Chemical Reactions as $\Gamma$-Limit of Diffusion

Mark A. Peletier\(^\dagger\)
Giuseppe Savaré\(^\ddagger\)
Marco Veneroni\(^\ddagger\)

Abstract. We study the limit of high activation energy of a special Fokker–Planck equation known as the Kramers–Smoluchowski equation (KS). This equation governs the time evolution of the probability density of a particle performing a Brownian motion under the influence of a chemical potential $H/\varepsilon$, having two wells corresponding to two chemical states $A$ and $B$. We prove that after a suitable rescaling the solution to the KS converges, in the limit of high activation energy ($\varepsilon \to 0$), to the solution of a simpler system modeling the spatial diffusion of $A$ and $B$ combined with the reaction $A \rightleftharpoons B$. With this result we give a rigorous proof of Kramers’s formal derivation, and we show how chemical reactions and diffusion processes can be embedded in a common framework. This allows one to see a chemical reaction as a singular limit of a diffusion process, thus establishing a connection between two worlds often regarded as separate. The proof rests on two main ingredients. One is the formulation of the two disparate equations as evolution equations for measures. The second is a variational formulation of both equations that allows us to use the tools of variational calculus and, specifically, $\Gamma$-convergence.

Key words. unification, scale-bridging, upscaling, high-energy limit, activation energy, Dirichlet forms, Mosco-convergence, variational evolution equations

AMS subject classifications. Primary, 35K57, 35Q84; Secondary, 49J45, 49S05, 80A30

DOI. 10.1137/110858781

In this paper we prove that in the limit $\varepsilon \to 0$ solutions of the parabolic partial differential equation in spatial variables $x$ and $\xi$,

\[ \begin{align*}
\partial_t \rho - \Delta_x \rho - \tau_x \partial_{\xi} \left( \partial_{\xi} \rho + \frac{1}{\varepsilon} \rho \partial_{\xi} H \right) &= 0 \quad \text{for } (x, \xi) \in \Omega \times [-1, 1] \text{ and } t > 0, \\
\text{with no-flux boundary conditions,}
\end{align*} \]

converge to solutions of the system of reaction-diffusion equations in spatial variable $x$,

\[ \begin{align*}
\partial_t \alpha - \Delta_x \alpha &= k(\beta - \alpha), \\
\partial_t \beta - \Delta_x \beta &= k(\alpha - \beta),
\end{align*} \quad \text{for } x \in \Omega \text{ and } t > 0.\]

\(\dagger\)Department of Mathematics and Institute for Complex Molecular Systems, Technische Universität Eindhoven, Eindhoven, The Netherlands (m.a.peletier@tue.nl). The research of this author was partially supported by the ITN “FIRST” of the Seventh Framework Programme of the European Community (grant 238702) and by the NWO VICI project 639.033.008.

\(\ddagger\)Dipartimento di Matematica Felice Casorati, Università degli studi di Pavia, Pavia, Italy (giuseppe.savare@unipv.it, marco.veneroni@unipv.it). The research of the second author was partially supported by a MIUR-PRIN 2008 grant for the project “Optimal Mass Transportation, Geometric and Functional Inequalities and Applications.”

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
what we macroscopically call both “diffusing” and “reacting.” First, a typical modeling argument leads to equations of the form (0.2) for a finite number of chemical species, with second-order diffusion terms and zero-order reaction terms. Here we limit ourselves to a simple $A \rightleftharpoons B$ system, which arises, for instance, when $A$ and $B$ are two forms of the same molecule, such that the molecule can change from one form into the other. A typical example is a molecule with spatial asymmetry, which might exist in two distinct, mirror-image spatial configurations (but see section 3.3 for more general reactions). The variables $\alpha$ and $\beta$ are the volume fractions of the two species and the constant $k$ is the reaction rate, which for simplicity we assume is the same for both reactions $A \rightarrow B$ and $B \rightarrow A$.

On the other hand, the discussion above suggests that the same system should be described by an equation of the form (0.1), in the following way. The variable $x$ models the spatial degrees of freedom of the molecule, i.e., the different positions in space where the molecule can be; the variable $\xi$ models the “chemical” degrees of freedom, corresponding to different arrangements of the atoms inside the molecule. For simplicity (and following Kramers) we assume $\xi$ to be one-dimensional. The energy of a state $(x, \xi)$ is given by a potential function $H$ that we assume to be independent of $x$ and that has a double-well structure as in Figure 1, corresponding to the two stable states $A$ and $B$. Since the particle undergoes an SDE in $(x, \xi)$, driven by Gaussian noise and the potential $H$, the probability distribution $\rho$ on $(x, \xi)$-space satisfies (0.1).

Therefore, the following questions arise: How are (0.1) and (0.2) related? Can they be seen as two signs of the same coin? Is it possible to take a limit (which will be the limit of large activation energy) such that solutions of (0.1) converge to (0.2)?

In this paper we give an answer to these questions by proving two convergence theorems that link the two descriptions. The limit of large activation energy is implemented by replacing $H$ by $H/\varepsilon$, leading to an activation energy of order $O(\varepsilon)$; this explains the factor $1/\varepsilon$ in (0.1). The convergence results are then in the limit $\varepsilon \to 0$.

In addition, the unusual aspects of this question prompted us to develop a method that not only allows us to address this question, but may be more generally useful. We describe this in the next section.

1.3. Structure of the Proof. Any attempt to prove a rigorous convergence result faces an important difficulty: the mathematical objects used in the two descriptions are of very different types.

Let us assume for simplicity that $\xi$ parametrizes an imaginary “optimal path” connecting the states $A$ and $B$ such that $\xi = -1$ corresponds to $A$ and $\xi = 1$ to $B$. At $\varepsilon > 0$, the system is described by a partial differential equation in the space $\Omega \times [-1, 1]$, while in the limit $\varepsilon = 0$ the chemical ($\xi$-) degrees of freedom are reduced to the two possibilities $A$ and $B$, which correspond to only $\xi = \pm 1$. Therefore, the question arises in what sense solutions of one could ever converge to the other.

As it turns out, both systems can be described by variational evolution equations in a common space of measures. Measures generalize functions and allow for a rigorous description of concentration phenomena: while functions in Lebesgue or Sobolev spaces cannot represent concentration of finite mass onto a point, in the space of measures this behavior can be described in terms of Dirac distributions $\xi \mapsto \delta(\xi \pm 1)$. Equation (0.1) is already a measure-valued equation; by also reformulating (0.2) as a measure-valued equation we unite the two equations in a common structure. The space of measures carries a natural concept of convergence, the convergence against test functions, and this will be the basis for the first of our convergence results.
2. The Model.

2.1. The Setup: Enthalpy. We now describe the systems of this paper in more detail. We consider the unimolecular reaction $A \rightleftharpoons B$. We assume that the observed forms $A$ and $B$ correspond to the wells of a double-well enthalpy function $H$. (Since it is common in the chemical literature to denote by “enthalpy difference” the release or uptake of heat as a particle $A$ is converted into a particle $B$, we shall adopt the same language.)

While the domain of definition of $H$ should be high-dimensional, corresponding to the many degrees of freedom of the atoms of the molecule, we will here make the standard reduction to a one-dimensional dependence. As we mentioned before, the states $A$ and $B$ correspond to $\xi = -1$ and $\xi = +1$, respectively, and the variable $\xi$ takes its values in $[-1, 1]$. A transition between $-1$ and $+1$ should pass through the “mountain pass,” the point which separates the basins of attraction of $A$ and $B$, and we arbitrarily choose that mountain pass to be at $\xi = 0$, with $H(0) = 1$. We also assume for simplicity that the wells are at equal depth, which we choose to be zero. A typical example of the function $H$ is shown in Figure 3.

![Figure 3](https://example.com/fig3.png)

Fig. 3  A typical function $H$.

Specifically, we make the following assumptions about $H$: $H \in C^\infty([-1, 1])$, and $H$ is even in $\xi$, maximal at $\xi = 0$ with value 1, and minimal at $\xi = \pm 1$ with value 0; $H(\xi) > 0$ for any $-1 < \xi < 1$; $H'(\pm 1^\mp) = 0$. The assumption of equal depth for the two wells corresponds to an assumption about the rate constants of the two reactions; we comment on this in section 3.3.

2.2. Diffusion in the Chemical Landscape. We now describe the diffusion process, starting with the state space. The “chemical variable” $\xi$ should be interpreted as an internal degree of freedom of the particle, associated with internal changes in configuration. In the case of two alternative states of a molecule, $\xi$ parametrizes all the intermediate states along a connecting path.

In this view the total state of a particle consists of this chemical state $\xi$ together with the spatial position of the particle, represented by a $d$-dimensional spatial variable $x$ in a Lipschitz, bounded, and open domain $\Omega \subset \mathbb{R}^d$, so that the full state space for the particle is the closure $\overline{D}$ of

$$D := \Omega \times (-1, 1) \quad \text{with variables } (x, \xi).$$

Taking a probabilistic point of view, and following Kramers, the motion of the particle will be described in terms of its probability density $\rho \in \mathcal{P}(\overline{D})$ in the sense that for Borel sets $X \subset \overline{\Omega}$ and $\Xi \subset [-1, 1]$ the number $\rho(X \times \Xi)$ is the probability of finding the particle at a position $x \in X$ and with a “chemical state” $\xi \in \Xi$.

The particle is assumed to perform a Brownian motion in $D$, under the influence of the potential landscape described by $H$. This assumption corresponds to the “large-friction limit” discussed by Kramers. The time evolution of the probability
We should interpret the behavior of $\gamma_\varepsilon$ as follows. In the limit $\varepsilon \to 0$, the deep wells at $\xi = \pm 1$ force particles to stay increasingly close to the bottom of the wells. However, at any given $\varepsilon > 0$, there is a positive probability that a particle switches from one well to the other in any given period of time. The rate at which this happens is governed by the local structure of $H$ near $\xi = \pm 1$ and near $\xi = 0$ and becomes very small, of order $\varepsilon^{-1} \exp(-1/\varepsilon)$, as we shall see below.

In the limit $\varepsilon = 0$, the behavior of particles in the $\xi$-direction is no longer recognizable as diffusional in nature. In the $\xi$-direction a particle can be in only one of two states, $\xi = \pm 1$, which we have interpreted as the $A$ and $B$ states. Of the diffusional movement in the $\xi$-direction only a jump process remains, in which a particle at $\xi = -1$ jumps with a certain rate to position $\xi = 1$, or vice versa.

At each time $t \geq 0$ the limit system can thus be described by the nonnegative functions $\alpha(\cdot; t), \beta(\cdot; t) : \Omega \to \mathbb{R}$ representing the densities of particles in the states $A$ and $B$, respectively; this means that $\int_X \alpha(x; t) \, dx$ is the (normalized) number of particles in the state $A$ contained in the region $X \subset \Omega$, and a similar formula holds for $\beta = 1 - \alpha$.

The time-dependent measures

\begin{equation}
\rho(x, \xi; t) = \alpha(x; t) \mathcal{L}^d(x) \otimes \delta_{-1}(\xi) + \beta(x; t) \mathcal{L}^d(x) \otimes \delta_{1}(\xi)
\end{equation}

are thus the limit distributions of particles, and we can expect that $\rho_\varepsilon(\cdot; t)$ converges weakly-$*$ to $\rho(\cdot; t)$ as $\varepsilon \downarrow 0$ in the space $\mathcal{M}(\mathbb{D})$, i.e.,

$$
\lim_{\varepsilon \downarrow 0} \int_{\mathbb{D}} \phi(x, \xi) \, d\rho_\varepsilon(x, \xi; t) = \int_{\Omega} \left( \phi(x, -1) \alpha(x; t) + \phi(x, 1) \beta(x; t) \right) \, dx
$$

for any $\phi \in C^0(\mathbb{D})$ and $t > 0$. The remarkable fact is that, under a suitable choice of $\tau = \tau_\varepsilon$ which we will explain in the next section, the limit functions $\alpha, \beta$ satisfy the reaction-diffusion system (2.2).

**2.4. Spatiochemical Rescaling.** Since the jumping (chemical reaction) rate at finite $\varepsilon > 0$ is of order $\varepsilon^{-1} \exp(-1/\varepsilon)$, the limiting reaction rate will be zero unless we rescale the system appropriately. This requires us to speed up time by a factor of $\varepsilon \exp(1/\varepsilon)$. At the same time, the diffusion rate in the $x$-direction remains of order 1 as $\varepsilon \to 0$, and the rescaling should preserve this. In order to obtain a limit in which both diffusion in $x$ and chemical reaction in $\xi$ enter at rates that are of order 1, we use the freedom of choosing the parameter $\tau$ that we introduced above.

We therefore choose $\tau$ equal to

\begin{equation}
\tau_\varepsilon := \varepsilon \exp(1/\varepsilon),
\end{equation}

and we then find the differential equation

\begin{equation}
\partial_t \rho_\varepsilon - \Delta_x \rho_\varepsilon - \tau_\varepsilon \partial_\xi \rho_\varepsilon (\partial^2 \rho_\varepsilon + \frac{1}{\varepsilon} \partial_\xi H) = 0 \quad \text{in } \mathcal{D}'(\mathbb{D} \times (0, \infty)),
\end{equation}

which clearly highlights the different treatment of $x$ and $\xi$: the diffusion in $x$ is independent of $\tau_\varepsilon$, while the diffusion and convection in the $\xi$-variable are accelerated by a factor $\tau_\varepsilon$.

**2.5. Switching to the Density Variable.** As is already suggested by the behavior of the invariant measure $\gamma_\varepsilon$, the solution $\rho_\varepsilon$ will become strongly concentrated at the extremities $\{ \pm 1 \}$ of the $\xi$-domain $(-1, 1)$. This is the reason why it is useful to
\( \rho = u \gamma \) with \( u \in H \) we set \( u^\pm(x) := u(x, \pm 1) \in L^2(\Omega, \lambda_0) \). Note that for a function \( u \in L^2(D, \gamma) \) these traces are well defined (in fact, the map \( u \mapsto (u^-, u^+) \) is an isomorphism between \( L^2(D, \gamma) \) and \( L^2(\Omega, \frac{1}{2} \lambda_0; \mathbb{R}^2) \)).

We define
\[
(2.10) \quad b(u, v) := \int_D u(x, \xi)v(x, \xi) \, d\gamma(x, \xi) = \frac{1}{2} \int_\Omega \left( u^+v^+ + u^-v^- \right) \, d\lambda_0.
\]

Similarly, we set \( V := \{ u \in H : u^\pm \in W^{1,2}(\Omega) \} \), which is continuously and densely imbedded in \( H \), and
\[
(2.11) \quad a(u, v) := \frac{1}{2} \int_\Omega \left( \nabla_x u^+ \nabla_x v^+ + \nabla_x u^- \nabla_x v^- + k(u^+ - u^-)(v^+ - v^-) \right) \, d\lambda_0.
\]

Then the system (0.2) can be formulated as
\[
(2.12) \quad b(\partial_t u(t), v) + a(u(t), v) = 0 \quad \text{for every } t > 0 \text{ and } v \in V,
\]
which has the same structure as (2.8).

3. Main Results and Discussion.

3.1. Main Result I: Weak Convergence of \( \rho_\varepsilon \) and \( u_\varepsilon \). The following theorem is the first main result of this paper. It states that for every time \( t \geq 0 \) the measures \( \rho_\varepsilon(t) \) that solve (2.6) weakly-* converge as \( \varepsilon \downarrow 0 \) to a limiting measure \( \rho(t) \) in \( \mathcal{M}(D) \), whose density \( u(t) = \frac{d\rho(t)}{d\lambda_0} \) is the solution of the limit system (0.2).

We state our result in a general form, which holds even for signed measures in \( \mathcal{M}(D) \).

Theorem 3.1. Let \( \rho_\varepsilon = u_\varepsilon \gamma_\varepsilon \in C^0([0, +\infty); \mathcal{M}(D)) \) be the solution of (2.6)–(2.7c) with initial datum \( \rho_0^\varepsilon \). If
\[
\sup_{\varepsilon > 0} \int_D \frac{1}{2} \left| u_\varepsilon \right|^2 \, d\gamma_\varepsilon < +\infty
\]
and \( \rho_0^\varepsilon \) weakly-* converges to
\[
\rho^0 = u_0 \gamma = \frac{1}{2} u^0^- \lambda_\Omega \otimes \delta_{-1} + \frac{1}{2} u^0^+ \lambda_\Omega \otimes \delta_{-1} \text{ as } \varepsilon \downarrow 0,
\]
then \( u_\varepsilon \in L^2(D; \gamma) \), \( u_\varepsilon \in L^2(\Omega) \), and, for every \( t \geq 0 \), the solution \( \rho_\varepsilon(t) \) weakly-* converges to
\[
\rho(t) = u(t) \gamma = \frac{1}{2} u^-(t) \lambda_\Omega \otimes \delta_{-1} + \frac{1}{2} u^+(t) \lambda_\Omega \otimes \delta_{1},
\]
whose densities \( u^\pm \) belong to \( C^0([0, +\infty); L^2(\Omega)) \cap C^1((0, +\infty); W^{1,2}(\Omega)) \) and solve the system
\[
(3.4a) \quad \partial_t u^+ - \Delta_x u^+ = k(u^+ - u^-) \quad \text{in } \Omega \times (0, +\infty),
\]
\[
(3.4b) \quad \partial_t u^- - \Delta_x u^- = k(u^+ - u^-) \quad \text{in } \Omega \times (0, +\infty),
\]
\[
(3.4c) \quad u^\pm(0) = u_0^\pm \quad \text{in } \Omega.
\]
The positive constant \( k \) in (3.4a)–(3.4b) can be characterized as the asymptotic minimal transition cost
\[
(3.5) \quad k = \frac{1}{\pi} \sqrt{|H''(0)|H''(1)}
\]
\[
= \lim_{\varepsilon \downarrow 0} \min_{\tau \in |0, 1|} \left\{ \tau \int_{-1}^1 \left( \varphi'(\xi) \right)^2 \, d\gamma_\varepsilon : \varphi \in W^{1,2}(-1, 1), \varphi(\pm 1) = \pm \frac{1}{2} \right\}.
\]
as an upscaled version of the movement of a Brownian particle in the same augmented phase space.

At the same time, they generalize the work of Kramers by adding the spatial dimension, resulting in a limit system which, for this choice of \( \tau_\varepsilon \) (see below for more on this choice), captures both reaction and diffusion effects.

**Measures versus densities.** It is interesting to note the roles of the measures \( \rho_\varepsilon, \rho \) and their densities \( u_\varepsilon, u \) with respect to \( \gamma_\varepsilon, \gamma \). The variational formulations of the equations are given in terms of the densities \( u_\varepsilon, u \), but the limit procedure is better understood in terms of the measures \( \rho_\varepsilon, \rho \), since a weak-* convergence is involved. This also allows for a unification of two problems with a different structure (a Fokker–Planck equation for \( u_\varepsilon \) and a reaction-diffusion system for the couple \( u^-, u^+ \)).

**Gradient flows.** The weak formulation (2.8) also shows that a solution \( u_\varepsilon \) can be interpreted as a gradient flow of the quadratic energy \( \frac{1}{2} u_\varepsilon^2(u, u) \) with respect to the \( L^2(D; \gamma_\varepsilon) \) distance. Another gradient-flow structure for the solutions of the same problem could be obtained by a different choice of energy functional and distance: for example, as proved in [16], Fokker–Planck equations like (2.6) can be interpreted also as the gradient flow of the relative entropy functional

\[
\mathcal{H}(\rho|\gamma_\varepsilon) := \int_D \frac{d\rho}{d\gamma_\varepsilon} \log \left( \frac{d\rho}{d\gamma_\varepsilon} \right) d\gamma_\varepsilon
\]

in the space \( \mathcal{P}(D) \) of probability measures endowed with the so-called \( L^2 \)-Wasserstein distance (see, e.g., [2]). Initiated by the work of Otto [16, 21] and extended in many directions since, this framework provides an appealing variational structure for very general diffusion processes.

Two recent results point toward a connection between the results of this paper and Wasserstein gradient flows. In [1] it was shown how the Wasserstein setting may be the most natural for understanding diffusion as a limit of the motion of Brownian particles. In addition, Mielke uncovered a Wasserstein-type gradient-flow structure for chemical reactions [19].

Fueled by these observations, in the first publication [22] of our results we asked whether a similar convergence result could also be proved within the Wasserstein gradient-flow framework. This question was answered affirmatively in two different ways [14, 3], and we refer the reader to [3] for further discussion of these issues.

**The choice of \( \tau_\varepsilon \).** In this paper the time scale \( \tau_\varepsilon \) is chosen to be equal to \( \varepsilon \exp(1/\varepsilon) \), and it is natural to ask about the limit behavior for different choices of \( \tau_\varepsilon \). If the scaling is chosen differently, i.e., if \( \tau_\varepsilon \varepsilon^{-1} \exp(-1/\varepsilon) \) converges to \( 0 \) or \( \infty \), then completely different limit systems are obtained:

- If \( \tau_\varepsilon \ll \varepsilon \exp(1/\varepsilon) \), then the reaction is not accelerated sufficiently as \( \varepsilon \to 0 \), and the limit system will contain only diffusion (i.e., \( k = 0 \) in (3.4)).
- If \( \tau_\varepsilon \gg \varepsilon \exp(1/\varepsilon) \), on the other hand, then the reaction becomes faster and faster as \( \varepsilon \to 0 \), resulting in a limit system in which the chemical reaction \( A \rightleftharpoons B \) is in continuous equilibrium. Because of this, both \( A \) and \( B \) have the same concentration \( u \), which \( u \) solves the diffusion problem

\[
\partial_t u = \Delta u \quad \text{for } x \in \Omega, \ t > 0,
\]

\[
u(0, x) = \frac{1}{2}(u^{0,+}(x) + u^{0,-}(x)) \quad \text{for } x \in \Omega.
\]

Note the instantaneous equilibration of the initial data in this system.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
Single particles versus multiple particles, and concentrations versus probabilities. The description in this paper of the system in terms of a probability measure $\rho$ on $\mathcal{D}$ is the description of the probability of a single particle. This implies that the limit object $(u^-, u^+)$ should be interpreted as the density (with respect to $\gamma$) of a limiting probability measure, again describing a single particle.

This is at odds with common continuum modeling philosophy, where the main objects are concentrations (mass or volume) that represent a large number of particles; in this philosophy the solution $(u^-, u^+)$ of (3.4) should be viewed as such a concentration, which is to say as the projection onto $x$-space of a joint probability distribution of a large number of particles.

For the simple reaction $A \rightleftharpoons B$ these two interpretations are actually equivalent. This arises from the fact that $A \rightarrow B$ reaction events in each of the particles are independent of each other; therefore, the joint distribution of a large number $N$ of particles factorizes into a product of $N$ copies of the distribution of a single particle.

For the case of this paper, therefore, the distinction between these two views is not important.

More general reactions. The remark above implies that the situation will be different for systems where reaction events cause differences in distributions between the particles, such as the reaction $A + B \rightleftharpoons C$. This can be recognized as follows: a particle $A$ that has just separated from a $B$ particle (in a reaction event of the form $C \rightarrow A + B$) has a position that is highly correlated with the corresponding $B$ particle, while this is not the case for all the other $A$ particles. Therefore, the $A$ particles will not have the same distribution. The best one can hope for is that in the limit of a large number of particles the distribution becomes the same in some weak way. This is one of the major obstacles in developing a similar connection as the one in this paper for more complex reaction equations.

Regarding possible extensions toward equations involving an arbitrary number of chemical species, as well as different reaction and diffusion rates, we point out that a formal gradient-flow structure has recently been established in [19], independently of this work.

4. Structure of the Proof: Formulation in Terms of Measures, Regularization Estimates, and $\Gamma$-Convergence of Quadratic Forms. We now explain the structure of the proof of Theorems 3.1 and 3.2 in more detail. This will also clarify the use of $\Gamma$-convergence and highlight the potential of the method for wider application.

We have already seen that the $\varepsilon$-problem (2.7) and the limit problem (0.2) can be formulated in the highly similar variational forms (2.8) and (2.12). The analogy between (2.8) and (2.12) suggests passing to the limit in these weak formulations, or even better, in their equivalent integrated forms

\begin{align}
 b_\varepsilon(u_\varepsilon(t), v_\varepsilon) + \int_0^t a_\varepsilon(u_\varepsilon(t), v_\varepsilon) \, dt = b(u^0_\varepsilon, v_\varepsilon) & \quad \text{for every } v_\varepsilon \in V_\varepsilon, \\
 b(u(t), v) + \int_0^t a(u(t), v) \, dt = b(u^0, v) & \quad \text{for every } v \in V.
\end{align}

Applying standard regularization estimates for the solutions to (2.8) (see the next section) and a weak coercivity property of $b_\varepsilon$, it is not difficult to prove that $u_\varepsilon(t)$ “weakly” converges to $u(t)$ for every $t > 0$, i.e.,

$$
\rho_\varepsilon(t) = \frac{u_\varepsilon(t)}{\gamma_\varepsilon} \rightharpoonup^* \rho(t) = \frac{u(t)}{\gamma} \quad \text{weakly-}^* \text{ in } \mathcal{M}(\mathcal{D}).
$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
Dom(\(b_\varepsilon\)) their proper domains, we still denote by \(a_\varepsilon(\cdot, \cdot)\) and \(b_\varepsilon(\cdot, \cdot)\) the corresponding bilinear forms defined on Dom(\(a_\varepsilon\)) and Dom(\(b_\varepsilon\)), respectively. Setting \(\rho_\varepsilon := u_\varepsilon \gamma_\varepsilon\) and \(\sigma := \nu_\gamma_\varepsilon\), (4.7) is equivalent to the integrated form

\[
\begin{align*}
\left(\frac{1}{2}b_\varepsilon(\rho_\varepsilon(t), \rho_\varepsilon) + \int_0^t a_\varepsilon(\rho_\varepsilon(r), \sigma) \, dr \right) & = \frac{1}{2}b_\varepsilon(\rho_\varepsilon^0, \sigma) \\
& \quad \text{for every } \sigma \in \text{Dom}(a_\varepsilon).
\end{align*}
\]

### 4.2. Regularization Estimates.
We recall here the standard regularization estimates satisfied by solutions to (4.10):

\[
\begin{align*}
(4.11) & \quad \frac{1}{2}b_\varepsilon(\rho_\varepsilon(t)) + \int_0^t a_\varepsilon(\rho_\varepsilon(r)) \, dr = \frac{1}{2}b_\varepsilon(\rho_\varepsilon^0) \quad \text{for every } t \geq 0, \\
(4.12) & \quad t a_\varepsilon(\rho_\varepsilon(t)) + 2 \int_0^t r b_\varepsilon(\partial_t \rho_\varepsilon(r)) \, dr = \int_0^t a_\varepsilon(\rho_\varepsilon(r)) \, dr \quad \text{for every } t \geq 0, \\
(4.13) & \quad \frac{1}{2}b_\varepsilon(\rho_\varepsilon(t)) + t a_\varepsilon(\rho_\varepsilon(t)) + t^2 (\partial_t \rho_\varepsilon(t) \sigma) \leq \frac{1}{2}b_\varepsilon(\rho_\varepsilon^0) \quad \text{for every } t > 0.
\end{align*}
\]

Although versions of these expressions appear in various places, for the ease of the reader we briefly describe their proof, and we use the more conventional formulation in terms of the bilinear forms \(a_\varepsilon\) and \(b_\varepsilon\), the density \(u_\varepsilon\), and spaces \(H_\varepsilon\) and \(V_\varepsilon\); note that \(b_\varepsilon\) is an inner product for \(H_\varepsilon\), and \(b_\varepsilon + a_\varepsilon\) is an inner product for \(V_\varepsilon\).

When \(u_0\) is sufficiently smooth, standard results (e.g., [8, Chapter VII]) provide the existence of a solution \(u_\varepsilon \in C((0, \infty); V_\varepsilon) \cap C^\infty((0, \infty); V_\varepsilon)\), such that the functions \(t \mapsto a_\varepsilon(u_\varepsilon(t))\) and \(t \mapsto b_\varepsilon(\partial_t u_\varepsilon(t))\) are nonincreasing; in addition, the solution operator (semigroup) \(S_t\) is a contraction in \(H_\varepsilon\). For this case (4.11) follows from (4.7) by choosing \(v := u_\varepsilon\) and integrating in time. Estimate (4.12) can be obtained by choosing \(v := \partial_t u_\varepsilon\) and multiplying (4.7) by \(2t\): the identity

\[
2t a_\varepsilon(u_\varepsilon, \partial_t u_\varepsilon) = \partial_t \left( t a_\varepsilon(u_\varepsilon, u_\varepsilon) \right) - a(u_\varepsilon, u_\varepsilon)
\]

and a further integration in time yield (4.12).

Finally, (4.13) follows by writing the sum of (4.11) and (4.12) as

\[
\frac{1}{2}b_\varepsilon(u_\varepsilon(t)) + t a_\varepsilon(u_\varepsilon(t)) + 2 \int_0^t r b_\varepsilon(\partial_t u_\varepsilon(r)) \, dr = \frac{1}{2}b_\varepsilon(u_\varepsilon^0)
\]

and recalling that \(r \mapsto b_\varepsilon(\partial_t u_\varepsilon(r))\) is nonincreasing.

In order to extend them to all \(u_\varepsilon^0 \in H_\varepsilon\), we note that for fixed \(t > 0\) the two norms on \(H_\varepsilon\) given by (the square roots of)

\[
\begin{align*}
(4.14) & \quad \frac{1}{2}b_\varepsilon(u_\varepsilon^0) & \quad \text{and} \quad \frac{1}{2}b_\varepsilon(S_t u_\varepsilon^0) \quad \text{and} \quad \frac{1}{2}b_\varepsilon(S_t u_\varepsilon^0) + \int_0^t a_\varepsilon(S_t u_\varepsilon^0) \, dr
\end{align*}
\]

are identical by (4.11) on an \(H_\varepsilon\)-dense subset. If we approximate a general \(u_\varepsilon^0 \in H_\varepsilon\) by smooth \(u_{\varepsilon,n}^0\), then the sequence \(u_{\varepsilon,n}^0\) is a Cauchy sequence with respect to both norms; by copying the proof of completeness of the space \(L^2(0, \infty; V_\varepsilon)\) (see, e.g., [8, Theorem IV.8]) it follows that the integral in (4.15) converges. This allows us to pass to the limit in (4.11). The argument is similar for (4.14), which yields (4.13).

### 4.3. The Reaction-Diffusion Limit.
We now adopt the same point of view to formulate the limit reaction-diffusion system in the setting of measures. Recall that,
(Γ-2) For every $\sigma \in \mathcal{M}(\mathcal{D})$ with $f(\sigma) < \infty$ there exists $\sigma_\varepsilon \in \mathcal{M}(\mathcal{D})$ converging to $\sigma$ as $\varepsilon \downarrow 0$ such that

\[
\lim_{\varepsilon \downarrow 0} f_\varepsilon(\sigma_\varepsilon) = f(\sigma).
\]

(4.23)

Here we are mainly concerned with nonnegative quadratic forms: this means that $\text{Dom}(f_\varepsilon)$ is a linear space and $f_\varepsilon$ satisfy the parallelogram identity, so that $f_\varepsilon(\rho) = f_\varepsilon(\rho, \rho)$ for a (symmetric) bilinear form defined on $\text{Dom}(f_\varepsilon)$ by

\[
2f_\varepsilon(\rho, \sigma) := f_\varepsilon(\rho + \varepsilon) - f_\varepsilon(\rho) - f_\varepsilon(\sigma) \quad \text{for every } \rho, \sigma \in \text{Dom}(f_\varepsilon).
\]

(4.24)

It is worth noting that the $\Gamma$-limit of nonnegative quadratic forms is still a (weakly-* lower-semicontinuous) nonnegative quadratic form [10, Theorem 11.10]. In this case one of the most useful consequences of $\Gamma(\mathcal{M}(\mathcal{D}))$-convergence is contained in the next result (see, e.g., [23, Lemma 3.6]), which shows the link between $\Gamma$-convergence and (4.2).

**Lemma 4.2** (weak-strong convergence). Assume that $f_\varepsilon$ is a family of equicoercive nonnegative quadratic forms $\Gamma(\mathcal{M}(\mathcal{D}))$-converging to $f$ as $\varepsilon \downarrow 0$. Let $\rho_\varepsilon, \sigma_\varepsilon \in \mathcal{M}(\mathcal{D})$ be two families weakly-* converging to $\rho, \sigma$ as $\varepsilon \downarrow 0$ and satisfying the uniform bound

\[
\limsup_{\varepsilon \downarrow 0} f_\varepsilon(\rho_\varepsilon) < +\infty, \quad \limsup_{\varepsilon \downarrow 0} f_\varepsilon(\sigma_\varepsilon) < +\infty,
\]

(4.25)

so that $\rho, \sigma$ belong to the domain of the limit quadratic form $f$. We have

\[
\lim_{\varepsilon \downarrow 0} f_\varepsilon(\sigma_\varepsilon) = f(\sigma) \quad \Longrightarrow \quad \lim_{\varepsilon \downarrow 0} f_\varepsilon(\rho_\varepsilon, \sigma_\varepsilon) = f(\rho, \sigma).
\]

(4.26)

**Proof.** We reproduce here the proof of [23]. For every positive scalar $r > 0$ we have by (4.24) and the fact that $f_\varepsilon(\cdot, \cdot)$ is bilinear that

\[
2f_\varepsilon(\rho_\varepsilon, \sigma_\varepsilon) = 2f_\varepsilon(r \rho_\varepsilon, r^{-1} \sigma_\varepsilon) = f_\varepsilon(r \rho_\varepsilon + r^{-1} \sigma_\varepsilon) - r^2 f_\varepsilon(\rho_\varepsilon) - r^{-2} f_\varepsilon(\sigma_\varepsilon).
\]

Taking the inferior limit as $\varepsilon \downarrow 0$ and recalling (4.25), we get for $A := \limsup_{\varepsilon \downarrow 0} f_\varepsilon(\rho_\varepsilon)$

\[
\liminf_{\varepsilon \downarrow 0} 2f_\varepsilon(\rho_\varepsilon, \sigma_\varepsilon) \geq f(r \rho + r^{-1} \sigma) - r^2 A - r^{-2} f(\sigma) = 2f(\rho, \sigma) + r^2 (f(\rho) - A).
\]

Since $r > 0$ is arbitrary and $A$ is finite by (4.25), we obtain $\liminf_{\varepsilon \downarrow 0} f_\varepsilon(\rho_\varepsilon, \sigma_\varepsilon) \geq f(\rho, \sigma)$; inverting the sign of $\sigma$ we get (4.26). \[\square\]

If one wants to apply the previous lemma for passing to the limit in the integral formulation (4.10) of the Kramers-Smoluchowski equations to get (4.20), it is not hard to guess that for every testing measure $\sigma \in \text{Dom}(a)$ one needs to find a “joint” recovery family $\sigma_\varepsilon$ converging to $\sigma$ in $\mathcal{M}(\mathcal{D})$ such that

\[
\lim_{\varepsilon \downarrow 0} a_\varepsilon(\sigma_\varepsilon) = a(\sigma), \quad \lim_{\varepsilon \downarrow 0} b_\varepsilon(\sigma_\varepsilon) = b(\rho).
\]

(4.27)

This property is not guaranteed by the separate $\Gamma$-convergence of $a_\varepsilon$ and $b_\varepsilon$ to $a$ and $b$, respectively, since the property (Γ-2) would provide two a priori different families, say, $\sigma^a_\varepsilon, \sigma^b_\varepsilon$ satisfying (4.27) for $a$ and $b$.

The next result shows that one can overcome the above difficulty by studying the $\Gamma$-convergence of the quadratic forms $q^a_\varepsilon(\rho) := b_\varepsilon(\rho) + \kappa a_\varepsilon(\rho)$ depending on the parameter $\kappa > 0$.\[\square\]
4. $\Gamma$-convergence in the weak topology of $H$ of the quadratic forms $b + \kappa a_\varepsilon$ to $b + \kappa a$ for every $\kappa > 0$.

In the present case, where $b_\varepsilon$ does depend on $\varepsilon$, $\Gamma$-convergence of the extended quadratic forms $q^\varepsilon_\varepsilon = b_\varepsilon + \kappa a_\varepsilon$ with respect to the weak-* topology of $\mathcal{M}(\mathcal{D})$ is thus a natural extension of the latter condition; we will present in section 6 a simple and general argument showing how to derive the convergence of the evolution problems by the $\Gamma$-convergence of $q^\varepsilon_\varepsilon$, thus justifying the scheme of Figure 2.

Theorem 5.1 in the next section provides the crucial information, i.e., the $\Gamma$-convergence of $q^\varepsilon_\varepsilon$ to $q^\varepsilon$.

5. $\Gamma$-Convergence Result for the Quadratic Forms $q^\varepsilon_\varepsilon, a_\varepsilon, b_\varepsilon$. The aim of this section is to prove the following $\Gamma$-convergence result involving the quadratic forms $a_\varepsilon, b_\varepsilon, a, b$ defined in (4.8), (4.9) and (4.18), (4.19).

**Theorem 5.1.** The family of quadratic forms $b_\varepsilon$ is equicoercive, according to (4.21), and for every $\kappa > 0$ we have

\[
(5.1) \quad q^\varepsilon_\varepsilon(\rho) := b_\varepsilon(\rho) + \kappa a_\varepsilon(\rho) \quad \Gamma(\mathcal{M}(\mathcal{D}))\text{-converges to} \quad q^\varepsilon(\rho) := b(\rho) + \kappa a(\rho)
\]

as $\varepsilon \downarrow 0$, i.e., properties ($\Gamma$-1') and ($\Gamma$-2') of Lemma 4.3 hold.

We split the proof of Theorem 5.1 into various steps, focusing directly on the properties ($\Gamma$-1') and ($\Gamma$-2'), which are in fact equivalent to (5.1).

While the $\Gamma$-convergence of $b_\varepsilon$ is a direct consequence of the weak convergence of $\gamma_\varepsilon$ to $\gamma$, the convergence of $a_\varepsilon$ is more subtle, since the convergence of $a_\varepsilon$ and the structure of the limit depend critically on the choice of $\tau_\varepsilon$ (defined in (2.5)); as we will show in section 5.3, the scaling of $\tau_\varepsilon$ in terms of $\varepsilon$ is chosen exactly such that the strength of the “connection” between $\xi = -1$ and $\xi = 1$ is of order $O(1)$ as $\varepsilon \to 0$.

5.1. Equicoercivity. Let us first prove that the quadratic forms $b_\varepsilon$ satisfy the equicoercivity condition (4.21).

**Lemma 5.2** (equicoercivity of $b_\varepsilon$). Every family of measures $\rho_\varepsilon \in \mathcal{M}(\mathcal{D}), \varepsilon > 0$, satisfying

\[
(5.2) \quad \limsup_{\varepsilon \to 0} b_\varepsilon(\rho_\varepsilon) < +\infty
\]

is bounded in $\mathcal{M}(\mathcal{D})$ and admits a weakly-* converging subsequence.

**Proof.** The proof follows immediately by the fact that $\gamma_\varepsilon$ is a probability measure, and therefore

\[
|\rho_\varepsilon(\mathcal{D})| \leq \left( b_\varepsilon(\rho_\varepsilon) \right)^{1/2}.
\]

Inequality (5.2) thus implies that the total mass of $\rho_\varepsilon$ is uniformly bounded, and we can apply the relative weak-* compactness of bounded sets in dual Banach spaces.

5.2. Estimates near $\Omega \times \{-1, 1\}$.

**Lemma 5.3.** If $\rho_\varepsilon = u_\varepsilon \gamma_\varepsilon$ satisfies the uniform bound $a_\varepsilon(\rho_\varepsilon) \leq C < +\infty$ for every $\varepsilon > 0$, then for every $\delta \in (0, 1)$

\[
(5.3) \quad \partial_\varepsilon u_\varepsilon \to 0 \quad \text{in } L^2(\Omega \times \omega_\delta) \quad \text{as } \varepsilon \to 0,
\]

where $\omega_\delta := (-1, -\delta) \cup (\delta, 1)$.
On the other hand, thanks to (5.5), we have

\[
\lim_{\varepsilon \downarrow 0} \int_{\Omega} \left| u_\varepsilon(x) - \tilde{u}_\varepsilon(x) \right|^2 d\lambda_\Omega(x)
\]
\[
= \lim_{\varepsilon \downarrow 0} J_\varepsilon \int_{\Omega} \int_{-1}^{1} \psi(\xi)(u_\varepsilon(x, \xi) - u^-(x)) d\gamma_\varepsilon(\xi) d\lambda_\Omega(x)
\]
\[
\leq \lim_{\varepsilon \downarrow 0} \int_{\Omega} \psi(\xi)\omega_\varepsilon^2(x) d\gamma_\varepsilon(x, \xi) = 0,
\]
which yields (5.4).

Remark. A completely analogous argument shows that if \( \rho_\varepsilon \) satisfies a \( W^{1,1}(D; \gamma_\varepsilon) \)-uniform bound

\[
\int_D \| \nabla_x u_\varepsilon \|_C d\gamma_\varepsilon(x, \xi) \leq C < +\infty
\]

instead of \( a_\varepsilon(\rho_\varepsilon) \leq C \), then \( u^\pm_\varepsilon \to u^\pm \) in \( L^1(\Omega) \).

5.3. Asymptotics for the Minimal Transition Cost. Given \( (\varphi^-, \varphi^+) \in \mathbb{R}^2 \), let us set

\[
K_\varepsilon(\varphi^-, \varphi^+) := \min \left\{ \kappa_\varepsilon \int_{-1}^{1} \left( \varphi'(\xi) \right)^2 d\gamma_\varepsilon : \varphi \in W^{1,2}(-1, 1), \varphi(\pm 1) = \varphi^\pm \right\}.
\]

It is immediate to check that \( K_\varepsilon \) is a quadratic form depending only on \( \varphi^+ - \varphi^- \); i.e.,

\[
K_\varepsilon(\varphi^-, \varphi^+) = K_\varepsilon(\varphi^+ - \varphi^-)^2, \quad K_\varepsilon = K_\varepsilon(-1/2, 1/2).
\]

We call \( \mathcal{T}_\varepsilon(\varphi^-, \varphi^+) \) the solution of the minimum problem (5.7): it admits the simple representation

\[
\mathcal{T}_\varepsilon(\varphi^-, \varphi^+) = \frac{1}{2}(\varphi^- + \varphi^+) + (\varphi^+ - \varphi^-)\phi_\varepsilon,
\]

where \( \phi_\varepsilon = \mathcal{T}_\varepsilon(-1/2, 1/2) \). We also set

\[
Q_\varepsilon(\varphi^-, \varphi^+) := \int_{-1}^{1} \left( \mathcal{T}_\varepsilon(\varphi^-, \varphi^+) \right)^2 d\gamma_\varepsilon = \frac{1}{2}((\varphi^-)^2 + (\varphi^+)^2) + (q_\varepsilon - \frac{1}{4})(\varphi^+ - \varphi^-)^2,
\]

where

\[
q_\varepsilon := \int_{-1}^{1} |\phi_\varepsilon(\xi)|^2 d\gamma_\varepsilon(\xi) = Q_\varepsilon(-1/2, 1/2).
\]

Lemma 5.5. We have

\[
\lim_{\varepsilon \downarrow 0} k_\varepsilon = \frac{k}{2} = \frac{\sqrt{-H''(0)H''(1)}}{2\pi}
\]

and

\[
\lim_{\varepsilon \downarrow 0} q_\varepsilon = \frac{1}{4}, \quad \text{so that} \quad \lim_{\varepsilon \downarrow 0} Q_\varepsilon(\varphi^-, \varphi^+) = \frac{1}{2}(\varphi^-)^2 + \frac{1}{2}(\varphi^+)^2.
\]
and it is easy to check that

\[(5.17) \quad \tilde{u}_e^\pm \to \frac{1}{2} u^\pm \text{ in } \mathscr{G}^r(\Omega). \]

We also set \( \theta_e := \int_{-1}^{1} \eta^+(\xi) \, d\tilde{\gamma}_e(\xi) \) \((= \int_{-1}^{1} \eta^-(\xi) \, d\tilde{\gamma}_e(\xi))\), observing that \( \theta_e \to 1/2 \).

We then have by the Jensen inequality and the assumption on the support of \( \eta^+ \)

\[ a^+_e(\rho_e) \geq \int_{\Omega} \int_{-1}^{1} (\eta^-(\xi)+\eta^+(\xi)) |\nabla_x u_e(x,\xi)|^2 \, d\tilde{\gamma}_e(\xi) \, d\lambda_\Omega \geq \theta_e^{-1} \int_{\Omega} |\nabla \tilde{u}_e^-|^2 + |\nabla \tilde{u}_e^+|^2 \, d\lambda_\Omega, \]

and, passing to the limit,

\[ \liminf_{\varepsilon \to 0} a^+_e(\rho_e) \geq \frac{1}{2} \int_{\Omega} |\nabla u^-|^2 + |\nabla u^+|^2 \, d\lambda_\Omega. \]

Let us now consider the behavior of \( a^-_e \): applying (5.7) and (5.8) we get

\[ a^-_e(\rho_e) = \int_{\Omega} \left( \tau_e \int_{-1}^{1} (\partial_k u_e(x,\xi))^2 \, d\tilde{\gamma}_e(\xi) \right) \, d\lambda_\Omega \geq \int_{\Omega} k_e(\tilde{u}_e^- - \tilde{u}_e^+)^2 \, d\lambda_\Omega, \]

so that by (5.12) and (5.4) we obtain

\[ \liminf_{\varepsilon \to 0} a^-_e(\rho_e) \geq \frac{k}{2} \int_{\Omega} (u^-(x) - u^+(x))^2 \, d\lambda_\Omega. \]

We want to prove the following property (4.31) of Lemma 4.3. We fix \( \sigma = u^\gamma \) with \( u \) in the domain of the quadratic forms \( a \) and \( b \) so that \( u^\pm = u(\cdot, \pm 1) \) belong to \( W^{1,2}(\Omega) \), and we set \( \sigma_e = u_e \gamma_e \), where \( u_e(x,\xi) = \mathcal{F}_e(u^-(x), u^+(x)) \) as in (5.9). We easily have, by (5.13) and the Lebesgue dominated convergence theorem,

\[ \lim_{\varepsilon \to 0} b_e(\sigma_e) = \lim_{\varepsilon \to 0} \int_{\Omega} Q_e(u^-(x), u^+(x)) \, d\lambda_\Omega = \int_{\Omega} \left( \frac{1}{2} |u^-(x)|^2 + \frac{1}{2} |u^+(x)|^2 \right) \, d\lambda_\Omega = b(\sigma). \]

Similarly, since for every \( j = 1, \ldots, d \) and almost every \( x \in \Omega \)

\[ \partial_{x_j} u_e(x,\xi) = \mathcal{F}(\partial_{x_j} u^-(x), \partial_{x_j} u^+), \]

we have

\[ \lim_{\varepsilon \to 0} a_e(\sigma_e) = \lim_{\varepsilon \to 0} \int_{\Omega} \left( \sum_{j=1}^{d} Q_e(\partial_{x_j} u^-(x), \partial_{x_j} u^+(x)) + K_e(u^-(x), u^+(x)) \right) \, d\lambda_\Omega \]

\[ = \int_{\Omega} \left( \frac{1}{2} |\nabla u^-(x)|^2 + \frac{1}{2} |\nabla u^+(x)|^2 + \frac{k}{2} (u^-(x) - u^+(x)) \right) \, d\lambda_\Omega = a(\sigma). \]

6. From \( \Gamma \)-Convergence to Convergence of the Evolution Problems: Proof of Theorems 3.1 and 3.2. Having at our disposal the \( \Gamma \)-convergence result of Theorem 5.1 and Lemma 4.2, it is not difficult to pass to the limit in the integrated evolution equation (4.10).

The proof of Theorems 3.1 and 3.2 is a consequence of the following general result.

**Theorem 6.1 (convergence of evolution problems).** Let us consider weakly-* lower-semicontinuous, nonnegative, and extended-valued quadratic forms \( a_e, b_e, a, b \) defined on \( \mathcal{M}(\Omega) \), and let us suppose the following.
so that $\rho$ is a solution of the limit equation. Since the limit is uniquely identified by
the nondegeneracy and density assumption (1), we conclude that the whole family $\rho_\varepsilon$
converges to $\rho$ as $\varepsilon \downarrow 0$. In particular, $\rho$ satisfies the identity

$$
\frac{1}{2} b(\rho(t)) + \int_0^t a(\rho(r)) \, dr = \frac{1}{2} b(\rho^0) \quad \text{for every } t \geq 0.
$$

This concludes the proof of (6.2) (and of Theorem 3.1).

In order to prove (6.3) (and Theorem 3.2), we note that by (4.11) and (6.6) we easily get

$$
\limsup_{\varepsilon \downarrow 0} \frac{1}{2} b_\varepsilon(\rho_\varepsilon(t)) + \int_0^t a_\varepsilon(\rho_\varepsilon(r)) \, dr \leq \frac{1}{2} b(\rho(t)) + \int_0^t a(\rho(r)) \, dr.
$$

The lower-semicontinuity property (4.30) and Fatou’s lemma yield

$$
\lim_{\varepsilon \downarrow 0} b_\varepsilon(\rho_\varepsilon(t)) = b(\rho(t)), \quad \lim_{\varepsilon \downarrow 0} \int_0^t a_\varepsilon(\rho_\varepsilon(r)) \, dr = \int_0^t a(\rho(r)) \, dr
$$

for every $t \geq 0$. Applying the same argument to (4.12) and its "$\varepsilon = 0$" analogue, we conclude that $a_\varepsilon(\rho_\varepsilon(t)) \to a(\rho(t))$ for every $t > 0$. \qed

Remark (more general ambient spaces). The particular structure of $\mathcal{M}(\overline{D})$ did not play any role in the previous argument, so the validity of the above result can be easily extended to general topological vector spaces (e.g., the dual of separable Banach spaces with their weak-$*$ topology) once the equicoercivity condition of $b_\varepsilon$ (as in Lemma 5.2) is satisfied.

REFERENCES


Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.