Then,  $\partial\Omega_1$  contains the Dirichlet boundary. Therefore, the following Poincaré's inequality holds true:

$$\|w_1\|_{L_2(\Omega_1)} \leq C \text{diam}(\Omega_1) \|\nabla w_1\|_{L_2(\Omega_1)}. \quad (18)$$

Write  $w_2$  as  $(w_2 - \Pi_0 w_2|_\Gamma) + \Pi_0 w_2|_\Gamma$ , where  $\Pi_0 w_2|_\Gamma$  is the constant function on  $\Omega_2$  equal to the mean value of the trace of  $w_2$  on  $\Gamma$ . The Poincaré's inequality

$$\|w_2 - \Pi_0 w_2|_\Gamma\|_{L_2(\Omega_2)} \leq C \text{diam}(\Omega_2) \|\nabla w_2\|_{L_2(\Omega_2)}$$

follows from Bramble-Hilbert's lemma applied to the operator  $\pi : H^1(\Omega_2) \rightarrow L_2(\Omega_2)$  defined by  $\pi(w) = w - \Pi_0 w|_\Gamma$ , which is zero on  $P^0(\Omega_2)$ , the space of constants on  $\Omega_2$ .

Therefore,

$$\begin{aligned} \|w_2\|_{L_2(\Omega_2)} &\leq \|w_2 - \Pi_0 w_2|_\Gamma\|_{L_2(\Omega_2)} + \|\Pi_0 w_2|_\Gamma\|_{L_2(\Omega_2)} \\ &\leq C \text{diam}(\Omega_2) \|\nabla w_2\|_{L_2(\Omega_2)} + (|\Omega_2|/|\Gamma|)^{1/2} \|\Pi_0 w_2\|_{L_2(\Gamma)} \\ &\leq C \text{diam}(\Omega_2) (\|\nabla w_2\|_{L_2(\Omega_2)} + \|\Pi_0 w_2 - \Pi_0 w_1\|_{L_2(\Gamma)} + \|\Pi_0 w_1\|_{L_2(\Gamma)}) \\ &\leq C \text{diam}(\Omega_2) (\|\nabla w_2\|_{L_2(\Omega_2)} + \|\Pi_0[w]\|_{L_2(\Gamma)} + \|\nabla w_1\|_{L_2(\Omega_1)}), \end{aligned} \quad (19)$$

where in the last step we have used a trace theorem and (18). Estimates (18) and (19) immediately give (17).

*Remark 2* From the proof of Proposition 2, it is clear that  $\gamma_0$  in the definition of  $\gamma$  has to be chosen smaller than  $1/C_I$ , where  $C_I$  is the inverse inequality constant in (13), which only depends on the shape regularity of the meshes, on  $\max \kappa / \min \kappa$ , and on the polynomial approximation degree for the variable  $u$ .

### 3.2 Error Analysis

We derive error estimates for the method (7) in a standard way from the results proven in Proposition 1 and 2.

**Theorem 1** *Provided that condition (3) is satisfied, we have*

$$\begin{aligned} \|(u - u^h, \lambda - \lambda^p)\| \leq C \inf_{w^h \in V^h, \nu^p \in \Lambda^p} (\|(u - w^h, \lambda - \nu^p)\| \\ + \|\gamma^{-1/2}[u - w^h]\|_{L_2(\Gamma)} + \|\lambda - \nu^p\|_{-1/2, \Gamma}). \end{aligned} \quad (20)$$

*Proof* We decompose, as usual, the error  $(u - u^h, \lambda - \lambda^p)$  as  $(u - w^h, \lambda - \nu^p) + (w^h - u^h, \nu^p - \lambda^p)$ . From triangle inequality, Proposition 2, Lemma 1 and Proposition 1, we obtain (20).

For the choice of global polynomial multipliers (see (4)) and under usual regularity assumptions for the analytical solution, we can prove the following result.

**Theorem 2** *Assume  $\Lambda^p$  given by (4),  $u_i \in H^{s+1}(\Omega_i)$ ,  $i = 1, 2$ , with  $s > 1/2$ . Denote by  $T_i^h$ ,  $i = 1, 2$ , the union of the elements contained in  $\Omega_i$  and having one side on  $\Gamma$  and, for any  $K \in T_1^h \cup T_2^h$ , define  $\gamma_K := \min_{\mathbf{x} \in \partial K \cap \Gamma} \gamma$ . We have*

$$\begin{aligned} \|(u - u^h, \lambda - \lambda^p)\| \leq C \left( \mathbf{k} \sum_{K \in \mathcal{T}^h} h_K^{2 \min\{k, s\}} |u|_{H^{s+1}(K)}^2 + \sum_{K \in T_1^h \cup T_2^h} \gamma_K^{-1} h_K^{2 \min\{k, s\} + 1} |u|_{H^{s+1}(K)}^2 \right. \\ \left. + \sum_j \left( \mathbf{k}^{-1} \ell_j^{2s} p_j^{-2s} + \sup_{\mathbf{x} \in \Gamma_j} \gamma \ell_j^{2s-1} p_j^{-(2s-1)} \right) |\lambda|_{H^{s-1/2}(\Gamma_j)}^2 \right)^{1/2}, \end{aligned} \quad (21)$$

with a positive constant  $C$  independent of the mesh-size and of  $\underline{p}$ .

We briefly comment on the result of Theorem 2 before showing its proof.

- Assume that  $\mathcal{T}^h$  is quasi-uniform and denote by  $h$  its characteristic mesh-size. Then, from (21), the optimal choice of the polynomial approximation seems to be  $p_j \approx \ell_j/h$ ,  $j = 1, \dots, n_\Gamma$ , when  $k \geq s$ , or  $p_j \approx \ell_j/h^{k/s}$ ,  $j = 1, \dots, n_\Gamma$ , when  $k < s < +\infty$ . Estimate (21) then becomes

$$\| (u - u^h, \lambda - \lambda^p) \| \leq C \mathbf{k}^{1/2} h^{\min\{k, s\}} \sum_i |u_i|_{H^{s+1}(\Omega_i)}.$$

For analytic solutions, from Theorem 3.20 in [18], we have that the approximation error for  $\lambda$  depends on  $p$  exponentially. Therefore,  $p_j$  can be chosen as  $p_j \approx |\log h|$ ,  $j = 1, \dots, n_\Gamma$ , which allows for a saving in the number of degrees of freedom for the approximation of  $\lambda$  (we give an example in Section 4).

- Assume that  $\mathcal{T}_1^h$  and  $\mathcal{T}_2^h$  are quasi-uniform and denote by  $h_1$  and  $h_2$  their characteristic mesh-sizes (here, we are not assuming any bound of  $h_1$  and  $h_2$  in terms of each other). Then, from the definition of the parameter  $\gamma$ , it is clear that, if  $h_1$  and  $h_2$  are very different in size, estimate (21) significantly depends on  $\alpha$ . If  $h_1 \ll h_2$  (resp.  $h_2 \ll h_1$ ), the best result is given for  $\alpha = 0$  (resp.  $\alpha = 1$ ), which gives the symmetric formulation (9). We point out that the same strategy was adopted in [10, 11] for an interior penalty treatment of the interface between non-matching meshes.

If we assume  $s = 1$ , for the sake of simplicity, and the optimal choice of the polynomial approximation orders, namely  $p_j \approx \ell_j / \max\{h_1, h_2\}$ ,  $j = 1, \dots, n_\Gamma$ , estimate (21) becomes

$$\| (u - u^h, \lambda - \lambda^p) \| \leq C \mathbf{k}^{1/2} \sum_i h_i |u_i|_{H^2(\Omega_i)}.$$

*Proof of Theorem 2* Let  $\tilde{u}_i^h \in V_i^h$  be the nodal interpolant of  $u_i$  in  $\Omega_i$ ,  $i = 1, 2$ , and let  $\tilde{\lambda}^p$  denote the  $L_2(\Gamma)$ -projection of  $\lambda$  on  $\Lambda^p$ : from (20), we have

$$\| (u - u^h, \lambda - \lambda^p) \| \leq C \left( \| (u - \tilde{u}^h, \lambda - \tilde{\lambda}^p) \| + \|\gamma^{-1/2}[u - \tilde{u}^h]\|_{L_2(\Gamma)} + \|\lambda - \tilde{\lambda}^p\|_{-1/2, \Gamma} \right). \quad (22)$$

Then the error bound (21) follows from the approximation estimates of  $u - \tilde{u}^h$  and  $\lambda - \tilde{\lambda}^p$ .

In fact, for the terms involving  $u - \tilde{u}^h$ , the standard interpolation estimates give

$$\sum_i \|\kappa_i^{1/2} \nabla(u - \tilde{u}^h)\|_{L_2(\Omega_i)}^2 + \|a(u - \tilde{u}^h)\|_{L_2(\Omega)}^2 \leq C\mathbf{k} \sum_{K \in \mathcal{T}^h} h_K^{2\min\{k,s\}} |u|_{H^{s+1}(K)}^2, \quad (23)$$

and

$$\begin{aligned} \|\gamma^{-1/2}[u - \tilde{u}^h]\|_{L_2(\Gamma)}^2 &\leq 2\|\gamma^{-1/2}(u_1 - \tilde{u}_1^h)\|_{L_2(\Gamma)}^2 + 2\|\gamma^{-1/2}(u_2 - \tilde{u}_2^h)\|_{L_2(\Gamma)}^2 \\ &\leq 2 \sum_{K \in \mathcal{T}_1^h} \|\gamma_K^{-1/2}(u_1 - \tilde{u}_1^h)\|_{L_2(\partial K)}^2 + 2 \sum_{K \in \mathcal{T}_2^h} \|\gamma_K^{-1/2}(u_2 - \tilde{u}_2^h)\|_{L_2(\partial K)}^2 \\ &\leq C \sum_{K \in \mathcal{T}_1^h \cup \mathcal{T}_2^h} \gamma_K^{-1} h_K^{2\min\{k,s\}+1} |u|_{H^{s+1}(K)}^2. \end{aligned} \quad (24)$$

For the terms with  $\lambda - \tilde{\lambda}^p$ , since  $\|\lambda - \tilde{\lambda}^p\|_{L_2(\Gamma_j)} = \inf_{\mu^p \in \Lambda^p} \|\lambda - \mu^p\|_{L_2(\Gamma_j)}$ , from standard  $hp$ -approximation estimates we have

$$\|\lambda - \tilde{\lambda}^p\|_{L_2(\Gamma_j)} \leq C \ell_j^t p_j^{-t} |\lambda|_{H^t(\Gamma_j)}, \quad 0 \leq t \leq p_j + 1, \quad (25)$$

which immediately gives

$$\|\gamma^{1/2}(\lambda - \tilde{\lambda}^p)\|_{L_2(\Gamma)} \leq C \left( \sum_j \sup_{\mathbf{x} \in \Gamma_j} (\gamma) \ell_j^{2s-1} p_j^{-(2s-1)} |\lambda|_{H^{s-1/2}(\Gamma_j)}^2 \right)^{1/2}. \quad (26)$$

It remains to deal with the third term appearing at right-hand side of (22). We derive an estimate of  $\lambda - \tilde{\lambda}^p$  in the norm  $\llbracket \cdot \rrbracket_{-1/2, \Gamma}$  by means of interpolation theory (see, e.g., [15, Theorem 12.2]) between the norms  $\|b^{-1} \cdot\|_{L_2(\Gamma)}$  (which is the dual of  $\|b \cdot\|_{L_2(\Gamma)}$ ) and  $\llbracket \cdot \rrbracket_{-1, \Gamma}$ , where  $\llbracket \cdot \rrbracket_{-1, \Gamma}$  denotes the dual norm of  $\left( \|b \cdot\|_{L_2(\Gamma)}^2 + \text{diam}(\Omega)^{1/2} |\mathbf{k}^{1/2} \cdot|_{H_0^1(\Gamma)}^2 \right)^{1/2}$ . We have

$$\begin{aligned} \llbracket \lambda - \tilde{\lambda}^p \rrbracket_{-1, \Gamma} &\leq \sup_{\mu \in H_0^1(\Gamma)} \frac{(\lambda - \tilde{\lambda}^p, \mu)_\Gamma}{\text{diam}(\Omega)^{1/2} |\mathbf{k}^{1/2} \mu|_{H^1(\Gamma)}} \\ &= \text{diam}(\Omega)^{-1/2} \mathbf{k}^{-1/2} \sup_{\mu \in H_0^1(\Gamma)} \frac{(\lambda - \tilde{\lambda}^p, \mu - \tilde{\mu}^p)_\Gamma}{|\mu|_{H^1(\Gamma)}}, \end{aligned} \quad (27)$$

where  $\tilde{\mu}^p$  denotes the  $L_2$ -projection of  $\mu$  on  $A^p$ , and  $(\cdot, \cdot)_\Gamma$  denotes the usual inner product of  $L_2(\Gamma)$ . Furthermore,

$$\begin{aligned}
(\lambda - \tilde{\lambda}^p, \mu - \tilde{\mu}^p)_\Gamma &= \sum_j (\lambda - \tilde{\lambda}^p, \mu - \tilde{\mu}^p)_{\Gamma_j} \leq \sum_j \|\lambda - \tilde{\lambda}^p\|_{L_2(\Gamma_j)} \|\mu - \tilde{\mu}^p\|_{L_2(\Gamma_j)} \\
&= \sum_j \ell_j p_j^{-1} \|\lambda - \tilde{\lambda}^p\|_{L_2(\Gamma_j)} \ell_j^{-1} p_j \|\mu - \tilde{\mu}^p\|_{L_2(\Gamma_j)} \\
&\leq \left( \sum_j \ell_j^2 p_j^{-2} \|\lambda - \tilde{\lambda}^p\|_{L_2(\Gamma_j)}^2 \right)^{1/2} \left( \sum_j \ell_j^{-2} p_j^2 \|\mu - \tilde{\mu}^p\|_{L_2(\Gamma_j)}^2 \right)^{1/2}.
\end{aligned} \tag{28}$$

We exploit the estimate (25) with  $t = s - 1/2$  (resp.  $t = 1$ ) to bound the first (resp. the second) term at right-hand side of (28) and obtain

$$\begin{aligned}
(\lambda - \tilde{\lambda}^p, \mu - \tilde{\mu}^p)_\Gamma &\leq C \left( \sum_j \ell_j^{2s+1} p_j^{-(2s+1)} |\lambda|_{H^{s-1/2}(\Gamma_j)}^2 \right)^{1/2} \left( \sum_j |\mu|_{H^1(\Gamma_j)}^2 \right)^{1/2} \\
&\leq C \left( \sum_j \ell_j^{2s+1} p_j^{-(2s+1)} |\lambda|_{H^{s-1/2}(\Gamma_j)}^2 \right)^{1/2} |\mu|_{H^1(\Gamma)}.
\end{aligned} \tag{29}$$

From (27) and (29) we easily get

$$\|\lambda - \tilde{\lambda}^p\|_{-1,\Gamma} \leq C \text{diam}(\Omega)^{-1/2} \mathbf{k}^{-1/2} \left( \sum_j \ell_j^{2s+1} p_j^{-(2s+1)} |\lambda|_{H^{s-1/2}(\Gamma_j)}^2 \right)^{1/2}. \tag{30}$$

Moreover, from estimate (25), evaluated with  $t = s - 1/2$ , we get immediately

$$\left\| b^{-1} (\lambda - \tilde{\lambda}^p) \right\|_{L^2(\Gamma)} \leq C \text{diam}(\Omega)^{1/2} \mathbf{k}^{-1/2} \left( \sum_j \ell_j^{2s-1} p_j^{-(2s-1)} |\lambda|_{H^{s-1/2}(\Gamma_j)}^2 \right)^{1/2}. \tag{31}$$

Finally, from (30) and (31) we can use the interpolation theory between function spaces: note that the right hand sides of (30)–(31) may be viewed as seminorms on the products (with different weights) of the  $H^{s-1/2}(\Gamma_j)$  spaces, and therefore the interpolation and the summation can be interchanged. This leads to

$$\|\lambda - \tilde{\lambda}^p\|_{-1/2,\Gamma} \leq C \left( \sum_j \mathbf{k}^{-1} \ell_j^{2s} p_j^{-2s} |\lambda|_{H^{s-1/2}(\Gamma_j)}^2 \right)^{1/2}. \tag{32}$$

Inserting (23), (24), (26) and (32) in (22) gives the result.  $\square$

## 4 Numerical examples

### 4.1 Implementation issues

We now describe how to implement the method (7) with the choice (4) for two-dimensional problems. Some care has to be taken to avoid ill conditioning of the polynomial approximation on the interface. The mass matrix corresponding to the Taylor polynomial, for example, is the Vandermonde matrix which is notoriously ill conditioned. We have chosen to instead work with Legendre polynomials which are orthogonal in the  $L_2([-1, 1])$  product. Instead of programming each Legendre polynomial  $P_n(x)$  separately for  $n = 0, 1, \dots$ , we have used the summation formula

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n - 2k)!}{k! (n - k)! (n - 2k)!} x^{n-2k},$$

where  $\lfloor \cdot \rfloor$  is the floor function, and integrated (analytically, in advance) products of polynomials and (traces of) test functions, and derivatives thereof, term-wise.

As for three-dimensional problems, we remark that the orthogonality property of the Legendre polynomials in one dimension can no longer be used; a 3D implementation requires the computation of integrals of general high order (piecewise) polynomials on a regular mesh, also for the products of the multipliers. Nevertheless, for this integration problem, assuming affine trace meshes, simple closed formulas exist for each element integral, e.g., for  $K \cap \Gamma$  a triangle in a local coordinate system  $(x, y)$ ,

$$\int_{K \cap \Gamma} x^m y^n dx dy = 2 \operatorname{meas}(K \cap \Gamma) \frac{m! n!}{(2 + m + n)!},$$

if  $m, n$  are positive integers. This is in contrast to the more difficult computation of integrals of products of low order piecewise polynomials emanating from different (non matching) grids.

We discuss now possible iterative algorithms for the solution of the algebraic system arising from the symmetric method (9). Assume that bases for the discrete spaces  $V^h$  and

$\Lambda^p$  are given; then we can rewrite (9) in algebraic form:

$$\begin{aligned} A\mathbf{u}^h + B^t\boldsymbol{\lambda}^p &= \mathbf{f} \\ B\mathbf{u}^h - C\boldsymbol{\lambda}^p &= 0, \end{aligned} \tag{33}$$

where  $\mathbf{u}^h$  and  $\boldsymbol{\lambda}^p$  are the vector coefficients for  $u^h$  and  $\lambda^p$  referred to the chosen bases,  $\mathbf{f}$  is the  $V^h$  expansion of the source term, while the matrices  $A$ ,  $B$  and  $C$  are the representation of the the bilinear forms

$$\begin{aligned} (w, v) &\mapsto a(w, v) := \sum_i \int_{\Omega_i} \kappa_i \nabla w_i \cdot \nabla v_i \, dx - \int_{\Gamma} \gamma (\mathbf{n} \cdot \kappa_j \nabla w_j) (\mathbf{n} \cdot \kappa_j \nabla v_j) \, ds, \\ (v, \mu) &\mapsto b(v, \mu) := \int_{\Gamma} \mu [v] \, ds - \int_{\Gamma} \gamma \mu (\mathbf{n} \cdot \kappa_j \nabla v_j) \, ds, \end{aligned}$$

and

$$(v, \mu) \mapsto c(v, \mu) := \int_{\Gamma} \gamma v \mu \, ds,$$

respectively. Taking  $\gamma_0$  small enough, as stated in Proposition 2 and Remark 2, the bilinear form  $a(\cdot, \cdot)$  is positive semi-definite:

$$a(v^h, v^h) \geq C \sum_i |v_h|_{H^1(\Omega_i)}^2 \quad \forall v^h \in V^h.$$

If we are in case of Fig.1, left, where the boundary of each subdomain intersects  $\partial\Omega$ , then  $a(\cdot, \cdot)$  is actually positive definite, whence  $A$  is invertible, and we can easily obtain from (33) the equations for  $\boldsymbol{\lambda}^p$

$$(BA^{-1}B^t + C)\boldsymbol{\lambda}^p = BA^{-1}\mathbf{f}. \tag{34}$$

Note that the evaluation of  $(BA^{-1}B^t + C)\boldsymbol{\lambda}^p$ , as well as the evaluation of  $BA^{-1}\mathbf{f}$ , which requires the solution of a linear system for  $A$ , can be performed in parallel on the two subdomains, since  $A$  is block diagonal. Therefore an iterative algorithm for solving (34) can be carried out in parallel.

In the situation of Fig.1, right, when the boundary of one subdomain does not intersect  $\partial\Omega$ , the bilinear form  $a(\cdot, \cdot)$  is only positive semidefinite. Then we cannot proceed as in (34), i.e., we can not easily eliminate  $\mathbf{u}^h$  from (33) in parallel. To overcome this difficulty, more sophisticated iterative procedures, which are based on a suitable modification of

$a(\cdot, \cdot)$ , have been proposed in other contexts (see, e.g., [12]) and could be adapted to (33) as well. Nevertheless in this context, due to the presence of a stabilizing term (i.e., to the presence of the positive definite matrix  $C$  in (33)) we can adopt a more standard approach. Starting from (33), we can eliminate  $\boldsymbol{\lambda}^p$ , obtaining

$$(A + B^t C^{-1} B) \mathbf{u}^h = \mathbf{f}. \quad (35)$$

Note that  $C^{-1}$  is associated with the  $\gamma$ -weighted  $L^2$ -projection on each straight line  $\Gamma_j$  of the internal interface  $\Gamma$ ; therefore, it can be easily computed when an orthogonal basis for  $A^p$  is used, as above discussed. Since  $A$  is positive semidefinite,  $C$  is positive definite and  $\text{Ker}(A) \cap \text{Ker}(B) = \{0\}$ , then  $M := A + B^t C^{-1} B$  is positive definite. The splitting of  $\mathbf{u}^h$  as  $[\mathbf{u}_\Gamma^h, \mathbf{u}_{\Omega \setminus \Gamma}^h]$ , where  $\mathbf{u}_\Gamma^h$  are the coefficients associated with the degrees of freedom located at the interface  $\Gamma$ , while  $\mathbf{u}_{\Omega \setminus \Gamma}^h$  are the remaining coefficients, induces a corresponding splitting of  $M$  and  $\mathbf{f}$ :

$$\begin{aligned} M_{\Gamma, \Gamma} \mathbf{u}_\Gamma^h + M_{\Gamma, \Omega \setminus \Gamma} \mathbf{u}_{\Omega \setminus \Gamma}^h &= \mathbf{f}_\Gamma \\ M_{\Omega \setminus \Gamma, \Gamma} \mathbf{u}_\Gamma^h + M_{\Omega \setminus \Gamma, \Omega \setminus \Gamma} \mathbf{u}_{\Omega \setminus \Gamma}^h &= \mathbf{f}_{\Omega \setminus \Gamma}. \end{aligned} \quad (36)$$

Since  $a(\cdot, \cdot)$  is positive definite when its arguments vanish on  $\Gamma$ , the submatrix  $M_{\Omega \setminus \Gamma, \Omega \setminus \Gamma}$  turns out to be invertible, which allow us to eliminate the *internal* degrees of freedoms  $\mathbf{u}_{\Omega \setminus \Gamma}^h$  from (36):

$$\left( M_{\Gamma, \Gamma} - M_{\Gamma, \Omega \setminus \Gamma} M_{\Omega \setminus \Gamma, \Omega \setminus \Gamma}^{-1} M_{\Omega \setminus \Gamma, \Gamma} \right) \mathbf{u}_\Gamma^h = \mathbf{f}_\Gamma - M_{\Gamma, \Omega \setminus \Gamma} M_{\Omega \setminus \Gamma, \Omega \setminus \Gamma}^{-1} \mathbf{f}_{\Omega \setminus \Gamma}. \quad (37)$$

Therefore, we can implement an iterative solver for (37), which can be carried out in parallel due to the block diagonal structure of  $M_{\Omega \setminus \Gamma, \Omega \setminus \Gamma}$ .

We observe that the above discussion remains valid when  $\Omega$  is partitioned into more than two subdomains.

We also note that efficient iterative solvers for (34) or (37) need suitable preconditioning; this is a relevant topic which is nonetheless beyond the scope of the present work.

In the examples below, unless otherwise stated, we doubled the polynomial degree for the approximation of  $\lambda$ , for every doubling of the number of nodes on the interface,

starting with  $p_j = 1$  on the coarsest mesh. The parameter  $\gamma$  was chosen as  $\gamma|_{\Gamma_j} = h_{\min}/\sqrt{3}$  where  $h_{\min}$  denotes the smallest element size along  $\Gamma_j$ .

#### 4.2 Interior domain

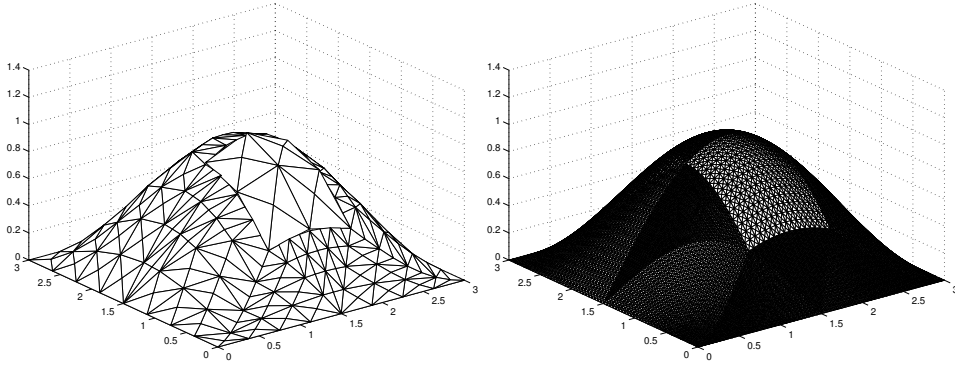
We considered the domain  $\Omega := (0, 3) \times (0, 3)$  divided into one interior domain  $\Omega_1 := (1, 2) \times (1/2, 3/2)$  and one exterior  $\Omega_2 := \Omega \setminus \Omega_1$ . We set  $\kappa_1 = \kappa_2 = 1$ ,

$$f = \frac{2\pi^2}{9} \sin(\pi x/3) \sin(\pi y/3),$$

and boundary conditions such that the analytical solution is

$$u = \sin(\pi x/3) \sin(\pi y/3).$$

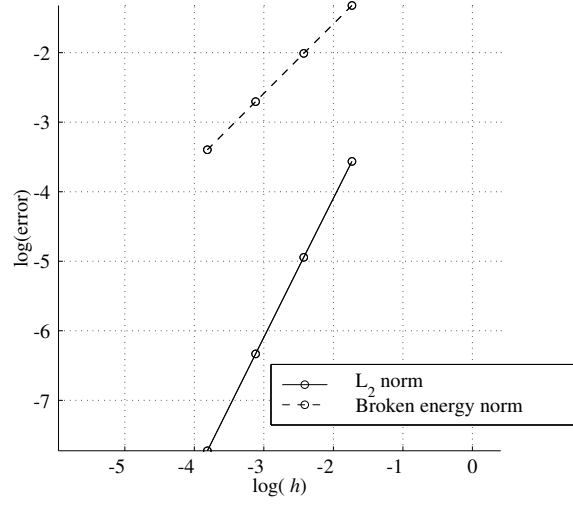
The initial and final meshes, with the elevation of the corresponding approximate solution, are shown in Fig. 2, and in Fig. 3 (see also Table 1) we show the corresponding convergence behavior, second order convergence in  $L_2$ -norm and first order convergence in broken energy norm.



**Fig. 2.** First and last mesh in the sequence corresponding to Fig. 3.

In contrast, the inconsistent method (6) gives optimal convergence in broken energy norm but not in  $L_2$ -norm, see Fig. 4 and Table 2.

We remark that, in more general cases (e.g., when  $\kappa_1 \neq \kappa_2$  and  $\Gamma$  is only Lipschitz continuous), the model problem (1) lacks of elliptic regularity. Typically, full elliptic regularity can only be expected if  $\Gamma$  is sufficiently smooth, cf. [9]. Therefore, in general, we cannot expect optimal convergence in  $L_2$ -norm even with the consistent method.



**Fig. 3.** Convergence in  $L_2$  and in broken energy norm.

| $h$    | Convergence                      |      |                            |      |
|--------|----------------------------------|------|----------------------------|------|
|        | $L_2 - \text{error} \times 10^4$ | Rate | Energy error $\times 10^2$ | Rate |
| 0.177  | 283.                             | –    | 27.4                       | –    |
| 0.0884 | 71.1                             | 1.99 | 13.4                       | 1.03 |
| 0.0442 | 17.8                             | 2.00 | 6.70                       | 1.00 |
| 0.0221 | 4.41                             | 2.01 | 3.35                       | 1.00 |

**Table 1.** Numbers underlying Fig. 3. The rates are based on the current and preceding values.

| $h$    | Convergence                      |      |                            |      |
|--------|----------------------------------|------|----------------------------|------|
|        | $L_2 - \text{error} \times 10^4$ | Rate | Energy error $\times 10^2$ | Rate |
| 0.177  | 255.                             | –    | 27.4                       | –    |
| 0.0884 | 61.5                             | 2.05 | 13.4                       | 1.03 |
| 0.0442 | 19.8                             | 1.63 | 6.72                       | 1.00 |
| 0.0221 | 9.46                             | 1.07 | 3.36                       | 1.00 |

**Table 2.** Numbers underlying Fig. 4.

Finally, we give an example showing that the polynomial degree of the multiplier can grow much slower than the number of degrees of freedom on the interface without loss of convergence. In Fig. 5 (see also Table 3), we give the convergence resulting from letting  $p_j = |\log h|$ , in accordance with the discussion after Theorem 2, which gives  $p_j = 4$  on

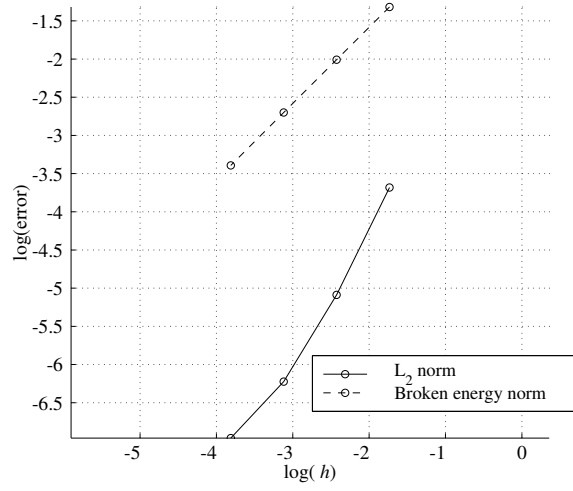


Fig. 4. Convergence in  $L_2$  and in broken energy norm for the inconsistent method.

the last mesh in the sequence, much less than the number of nodes on each segment in the final mesh (cf. Fig. 3). Nevertheless, comparable errors are obtained.

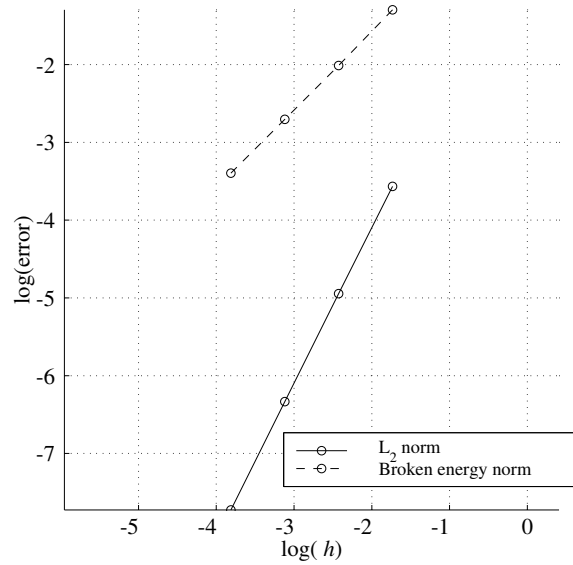


Fig. 5. Convergence in  $L_2$  and in broken energy norm using  $p_j = |\log h|$ .

#### 4.3 The case $\kappa_1 \neq \kappa_2$

Consider solutions to the ordinary differential equation

$$-\sum_i \frac{d}{dx} \left( \kappa_i \frac{du_i}{dx} \right) = 1; [u(1/2)] = 0; \kappa_1 \frac{du_1}{dx}(1/2) = \kappa_2 \frac{du_2}{dx}(1/2).$$

| $h$    | Convergence                 |      |                            |      |
|--------|-----------------------------|------|----------------------------|------|
|        | $L_2$ - error $\times 10^4$ | Rate | Energy error $\times 10^2$ | Rate |
| 0.177  | 283.                        | –    | 27.4                       | –    |
| 0.0884 | 71.1                        | 1.99 | 13.4                       | 1.03 |
| 0.0442 | 17.8                        | 2.00 | 6.70                       | 1.00 |
| 0.0221 | 4.41                        | 2.01 | 3.35                       | 1.00 |

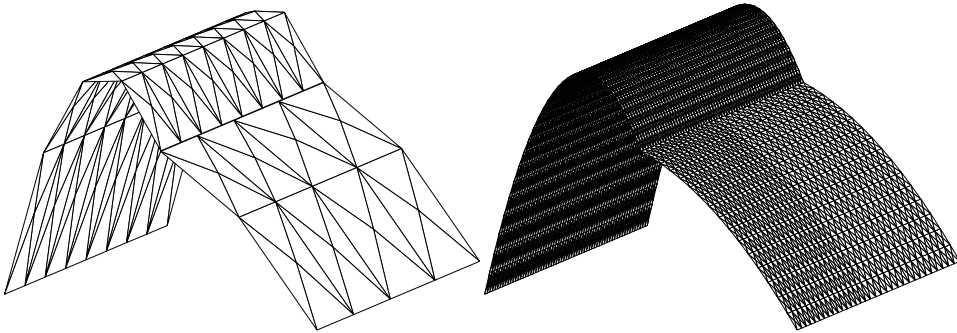
**Table 3.** Numbers underlying Fig. 5.

The domain is  $(0, 1)$ , with an interface at  $x = 1/2$ . While this is a one-dimensional problem, we solved it numerically in 2D on the domain  $(0, 1) \times (0, 1)$ , with zero Neumann boundary conditions at  $y = 0$  and  $y = 1$ . The equation has a closed-form solution that, for homogeneous Dirichlet boundary conditions at  $x = 0$  and  $x = 1$ , is given by

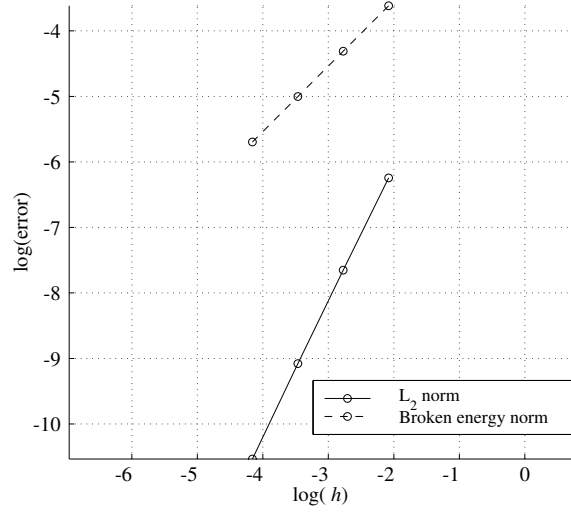
$$u_1(x) = \frac{(3\kappa_1 + \kappa_2)x}{4\kappa_1^2 + 4\kappa_1\kappa_2} - \frac{x^2}{2\kappa_1}, \quad u_2(x) = \frac{\kappa_2 - \kappa_1 + (3\kappa_1 + \kappa_2)x}{4\kappa_2^2 + 4\kappa_1\kappa_2} - \frac{x^2}{2\kappa_2}.$$

We chose  $\kappa_1 = 1/2$ ,  $\kappa_2 = 3$ .

The initial and final meshes, with the elevation of the corresponding approximate solution, are shown in Fig. 6, and in Fig. 7 we show the corresponding convergence behavior, again with second order convergence in  $L_2$ -norm and first order convergence in broken energy norm (see also Table 4).



**Fig. 6.** First and last mesh in the sequence corresponding to Fig. 7.



**Fig. 7.** Convergence in  $L_2$  and in broken energy norm.

| $h$      | Convergence                      |      |                            |      |
|----------|----------------------------------|------|----------------------------|------|
|          | $L_2 - \text{error} \times 10^5$ | Rate | Energy error $\times 10^3$ | Rate |
| 0.125    | 194.                             | –    | 26.8                       | –    |
| 0.0625   | 47.5                             | 2.03 | 13.4                       | 1.00 |
| 0.03125  | 11.4                             | 2.06 | 6.72                       | 1.00 |
| 0.015625 | 2.66                             | 2.10 | 3.36                       | 1.00 |

**Table 4.** Numbers underlying Fig. 7.

#### 4.4 Non-constant conductivity

In the last numerical example, we use the same domains as in Example 4.2, but with quadratic approximation, nonconstant conductivity,

$$\kappa = (1 + x)^2(2 + \sin(1 + y^2)),$$

and also with the symmetric variant of the method (choosing the normal fluxes from the exterior domain). We used a source term

$$f = (2\pi(1+x)(-3(1+x)y \cos(\pi y/3) \cos(1+y^2) \sin(\pi x/3) - 3 \cos(\pi x/3) \sin(\pi y/3)(2 + \sin(1+y^2)) + \pi(1+x) \sin(\pi x/3) \sin(\pi y/3)(2 + \sin(1+y^2))))/9,$$

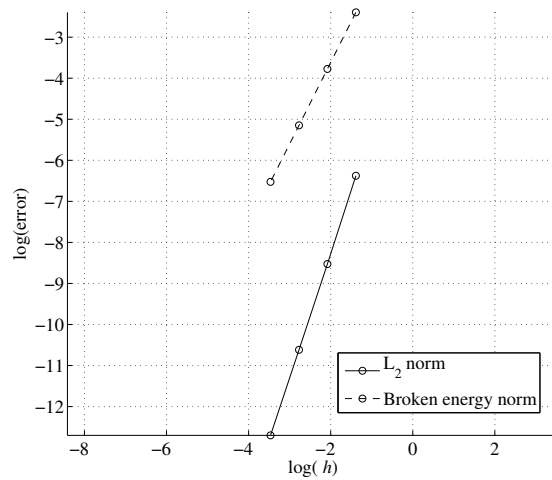
corresponding to the exact solution

$$u = \sin(\pi x/3) \sin(\pi y/3).$$

| $h$     | Convergence                 |      |                            |      |
|---------|-----------------------------|------|----------------------------|------|
|         | $L_2$ - error $\times 10^6$ | Rate | Energy error $\times 10^3$ | Rate |
| 0.25    | 1700.                       | –    | 91.2                       | –    |
| 0.125   | 198.                        | 3.10 | 22.9                       | 1.99 |
| 0.0625  | 24.5                        | 3.01 | 5.82                       | 1.98 |
| 0.03125 | 3.05                        | 3.00 | 1.47                       | 1.99 |

**Table 5.** Numbers underlying Fig. 8.

Again, we used  $p_j = |\log h|$ , and for this example we also used high order quadrature instead of exact integration. In Fig. 8 (see also Table 5) we give the obtained, and expected, convergence behavior: third order convergence in  $L_2$ -norm and second order convergence in broken energy norm.



**Fig. 8.** Convergence in  $L_2$  and in broken energy norm.

## 5 Concluding remarks

We have introduced and analyzed a new stabilized Lagrange multiplier method for interface problems. The basic idea in this method is to avoid integrating products of piecewise function from the two trace meshes, and instead use a global polynomial for the multiplier. It should be pointed out that the stability of our method is not related to the globalness

of the polynomial; if a global polynomial is not feasible, some other simple approximation on the interface could be chosen (though we have not analyzed the convergence of other choices). A typical example could be a multiplier space consisting of piecewise constant multipliers on a structured Cartesian grid. The main point is avoiding  $L_2$ -projections between unrelated and unstructured meshes.

Future work will focus on elastic contact problems where, typically, discrete Lagrange multipliers in the nodes of the trace mesh(es) are used. With a distributed multiplier approach, such as the one suggested here, the need for carefully selecting a master surface (from the point of view of the discretization of the surfaces) will be alleviated since the multiplier space will not be tied directly to the discretization of the contact surfaces.

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