

A robust a posteriori estimator for the residual-free bubbles method applied to advection-diffusion problems

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We develop the a posteriori error analysis for the RFB method, applied to the linear advection-diffusion problem: the numerical error, measured in suitable norms, is estimated in terms of the numerical residual. The robustness is investigated, in the sense that we prove uniform equivalence between a norm of the numerical residual and a particular norm of the error.

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1 Introduction

We consider the linear advection-diffusion operator

$$\mathcal{L} := -\varepsilon\Delta + \mathbf{c} \cdot \nabla, \quad (1)$$

and the related p.d.e. problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where the unknown u is a real function defined on a bounded polygonal domain $\Omega \in \mathbb{R}^N$ and $f \in L^2(\Omega)$ is the source term. We take the diffusion coefficient ε strictly positive and, for the moment, the advection velocity \mathbf{c} in \mathbb{R}^N (we shall generalize this last hypothesis in the following).

It is well known that standard numerical methods (like central finite difference or standard Galerkin finite element methods) are not adequate when the quantity $\varepsilon/|\mathbf{c}|$ is small compared to the discretization step size, since the numerical solutions exhibit unphysical oscillatory behavior.

The *residual-free bubbles* (RFB) approach for problem (2), proposed by Brezzi and Russo in [5], is in some sense different from the usual stabilized methods such as artificial diffusion or SUPG (see [6]). Following the abstract point of view of [3], we write the Galerkin variational formulation for (2) without a stabilizing term

$$\begin{cases} \text{Find } u_{\text{RFB}} \in V_h \text{ such that} \\ a(u_{\text{RFB}}, v) = (f, v) \quad \forall v \in V_h, \end{cases} \quad (3)$$

where (\cdot, \cdot) denotes the usual $L^2(\Omega)$ scalar product and

$$a(w, v) := \varepsilon(\nabla w, \nabla v) + (\mathbf{c} \cdot \nabla w, v);$$

actually the RFB method is founded only on an unusual approximating space V_h : let $\{\mathcal{T}_h\}_h$ be a family of partitions of Ω into open N -simplices (termed as *elements*) and k be a positive integer; we set:

$$V_h \equiv V_h^k(\mathcal{T}_h, \Omega) := \left\{ v \in H_0^1(\Omega) : v \text{ is polynomial of degree } \leq k \right. \\ \left. \text{on each } (N-1)\text{-face of elements in } \mathcal{T}_h \right\}. \quad (4)$$

The formulation (3)–(4) cannot be used in order to obtain a numerical scheme, because the space V_h is of infinite dimension. In fact the final algorithm does not compute the whole solution. Let us suppose, for the moment, that $k \leq 2$; then we can split V_h into the direct sum of the two subspaces W_h and B_h ,

$$V_h = W_h \oplus B_h, \quad (5)$$

where W_h is the usual finite element space

$$W_h \equiv W_h^k(\mathcal{T}_h, \Omega) := \left\{ \begin{array}{l} v \in H_0^1(\Omega) : v \text{ is polynomial} \\ \text{of degree } \leq k \text{ on each element in } \mathcal{T}_h \end{array} \right\}, \quad (6)$$

and B_h contains generic functions confined to the interior of the elements

$$B_h := \bigoplus_{T \in \mathcal{T}_h} H_0^1(T).$$

Consequently the solution u_{RFB} of (3) admits a unique decomposition into the sum of the k -order piecewise polynomial part $u_k \in W_h$ and the *bubble* part $u_b \in B_h$

$$u_{\text{RFB}} = u_k \oplus u_b. \tag{7}$$

Note that u_k is the usual nodal interpolant of u_{RFB} , and we may think of u_k as representing the global part of the RFB approximation, while u_b reflects the local behavior which can be neglected without a significant loss of accuracy. In fact the numerical procedure based on the RFB approximation computes the u_k part only. Observe that using the splitting (5) into (3) we get

$$a(u_k, v_k) + a(u_b, v_k) = (f, v_k) \quad \forall v_k \in W_h, \tag{8}$$

$$a(u_k, v_b) + a(u_b, v_b) = (f, v_b) \quad \forall v_b \in B_h. \tag{9}$$

Then (8) is the problem we solve on a computer, once the local effect represented in (8) by $a(u_b, v_k)$ is written as a function of u_k and f . This last step is made using (9), which is equivalent in each element $T \in \mathcal{T}_h$ to

$$\begin{cases} \mathcal{L}u_b = f - \mathcal{L}u_k & \text{in } T \\ u_b = 0 & \text{on } \partial T. \end{cases} \tag{10}$$

We leave the details of the implementation and refer the interested reader to [5], [2] and [11] for the case $k = 1$, [14] and [16] for the case $k = 2$. The general framework is presented in [3].

Brezzi et al. proved in [4] an a priori error estimate for the u_k part when $k = 1$. Then in [3] Brezzi, Marini and Süli derived the a priori error estimate for u_{RFB} , using quite an original technique. This error analysis was extended and localized in [15].

The aim of this paper is the a posteriori error analysis of the RFB method. We derive error bounds on the L^2 -norm and the *energy* norm of the error, and on the H^{-1} -norm of the error in the advective flux in terms of the numerical residual. These kinds of estimates are closely related to the construction of an adaptive procedure, which allows to achieve a desired accuracy with an iterated refinement (or de-refinement) of the mesh. With this in mind and since we can compute numerically only the u_k part of the RFB solution, we have intentionally dealt with the associated error $u - u_k$ and its residual (i.e. $f - \mathcal{L}u_k$ on the interior of the elements and the jump of the diffusive flux of u_k across the boundary of elements). An aspect of our analysis is the introduction of a *robust* estimator, in the sense that the a posteriori estimator is equivalent to a particular norm of the error (a proper combination of the terms mentioned before). From the practical point of view this means that one could use this estimator to construct an optimal mesh in order to reach the prescribed accuracy with respect to that particular norm; from the

theoretical point of view this last result provides an insight into the structure of the estimator. This analysis will be extended in a forthcoming paper ([17]) to different stabilizing techniques, such as SUPG.

Our proofs make use of the duality technique introduced in the context of advective-diffusive problems by Johnson (see e.g. [10]). Among the more recent works on this subject the ones of Süli, Rannacher and Houston (in particular [12]) and Verfürth (see [20]) have inspired the present paper.

This is, as far as we know, the first attempt to develop the a posteriori error analysis of the RFB method. In Section 2 we introduce the notations and rewrite (8) in a form that is more convenient from the point of our analysis in the case of $k > 2$. The fundamental tools are the approximation results presented in Section 3. In Section 4 we prove the a posteriori error estimates and the optimality result mentioned above. Finally, in the Appendix, we prove some estimates we need in this work.

2 The RFB discretization

Given an open subset ω of Ω , we denote by $L^2(\omega) = H^0(\omega)$, $H^1(\omega), \dots, H^m(\omega)$, $m \in \mathbb{N}$ the usual Sobolev spaces equipped with the standard norms $\|\cdot\|_{L^2(\omega)}$, $\|\cdot\|_{H^m(\omega)}$ and seminorms $|\cdot|_{H^m(\omega)}$; $H_0^1(\omega)$ is the space of functions contained in $H^1(\omega)$ with zero trace on $\partial\omega$, equipped with the norm $|\cdot|_{H^1(\omega)}$; finally $H^{-1}(\omega)$ denotes the dual space of $H_0^1(\omega)$ equipped with the dual norm $\|\cdot\|_{H^{-1}(\omega)}$. The standard notation $(\cdot, \cdot)_\omega$ is used for the scalar product in $L^2(\omega)$ and we simply denote $(\cdot, \cdot)_\Omega$ by (\cdot, \cdot) . We follow similar notations when the domain is an $(N - 1)$ -dimensional piecewise regular manifold: for example $L^2(\partial\omega)$ is the space of square integrable functions defined on the boundary of ω , equipped with the scalar product $(\cdot, \cdot)_{\partial\omega}$ and the related norm $\|\cdot\|_{L^2(\partial\omega)}$. See [7] for further details.

In the sequel C will denote a generic strictly positive constant whose value, possibly different at any occurrence, does not depend on ε , f , h or the parameter δ appearing in the Appendix: in fact we allow C to depend only on the advection field \mathbf{c} . Moreover we adopt the notational convention

$$\begin{aligned} a \preceq b &\iff a \leq Cb, \\ a \simeq b &\iff a \preceq b \text{ and } b \preceq a. \end{aligned}$$

The family of partitions $\{\mathcal{T}_h\}_h$ must satisfy the conditions:

1. *admissibility*: any two elements either have disjoint closure or share a complete n -face, with $1 \leq n \leq N - 1$,
2. *shape regularity*: $\forall T \in \mathcal{T}_h$

$$\frac{h_T}{\rho_T} \preceq 1,$$

where h_T and ρ_T are the diameter of T and the diameter of the largest ball inscribed in T .

We now consider the differential operator \mathcal{L} defined in (1). This is a singularly perturbed operator with respect to ε . We assume that the advection field $\mathbf{c} : \Omega \mapsto \mathbb{R}^N$ obeys the following regularity conditions:

$$\begin{cases} \exists \eta : \Omega \mapsto \mathbb{R} \text{ smooth:} \\ \mathbf{c} \cdot \nabla \eta \geq 2 & \text{in } \Omega, \end{cases} \quad (11)$$

and

$$\operatorname{div}(\mathbf{c}) = 0 \quad \text{in } \Omega. \quad (12)$$

Note that (11) is actually equivalent to the hypothesis that any characteristic line of \mathbf{c} leaves the domain Ω in finite time (see [9] and [13] for details about characteristics and their geometric properties). By (12) the formal adjoint of \mathcal{L} is

$$\mathcal{L}^* = -\varepsilon \Delta - \mathbf{c} \cdot \nabla.$$

We assume moreover that f and each component of \mathbf{c} are piecewise polynomial (of bounded order) on each \mathcal{T}_h .

In the sequel we denote by $[\nabla u_{\text{RFB}} \cdot \mathbf{n}]$ the *jump* of the normal derivative of u_{RFB} across the internal boundary of elements in the normal direction: given a pair of elements T_- and T_+ having the $(N-1)$ -face $E := \partial T_- \cap \partial T_+$ in common, and denoting by \mathbf{n} the unit normal vector defined on E and directed from T_- to T_+ , we set

$$[\nabla u_{\text{RFB}} \cdot \mathbf{n}] := ((\nabla u_{\text{RFB}} \cdot \mathbf{n})_+ - (\nabla u_{\text{RFB}} \cdot \mathbf{n})_-) \quad \text{on } E, \quad (13)$$

where $(\cdot)_-$ and $(\cdot)_+$ denote the trace operators on E defined for functions on T_- and T_+ , respectively. Note that definition (13) does not depend on the order of T_- and T_+ .

Our aim is to develop the a posteriori analysis for a generic, but fixed, order k . In the case $k > 2$ there are more difficulties in the derivation of a scheme for u_k since the way sketched in the introduction is no longer effective. Actually the definition of u_k itself poses some difficulties since W_h , the space containing the part we compute in practice, and B_h , the space containing the local details we want to discharge, have non empty intersection: the reason is that, when $k > 2$, the functions in W_h have degrees of freedom located in the interior of elements. In general, instead of (5), we have

$$V_h = W_h + B_h,$$

and, in order to obtain a numerical procedure to compute a finite-dimensional part of u_{RFB} belonging to W_h , we have to choose a criterion to make the decomposition unique. This involves a reformulation of (8)–(10): it is not a trivial task at all, and at the moment we do not know any result on the subject.

Fortunately these difficulties do not affect our analysis. Remember that, from (2) and (3) we infer the Galerkin orthogonality property

$$a(u - u_{\text{RFB}}, v) = 0 \quad \forall v \in V_h. \quad (14)$$

Now suppose we write $u_{\text{RFB}} = u_k + u_b$ in any possible way, with $u_k \in W_h$ and $u_b \in B_h$; then we have from (14), integrating by parts element-wise,

$$a(u, v) - a(u_k, v) - \sum_{T \in \mathcal{T}_h} (u_b, \mathcal{L}^* v)_T = 0 \quad \forall v \in V_h; \quad (15)$$

note that the boundary terms arising in the integration by parts are equal to zero because u_b has zero trace on each ∂T ; in order to make zero the third term appearing in (15) we consider test functions belonging to

$$\widetilde{W}_h \equiv \widetilde{W}_h^k(\mathcal{T}_h, \Omega) := \{v \in V_h : \mathcal{L}^* v = 0 \text{ in each element } T \in \mathcal{T}_h\}. \quad (16)$$

The following proposition summarizes our observations.

Proposition 1. *Let u and u_{RFB} be solutions of (2) and (3) respectively; let $u_k \in W_h$ with $u_{\text{RFB}} - u_k \in B_h$. Then*

$$a(u - u_k, v) = 0 \quad \forall v \in \widetilde{W}_h. \quad (17)$$

This is all we need in the following error analysis: it does not matter how u_k is defined, provided that the difference $u_k - u_{\text{RFB}}$ belongs to B_h . This is not a paradox: if we have a bad procedure to manage the degree of freedom of u_k in the interior of the elements we will get a large error $u - u_k$, and the numerical residual will be large too.

3 Approximation in W_h and \widetilde{W}_h

In this section we introduce the operators Π_h and $\widetilde{\Pi}_h$, defined on $H_0^1(\Omega)$ with values in W_h and \widetilde{W}_h respectively. The first one is the usual quasi-interpolation operator (see [8] or [19]): for any function $v \in H_0^1(\Omega)$ and for any element $T \in \mathcal{T}_h$, denoting by $\mathcal{N}(T)$ the closure of the union of elements in \mathcal{T}_h having at least a vertex in common with T , we have

$$\|v - \Pi_h v\|_{L^2(T)} \preceq \|v\|_{L^2(\mathcal{N}(T))}, \quad (18)$$

$$|v - \Pi_h v|_{H^1(T)} \preceq |v|_{H^1(\mathcal{N}(T))}, \quad (19)$$

$$\|v - \Pi_h v\|_{L^2(T)} \preceq h_T^m |v|_{H^m(\mathcal{N}(T))} \quad 1 \leq m \leq k+1, \quad (20)$$

$$\|v - \Pi_h v\|_{L^2(\partial T)} \preceq h_T^{m-1/2} |v|_{H^m(\mathcal{N}(T))} \quad 1 \leq m \leq k+1. \quad (21)$$

Moreover, using a standard interpolation inequality (see for example [20]) and (18)–(20), we get the estimate

$$\begin{aligned} \|v - \Pi_h v\|_{L^2(\partial T)}^2 &\preceq \|v - \Pi_h v\|_{L^2(T)} |v - \Pi_h v|_{H^1(T)} \\ &\quad + h_T^{-1} \|v - \Pi_h v\|_{L^2(T)}^2 \\ &\preceq \|v\|_{L^2(\mathcal{N}(T))} |v|_{H^1(\mathcal{N}(T))}. \end{aligned} \quad (22)$$

Then we define $\widetilde{\Pi}_h v$ as the unique function in \widetilde{W}_h which coincides with $\Pi_h v$ on the element boundaries ∂T , $\forall T \in \mathcal{T}_h$; the next proposition states the approximation estimates we need in the sequel.

Proposition 2. *Let T be an element of \mathcal{T}_h and let v be a function defined on Ω . Then*

$$\|v - \tilde{\Pi}_h v\|_{L^2(T)} \preceq \|v\|_{L^2(\mathcal{N}(T))}, \quad (23)$$

$$\|v - \tilde{\Pi}_h v\|_{L^2(T)} \preceq h_T |v|_{H^1(\mathcal{N}(T))}, \quad (24)$$

$$\|v - \tilde{\Pi}_h v\|_{L^2(T)} \preceq h_T^2 |v|_{H^2(\mathcal{N}(T))} + \varepsilon^{-1} h_T^2 |v|_{H^1(\mathcal{N}(T))}, \quad (25)$$

$$\|v - \tilde{\Pi}_h v\|_{L^2(\partial T)} \preceq \|v\|_{L^2(\mathcal{N}(T))} |v|_{H^1(\mathcal{N}(T))}, \quad (26)$$

$$\|v - \tilde{\Pi}_h v\|_{L^2(\partial T)} \preceq h_T^{m-1/2} |v|_{H^m(\mathcal{N}(T))} \quad 1 \leq m \leq k+1. \quad (27)$$

Proof. The last two estimates (26) and (27) are obvious repetitions of (21) and (22).

Using the triangle inequality we split

$$\begin{aligned} \|v - \tilde{\Pi}_h v\|_{L^2(T)} &\leq \|v - \Pi_h v\|_{L^2(T)} + \|\Pi_h v - \tilde{\Pi}_h v\|_{L^2(T)} \\ &\leq I + II. \end{aligned}$$

Observe, from (18) and (20), that the term I agrees with our estimates (23)-(25). It remains to bound II . For this purpose we use the stability estimates proved in the Appendix. Note that in each $T \in \mathcal{T}_h$

$$\mathcal{L}^*(\Pi_h v - \tilde{\Pi}_h v) = \mathcal{L}^*(\Pi_h v).$$

From (55) applied in the element T we have

$$\max \{h_T^{-1}, \varepsilon h_T^{-2}\} \cdot II \preceq \|\mathcal{L}^*(\Pi_h v)\|_{L^2(T)}. \quad (28)$$

Using inverse inequalities and (18) we get

$$\begin{aligned} \|\mathcal{L}^*(\Pi_h v)\|_{L^2(T)} &\preceq \varepsilon |\Pi_h v|_{H^2(T)} + |\Pi_h v|_{H^1(T)} \\ &\preceq (\varepsilon h_T^{-2} + h_T^{-1}) \|\Pi_h v\|_{L^2(T)} \\ &\preceq \max \{h_T^{-1}, \varepsilon h_T^{-2}\} \|v\|_{L^2(\mathcal{N}(T))}, \end{aligned}$$

then, substituting in (28), we get

$$II \preceq \|v\|_{L^2(\mathcal{N}(T))},$$

and (23) is proved. In the same way we have, using (19)

$$\|\mathcal{L}^*(\Pi_h v)\|_{L^2(T)} \preceq h_T \max \{h_T^{-1}, \varepsilon h_T^{-2}\} |v|_{H^1(\mathcal{N}(T))},$$

and so

$$II \preceq h_T |v|_{H^1(\mathcal{N}(T))},$$

which proves (24). Finally

$$\|\mathcal{L}^*(\Pi_h v)\|_{L^2(T)} \preceq \varepsilon |\Pi_h v|_{H^2(T)} + |\Pi_h v|_{H^1(T)},$$

hence

$$II \preceq h_T^2 |v|_{H^2(\mathcal{N}(T))} + \varepsilon^{-1} h_T^2 |v|_{H^1(\mathcal{N}(T))},$$

so (25) follows. \square

4 A posteriori error estimate

The following theorems contain our fundamental error estimates.

Theorem 1. *Assume (11) and (12) hold true; let u be the solution of (2) and let u_k be as in Proposition 1. Set*

$$\alpha_T := \min\{1, \varepsilon^{-1/2} h_T\}.$$

Then

$$\begin{aligned} \|u - u_k\|_{L^2(\Omega)}^2 &\preceq \sum_{T \in \mathcal{T}_h} \left(\alpha_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right. \\ &\quad \left. + \varepsilon^{-1/2} \alpha_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right). \end{aligned} \quad (29)$$

Proof. Let ϕ be an arbitrary function in $L^2(\Omega)$, with $\|\phi\|_{L^2(\Omega)} = 1$. Let z be the solution of the dual b.v.p.

$$\begin{cases} \mathcal{L}^* z = \phi & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Integration by parts on Ω and (17) yield

$$\begin{aligned} (u - u_k, \phi) &= (u - u_k, \mathcal{L}^* z) \\ &= a(u - u_k, z) \\ &= a(u - u_k, z - \tilde{\Pi}_h z). \end{aligned} \quad (30)$$

One more integration by parts, this time element-wise, gives

$$\begin{aligned} a(u - u_k, z - \tilde{\Pi}_h z) &= \sum_{T \in \mathcal{T}_h} (f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k, z - \tilde{\Pi}_h z)_T \\ &\quad + \frac{1}{2} \sum_{T \in \mathcal{T}_h} (\varepsilon [\nabla u_k \cdot \mathbf{n}], z - \tilde{\Pi}_h z)_{\partial T \cap \Omega} \\ &= I + II. \end{aligned} \quad (31)$$

Using the Cauchy-Schwarz inequality twice we obtain

$$\begin{aligned} I &\leq \sum_{T \in \mathcal{T}_h} \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)} \|z - \tilde{\Pi}_h z\|_{L^2(T)} \\ &\leq \left(\sum_{T \in \mathcal{T}_h} \alpha_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right)^{1/2} \\ &\quad \cdot \left(\sum_{T \in \mathcal{T}_h} \alpha_T^{-2} \|z - \tilde{\Pi}_h z\|_{L^2(T)}^2 \right)^{1/2}. \end{aligned} \quad (32)$$

Using now estimates (23), (24), (55) and the shape regularity of $\{\mathcal{T}_h\}_h$ we get

$$\begin{aligned}
& \left(\sum_{T \in \mathcal{T}_h} \alpha_T^{-2} \|z - \tilde{\Pi}_h z\|_{L^2(T)}^2 \right)^{1/2} \\
& \preceq \left(\sum_{T \in \mathcal{T}_h} \varepsilon |z|_{H^1(\mathcal{N}(T))}^2 + \|z\|_{L^2(\mathcal{N}(T))}^2 \right)^{1/2} \\
& \preceq \varepsilon^{1/2} |z|_{H^1(\Omega)} + \|z\|_{L^2(\Omega)} \\
& \preceq \|\phi\|_{L^2(\Omega)} \\
& = 1.
\end{aligned} \tag{33}$$

We proceed in a similar way for II ; from the Cauchy-Schwarz inequality

$$\begin{aligned}
II & \leq \sum_{T \in \mathcal{T}_h} \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)} \|z - \tilde{\Pi}_h z\|_{L^2(\partial T \cap \Omega)} \\
& \leq \left(\sum_{T \in \mathcal{T}_h} \varepsilon^{-1/2} \alpha_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2} \\
& \quad \cdot \left(\sum_{T \in \mathcal{T}_h} \varepsilon^{1/2} \alpha_T^{-1} \|z - \tilde{\Pi}_h z\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2},
\end{aligned} \tag{34}$$

and from (26)–(27) and (55)

$$\begin{aligned}
& \left(\sum_{T \in \mathcal{T}_h} \varepsilon^{1/2} \alpha_T^{-1} \|z - \tilde{\Pi}_h z\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2} \\
& \preceq \left(\sum_{T \in \mathcal{T}_h} \varepsilon |z|_{H^1(\mathcal{N}(T))}^2 + \varepsilon^{1/2} |z|_{H^1(\mathcal{N}(T))} \|z\|_{L^2(\mathcal{N}(T))} \right)^{1/2} \\
& \preceq \varepsilon^{1/2} |z|_{H^1(\Omega)} + \|z\|_{L^2(\Omega)} \\
& \preceq \|\phi\|_{L^2(\Omega)} \\
& = 1.
\end{aligned} \tag{35}$$

In conclusion (29) follows from (30)–(35), by the arbitrariness of ϕ . \square

Theorem 2. *With the same hypotheses and notations of Theorem 1 we have*

$$\begin{aligned}
\varepsilon |u - u_k|_{H^1(\Omega)}^2 & \preceq \sum_{T \in \mathcal{T}_h} \left(\alpha_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right. \\
& \quad \left. + \varepsilon^{-1/2} \alpha_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right).
\end{aligned} \tag{36}$$

Proof. Integrating by parts,

$$\varepsilon^{1/2} |u - u_k|_{H^1(\Omega)} = \sup(u - u_k, -\varepsilon \Delta \phi),$$

where the sup is taken on the set of all smooth functions ϕ which are null on the boundary and verify $\varepsilon^{1/2}|\phi|_{H^1(\Omega)} = 1$. Given an arbitrary ϕ satisfying these conditions, we define z as the solution of

$$\begin{cases} \mathcal{L}^* z = -\varepsilon \Delta \phi & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof is the same of Theorem 1, with the only difference that now we have, using (68),

$$\varepsilon^{1/2}|z|_{H^1(\Omega)} + \|z\|_{L^2(\Omega)} \preceq \varepsilon^{1/2}|\phi|_{H^1(\Omega)} = 1.$$

Indeed, following (31)-(35), we infer

$$\begin{aligned} (u - u_k, -\varepsilon \Delta \phi) &= a(u - u_k, z - \tilde{\Pi}_h z) \\ &= \sum_{T \in \mathcal{T}_h} (f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k, z - \tilde{\Pi}_h z)_T \\ &\quad + \frac{1}{2} \sum_{T \in \mathcal{T}_h} (\varepsilon [\nabla u_k \cdot \mathbf{n}], z - \tilde{\Pi}_h z)_{\partial T \cap \Omega} \\ &\preceq \left(\sum_{T \in \mathcal{T}_h} \alpha_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right)^{1/2} \\ &\quad + \left(\sum_{T \in \mathcal{T}_h} \varepsilon^{-1/2} \alpha_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2}; \end{aligned} \tag{37}$$

then (36) follows, by the arbitrariness of ϕ . □

Theorem 3. *Assume the hypotheses of Theorem 1 and set*

$$\beta_T := \min\{h_T, \varepsilon^{-1}h_T^2\}.$$

Then

$$\begin{aligned} \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)}^2 &\preceq \sum_{T \in \mathcal{T}_h} \left(\beta_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right. \\ &\quad \left. + h_T^{-1} \beta_T^2 \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right). \end{aligned} \tag{38}$$

Proof. In this case we have

$$\|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)} = \sup(u - u_k, -\mathbf{c} \cdot \nabla \phi),$$

the sup being computed over all functions $\phi \in H_0^1(\Omega)$ with $|\phi|_{H^1(\Omega)} = 1$. Given such a function ϕ , we define now z as the solution of

$$\begin{cases} \mathcal{L}^* z = -\mathbf{c} \cdot \nabla \phi & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

and, as in (30)-(31), we obtain

$$\begin{aligned}
(u - u_k, -\mathbf{c} \cdot \nabla \phi) &= \sum_{T \in \mathcal{T}_h} (f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k, z - \tilde{\Pi}_h z)_T \\
&\quad + \frac{1}{2} \sum_{T \in \mathcal{T}_h} (\varepsilon [\nabla u_k \cdot \mathbf{n}], z - \tilde{\Pi}_h z)_{\partial T} \\
&= I + II.
\end{aligned} \tag{39}$$

Now z verifies the stronger estimate (75):

$$\varepsilon |z|_{H^2(\Omega)} + |z|_{H^1(\Omega)} \preceq |\phi|_{H^1(\Omega)} = 1;$$

so we obtain, using (24) and (25),

$$\begin{aligned}
I &\leq \left(\sum_{T \in \mathcal{T}_h} \beta_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} \beta_T^{-2} \|z - \tilde{\Pi}_h z\|_{L^2(T)}^2 \right)^{1/2} \\
&\leq \left(\sum_{T \in \mathcal{T}_h} \beta_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right)^{1/2},
\end{aligned}$$

and similarly, by (27)

$$\begin{aligned}
II &\leq \left(\sum_{T \in \mathcal{T}_h} h^{-1/2} \beta_T^2 \|\varepsilon \nabla u_k \cdot \mathbf{n}\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} h^{1/2} \beta_T^{-2} \|z - \tilde{\Pi}_h z\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2} \\
&\leq \left(\sum_{T \in \mathcal{T}_h} h^{-1/2} \beta_T^2 \|\varepsilon \nabla u_k \cdot \mathbf{n}\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2}.
\end{aligned}$$

For the arbitrariness of ϕ , the estimate (38) follows. \square

Now we face the question of the optimality of the previous result, that is we ask ourselves whether the two members of inequalities (29), (36) and (38) are actually equivalent or not. The answer is negative in every cases: this is not surprising because the norms on the first member of (29), (36) and (38) can be related only to a part of the residual, but the remaining one is “out of control”. This suggests to combine the previous terms in order to get an equivalence with the residual. In [20], for example, (29) and (36) are combined; using the same technique it is possible to prove that for every $T \in \mathcal{T}_h$ such that $\varepsilon \geq h_T$

$$\begin{aligned}
&\alpha_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 + \varepsilon^{-1/2} \alpha_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \\
&\preceq \varepsilon |u - u_k|_{H^1(\mathcal{N}(T))}^2 + \|u - u_k\|_{L^2(\mathcal{N}(T))}^2.
\end{aligned} \tag{40}$$

Then in critical regions (where boundary layers occur and then the mesh is very refined) the sum of (29) and (36) gives an optimal estimate for $\varepsilon|u - u_k|_{H^1(\Omega)}^2 + \|u - u_k\|_{L^2(\Omega)}^2$.

In the following theorem we prove a new optimality estimate that holds true in any regimes and on the entire domain. In some sense it ‘‘explains’’ how the error is measured via the numerical residual (of the kind of the one considered here and in many other papers).

Theorem 4. *Assume (11) and (12) hold true; let u be the solution of (2) and let u_k be as in Proposition 1. Then*

$$\begin{aligned} & \varepsilon|u - u_k|_{H^1(\Omega)} + \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)} \\ & \simeq \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 + h_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2}. \end{aligned} \quad (41)$$

Proof. We consider separately the two inequalities summarized by (41):

$$\text{first member} \preceq \text{second member} \quad (42)$$

$$\text{second member} \preceq \text{first member}. \quad (43)$$

Note that (42) is a simple consequence of (36) and (38) because, by definition, $\alpha_T \leq \varepsilon^{-1/2} h_T$ and $\beta_T \leq h_T$, $\forall T \in \mathcal{T}_h$. In order to prove (43) we define some auxiliary functions. Let $r_1 \in L^2(\Omega)$ defined on each element $T \in \mathcal{T}_h$ by

$$r_1 := h_T^2 \left(\prod_{i=1}^{N+1} \lambda_{i,T} \right) (f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k) \quad \text{in } T,$$

where, as usual, $\lambda_{i,T}$ (for $i = 1, \dots, N+1$) refers to the barycentric coordinate in T related to its i -th $(N-1)$ -face $E_i \in \partial T$, i.e. the only linear function null on E_i and assuming unitary value on the opposite vertex. Note that we have neglected in the definition of r_1 the distributional part of the numerical residual, located on the element’s boundary. This term comes in the function r_2 , defined for the moment only on the boundary of elements: we set r_2 null on $\partial\Omega$ and, for each element $T \in \mathcal{T}_h$ and for each of its $(N-1)$ -face E_j (for $j = 1, \dots, N+1$) which is not contained on $\partial\Omega$ we set

$$r_2 := h_{T,j} \left(\prod_{\substack{i=1 \\ i \neq j}}^{N+1} \lambda_{i,T} \right) (\varepsilon [\nabla u_k \cdot \mathbf{n}]) \quad \text{on } E_j,$$

where $h_{T,j}$ is the average value between h_T and $h_{T'}$ and $T' \in \mathcal{T}_h$ is the element sharing E_j with T . Remember that the shape regularity of $\{\mathcal{T}_h\}_h$ imply $h_T \simeq h_{T'}$. Note that the definition above is actually dependent only on the $(N-1)$ -face and not on the element (between the two adjacent). Note moreover r_2 is a continuous and piecewise polynomial function on each ∂T

and then, using a construction like in Corollary 1 of [15] or the different one proposed in [20], we can extend r_2 on the whole domain Ω with

$$h_T^{1/2}|r_2|_{H^1(T)} + h_T^{-1/2}\|r_2\|_{L^2(T)} \preceq \|r_2\|_{L^2(\partial T)} \quad \forall T \in \mathcal{T}_h. \quad (44)$$

Standard scaling arguments and the equivalence of norms on finite dimensional space yield on each element T the equivalences

$$\begin{aligned} & h_T^2\|f + \varepsilon\Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \\ & \simeq (f + \varepsilon\Delta u_k - \mathbf{c} \cdot \nabla u_k, r_1)_T \\ & \simeq h_T^{-2}\|r_1\|_{L^2(T)}^2, \end{aligned} \quad (45)$$

and

$$\begin{aligned} & h_T\|\varepsilon[\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \\ & \simeq (\varepsilon[\nabla u_k \cdot \mathbf{n}], r_2)_{\partial T \cap \Omega} \\ & \simeq h_T^{-1}\|r_2\|_{L^2(\partial T \cap \Omega)}^2. \end{aligned} \quad (46)$$

Then we get, using (45) and integrating by parts (note that there is no boundary terms because r_1 is null on each ∂T)

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} h_T^2\|f + \varepsilon\Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \\ & \preceq \sum_{T \in \mathcal{T}_h} (f + \varepsilon\Delta u_k - \mathbf{c} \cdot \nabla u_k, r_1)_T \\ & = a(u - u_k, r_1) \\ & \preceq (\varepsilon|u - u_k|_{H^1(\Omega)} + \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)}) |r_1|_{H^1(\Omega)}. \end{aligned} \quad (47)$$

Recall that r_1 is piecewise polynomial: a simple inverse inequality and again (45) yield

$$\begin{aligned} |r_1|_{H^1(\Omega)}^2 & \preceq h_T^{-2}\|r_1\|_{L^2(\Omega)}^2 \\ & \preceq \sum_{T \in \mathcal{T}_h} h_T^2\|f + \varepsilon\Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2. \end{aligned} \quad (48)$$

Inserting (48) in (47) we obtain the first part of (43)

$$\begin{aligned} & \left(\sum_{T \in \mathcal{T}_h} h_T^2\|f + \varepsilon\Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right)^{1/2} \\ & \preceq \varepsilon|u - u_k|_{H^1(\Omega)} + \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)}. \end{aligned} \quad (49)$$

Using (46) and integration by parts we get

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} h_T\|\varepsilon[\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \\ & \preceq \frac{1}{2} \sum_{T \in \mathcal{T}_h} (\varepsilon[\nabla u_k \cdot \mathbf{n}], r_2)_{\partial T \cap \Omega} \\ & = a(u - u_k, r_2) - \sum_{T \in \mathcal{T}_h} (f + \varepsilon\Delta u_k - \mathbf{c} \cdot \nabla u_k, r_2)_T \\ & = I + II. \end{aligned} \quad (50)$$

From (44) and (46) we get

$$\begin{aligned}
I &\preceq \left(\varepsilon |u - u_k|_{H^1(\Omega)} + \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)} \right) |r_2|_{H^1(\Omega)} \\
&\preceq \left(\varepsilon |u - u_k|_{H^1(\Omega)} + \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)} \right) \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} h_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2}. \tag{51}
\end{aligned}$$

Moreover, using (44), (46) and (49)

$$\begin{aligned}
II &\preceq \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|f + \varepsilon \Delta u_k - \mathbf{c} \cdot \nabla u_k\|_{L^2(T)}^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} h_T^{-2} \|r_2\|_{L^2(T)}^2 \right)^{1/2} \\
&\preceq \left(\varepsilon |u - u_k|_{H^1(\Omega)} + \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)} \right) \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} h_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2}. \tag{52}
\end{aligned}$$

In conclusion, from (50)–(52) we obtain

$$\begin{aligned}
&\left(\sum_{T \in \mathcal{T}_h} h_T \|\varepsilon [\nabla u_k \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \right)^{1/2} \\
&\preceq \varepsilon |u - u_k|_{H^1(\Omega)} + \|\mathbf{c} \cdot \nabla(u - u_k)\|_{H^{-1}(\Omega)}, \tag{53}
\end{aligned}$$

so (41) is proved. \square

5 Conclusion and extensions

We have developed our analysis assuming some non strictly necessary hypotheses. For example we may allow \mathcal{L} to be a more general: ε may be variable (but strictly positive) as well as \mathbf{c} may have a negative divergence, and it is possible (as in [20]) to consider non piecewise polynomial \mathbf{c} and f .

In the first member of (41) we can add a term like $\varepsilon^{1/2} \|u - u_k\|_{L^2(\Omega)}$, the estimate holding true anyway.

We derived a posteriori estimates which can be used in the mesh refinement procedure. Although our analysis holds true for a general order of elements k , the explicit dependence on k is not taken into consideration. Anyway, for the moment, the proposed algorithms based on the RFB approach allows us to use linear or quadratic elements only.

Our optimality estimate (41) gives some insight into the way we measure the error looking at the numerical residual of the kind proposed here.

A Some stability estimates

In this appendix we present results about the stability of the operator \mathcal{L}^* . Some of them are well known on a fixed domain; in the following the dependence on the domain (on its diameter δ) is explicitly taken into account.

Proposition 3. *Assume (11) and (12) hold true. Let ω be an open subset of Ω with $\delta := \text{diam}(\omega)$; let $\phi \in L^2(\omega)$ and $z \in H_0^1(\omega)$ with*

$$\mathcal{L}^* z = \phi. \quad (54)$$

Then the following estimate holds true

$$\max \{ (\varepsilon \delta^{-1})^{1/2}, \varepsilon \delta^{-1} \} |z|_{H^1(\omega)} + \max \{ \delta^{-1}, \varepsilon \delta^{-2} \} \|z\|_{L^2(\omega)} \preceq \|\phi\|_{L^2(\omega)}. \quad (55)$$

Proof. We have to prove that

$$\varepsilon \delta^{-1} |z|_{H^1(\omega)} + \varepsilon \delta^{-2} \|z\|_{L^2(\omega)} \preceq \|\phi\|_{L^2(\omega)} \quad \text{when } \varepsilon > \delta, \quad (56)$$

$$\varepsilon^{1/2} \delta^{-1/2} |z|_{H^1(\omega)} + \delta^{-1} \|z\|_{L^2(\omega)} \preceq \|\phi\|_{L^2(\omega)} \quad \text{when } \varepsilon \leq \delta. \quad (57)$$

Multiplying (54) by z , integrating over ω , integrating by parts and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} \varepsilon |z|_{H^1(\omega)} &= (\phi, z)_\omega \\ &\preceq \|\phi\|_{L^2(\omega)} \|z\|_{L^2(\omega)}. \end{aligned} \quad (58)$$

We recall now the Poincaré inequality: using our notations

$$\|z\|_{L^2(\omega)} \preceq \delta |z|_{H^1(\omega)}; \quad (59)$$

then (58) and (59) give (56).

Suppose now $\varepsilon \leq \delta$. We multiply (54) by $z e^{\gamma \eta(\mathbf{x})/\delta}$ (the positive constant γ will be chosen below) and integrate over ω . Then, integrating by parts we get

$$\begin{aligned} \int_{\omega} \phi(\mathbf{x}) z(\mathbf{x}) e^{\gamma \eta(\mathbf{x})/\delta} d\mathbf{x} &= \varepsilon \int_{\omega} |\nabla z(\mathbf{x})|^2 e^{\gamma \eta(\mathbf{x})/\delta} d\mathbf{x} \\ &\quad + \varepsilon \int_{\omega} \nabla z(\mathbf{x}) \cdot \nabla (e^{\gamma \eta(\mathbf{x})/\delta}) z(\mathbf{x}) d\mathbf{x} \\ &\quad - \frac{1}{2} \int_{\omega} \mathbf{c}(\mathbf{x}) \cdot \nabla (z^2(\mathbf{x})) e^{\gamma \eta(\mathbf{x})/\delta} d\mathbf{x} \\ &= I + II + III. \end{aligned} \quad (60)$$

Introducing the weighted norm

$$\|v\|^2 := \int_{\omega} v^2(\mathbf{x}) e^{\gamma \eta(\mathbf{x})/\delta} d\mathbf{x},$$

we have

$$I = \varepsilon \|\nabla z\|^2, \quad (61)$$

and, using (11) e (12), we have

$$\begin{aligned} III &= \frac{1}{2} \int_{\omega} \mathbf{c}(\mathbf{x}) \cdot \nabla (e^{\gamma\eta(\mathbf{x})/\delta}) z^2(\mathbf{x}) \, d\mathbf{x} \\ &\quad + \frac{1}{2} \int_{\omega} \operatorname{div}(\mathbf{c}(\mathbf{x})) e^{\gamma\eta(\mathbf{x})/\delta} z^2(\mathbf{x}) \, d\mathbf{x} \\ &= \frac{\gamma}{2\delta} \int_{\omega} \mathbf{c}(\mathbf{x}) \cdot \nabla \eta(\mathbf{x}) e^{\gamma\eta(\mathbf{x})/\delta} z^2(\mathbf{x}) \, d\mathbf{x} \\ &\geq \frac{\gamma}{\delta} \|z\|^2. \end{aligned} \quad (62)$$

On the other hand using the Cauchy-Schwarz inequality we have (remembering the assumption $\varepsilon \leq \delta$)

$$\begin{aligned} II &\leq \frac{\varepsilon\gamma}{\delta} \left(\sup_{\mathbf{x} \in \omega} |\nabla \eta| \right) \|\nabla z\| \|z\| \\ &\leq \gamma^{1/2} \left(\sup_{\mathbf{x} \in \omega} |\nabla \eta| \right) \left(\varepsilon \|\nabla z\|^2 + \frac{\gamma}{\delta} \|z\|^2 \right). \end{aligned} \quad (63)$$

Then, choosing γ small enough (dependent only on η , i.e. on \mathbf{c}), we get

$$I + III \preceq I + II + III. \quad (64)$$

Returning to (60) and using the Cauchy-Schwarz inequality we infer

$$\varepsilon \|\nabla z\|^2 + \delta^{-1} \|z\|^2 \preceq \|\phi\| \|z\|,$$

hence

$$\varepsilon^{1/2} \delta^{-1/2} \|\nabla z\| + \delta^{-1} \|z\| \preceq \|\phi\|. \quad (65)$$

We observe that

$$\min_{\mathbf{x} \in \omega} \{e^{\gamma\eta(\mathbf{x})/\delta}\} \simeq \max_{\mathbf{x} \in \omega} \{e^{\gamma\eta(\mathbf{x})/\delta}\}, \quad (66)$$

so we can remove the weight from the norms in (65) to obtain (57). \square

Proposition 4. *Assume (11) and (12) hold true. Let ω be an open regular¹ subset of Ω with $\delta := \operatorname{diam}(\omega)$; let $\phi \in H^2(\omega)$ and $z \in H_0^1(\omega)$ with*

$$\mathcal{L}^* z = -\varepsilon \Delta \phi. \quad (67)$$

Then the following estimate holds true

$$\varepsilon |z|_{H^1(\omega)} + \max \{(\varepsilon\delta^{-1})^{1/2}, \varepsilon\delta^{-1}\} \|z\|_{L^2(\omega)} \preceq \varepsilon |\phi|_{H^1(\omega)}. \quad (68)$$

¹regular means that the elliptic regularity properties hold true; C^2 or convex polygonal is enough.

Proof. We proceed similarly to the case of Proposition 3. We have to show that

$$\varepsilon|z|_{H^1(\omega)} + \varepsilon\delta^{-1}\|z\|_{L^2(\omega)} \preceq \varepsilon|\phi|_{H^1(\omega)} \quad \text{when } \varepsilon > \delta, \quad (69)$$

$$\varepsilon|z|_{H^1(\omega)} + \varepsilon^{1/2}\delta^{-1/2}\|z\|_{L^2(\omega)} \preceq \varepsilon|\phi|_{H^1(\omega)} \quad \text{when } \varepsilon \leq \delta. \quad (70)$$

To proof (69) simply multiply (67) by z and integrate by parts, obtaining

$$\varepsilon|z|_{H^1(\omega)}^2 \leq \varepsilon|\phi|_{H^1(\omega)}|z|_{H^1(\omega)},$$

hence

$$\varepsilon|z|_{H^1(\omega)} \leq \varepsilon|\phi|_{H^1(\omega)};$$

then, using (59), we get

$$\varepsilon\delta^{-1}\|z\|_{L^2(\omega)} \preceq \varepsilon|\phi|_{H^1(\omega)}. \quad (71)$$

Suppose now $\varepsilon \leq \delta$; proceeding as in (60)–(64) we get (for a suitable value of γ)

$$\varepsilon\|\nabla z\|^2 + \delta^{-1}\|z\|^2 \preceq -\varepsilon \int_{\omega} \Delta\phi(\mathbf{x})z(\mathbf{x})e^{\gamma\eta(\mathbf{x})/\delta} d\mathbf{x}. \quad (72)$$

Using again integration by parts and the Cauchy-Schwarz formulas, we obtain

$$\begin{aligned} -\varepsilon \int_{\omega} \Delta\phi(\mathbf{x})z(\mathbf{x})e^{\gamma\eta(\mathbf{x})/\delta} d\mathbf{x} &= \varepsilon \int_{\omega} \nabla\phi(\mathbf{x}) \cdot \nabla z(\mathbf{x}) e^{\gamma\eta(\mathbf{x})/\delta} d\mathbf{x} \\ &\quad + \frac{\varepsilon\gamma}{\delta} \int_{\omega} \nabla\phi(\mathbf{x}) \cdot \nabla\eta(\mathbf{x}) z(\mathbf{x})e^{\gamma\eta(\mathbf{x})/\delta} d\mathbf{x} \quad (73) \\ &\preceq (\varepsilon^{1/2}\|\nabla\phi\|_{L^2(\omega)}) (\varepsilon^{1/2}\|\nabla z\|_{L^2(\omega)}) \\ &\quad + (\varepsilon^{1/2}\|\nabla\phi\|_{L^2(\omega)}) (\delta^{-1/2}\|z\|_{L^2(\omega)}). \end{aligned}$$

Now collect (72), (73) and (66) to infer (70). \square

The last result is quite simple to prove but nevertheless interesting: it allows a strong estimate on the solution z of (74) (note in (75) the H^1 -norm term without any scaling on ε).

Proposition 5. *Assume (11) and (12) hold true. Let ω be an open regular subset of Ω ; let $\phi \in H_0^1(\omega)$ and $z \in H_0^1(\omega)$ with*

$$\mathcal{L}^*z = -\mathbf{c} \cdot \nabla\phi. \quad (74)$$

Then the following estimate holds true

$$\varepsilon|z|_{H^2(\omega)} + |z|_{H^1(\omega)} \preceq |\phi|_{H^1(\omega)}. \quad (75)$$

Proof. We multiply (74) by $z - \phi$ and integrate over ω . After integration by parts we get

$$\begin{aligned} \varepsilon \int_{\omega} \nabla z(\mathbf{x}) \cdot \nabla (z(\mathbf{x}) - \phi(\mathbf{x})) \, d\mathbf{x} &= \int_{\omega} \mathbf{c}(\mathbf{x}) \cdot \nabla \left(\frac{(z(\mathbf{x}) - \phi(\mathbf{x}))^2}{2} \right) \, d\mathbf{x} \\ &= -\frac{1}{2} \int_{\omega} \operatorname{div}(\mathbf{c}(\mathbf{x})) (z(\mathbf{x}) - \phi(\mathbf{x}))^2 \, d\mathbf{x} \\ &= 0. \end{aligned}$$

Hence, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \varepsilon |z|_{H^1(\omega)}^2 &= \varepsilon (\nabla z, \nabla \phi)_{\omega} \\ &\leq \varepsilon |z|_{H^1(\omega)} |\phi|_{H^1(\omega)}, \end{aligned}$$

that is

$$|z|_{H^1(\omega)} \leq |\phi|_{H^1(\omega)}. \quad (76)$$

Using (67), (76) and the elliptic regularity property we obtain

$$\begin{aligned} \varepsilon |z|_{H^2(\omega)} &\preceq \varepsilon \|\Delta z\|_{L^2(\omega)} \\ &= \|\mathbf{c} \cdot \nabla (z - \phi)\|_{L^2(\omega)} \\ &\preceq |\phi|_{H^1(\omega)}, \end{aligned}$$

that concludes the proof. □

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