# Long-time behavior and weak-strong uniqueness for incompressible viscoelastic flows

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# Outline

(1) Introduction

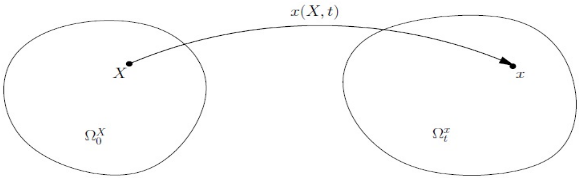
(2) Main results and ideas of their proof

- Decomposition of the viscoelastic system
- Well-posedness revisited
- Optimal $L^2$-decay of global solutions near equilibrium
- Weak-strong uniqueness

(3) Open problems

Joint work with X.-P. Hu (Courant Institute of Mathematical Sciences).

# The flow map




- Let $\Omega_0^X$ be the *reference domain*, $\Omega_t^x$ be the *deformed domain* at time $t$ with variables $X$ and $x$, respectively.
  - $X \sim$ the Lagrangian coordinate system
  - $x \sim$ the Eulerian coordinate system
- Flow map $x(X,t) : \Omega_0^X \to \Omega_t^x$

$$\begin{cases} \dfrac{\partial}{\partial t} x(X,t) = \mathbf{u}(x(X,t),t), & t > 0, \\ x(X,0) = X. \end{cases}$$

It links the two coordinate systems: Both sides describe the velocity of a particle labeled with $X$ at position $x$ and time $t$.

# The deformation tensor F

- Let $\widetilde{\mathsf{F}}$ be the Jacobian matrix of the map $X \to x(X,t)$ defined by

$$\widetilde{\mathsf{F}}(X,t) = \frac{\partial x(X,t)}{\partial X}.$$

- Push forward to the Eulerian coordinate:

$$\mathsf{F}(x(X,t),t) = \widetilde{\mathsf{F}}(X,t)$$

  By chain rule,

$$\mathsf{F}_t + \mathbf{u} \cdot \nabla_x \mathsf{F} = \nabla_x \mathbf{u} \mathsf{F}.$$

- The deformation tensor F carries all the information about how the configuration is deformed with respect to the reference configuration, including microstructures, patterns etc.

# Kinematic transport of liquid crystal

- $\mathbf{d}$: orientation director of nematic liquid crystal molecules
- Kinematic transport relation

$$\text{Rod-like molecule} \quad \mathbf{d}(x(X,t),t) = \mathsf{F}\mathbf{d}_0(X)$$
$$\text{General ellipsoidal shapes} \quad \mathbf{d}(x(X,t),t) = \mathbb{E}\mathbf{d}_0(X)$$

where $\mathbb{E}(x(X,t),t)$ satisfies

$$\mathbb{E}_t + \mathbf{u}\cdot\nabla_x\mathbb{E} = S\mathbb{E} + (2\alpha-1)A\mathbb{E},$$

$A = \frac{1}{2}(\nabla_x\mathbf{u} + \nabla_x^T\mathbf{u})$ and $S = \frac{1}{2}(\nabla_x\mathbf{u} - \nabla_x^T\mathbf{u})$,
$2\alpha - 1 = \frac{r^2-1}{r^2+1}$ with $r$ being the aspect ratio of the ellipsoids.

- The general transport equation for $\mathbf{d}$:

$$\mathbf{d}_t + u\cdot\nabla_x\mathbf{d} - S\mathbf{d} - (2\alpha-1)A\mathbf{d} = 0.$$

It includes the transport of the center of mass and the rotating/stretching effect of the director $\mathbf{d}$ under the flow.

# The incompressible viscoelastic system

Consider the Cauchy problem in for $\mathbb{R}^d$, $d = 2, 3$:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \nabla \cdot \left( \frac{\partial W(\mathsf{F})}{\partial \mathsf{F}} \mathsf{F}^T \right), \\ \nabla \cdot \mathbf{u} = 0, \\ \mathsf{F}_t + \mathbf{u} \cdot \nabla \mathsf{F} = \nabla \mathbf{u} \mathsf{F}, \\ \mathbf{u}(t, x)|_{t=0} = \mathbf{u}_0(x), \quad \mathsf{F}(t, x)|_{t=0} = \mathsf{F}_0(x), \end{cases} \tag{1}$$

- $W(\mathsf{F})$: the elastic energy functional
- $\frac{\partial W(\mathsf{F})}{\partial \mathsf{F}}$: the Piola–Kirchhoff stress tensor
- $\frac{\partial W(\mathsf{F})}{\partial \mathsf{F}} \mathsf{F}^T$: the Cauchy–Green tensor
  - The latter is the change variable form of the former one (from Lagrangian to Eulerian coordinates).
- For simplicity, we confine ourselves to the Hookean linear elasticity

$$W(\mathsf{F}) = \frac{1}{2} |\mathsf{F}|^2.$$

# Some remarks

- System (1) is an important model in complex fluids
  - It presents the competition between the elastic energy and the kinetic energy.
  - The deformation tensor F carries all the kinematic transport information of the micro-structures and configurations.
- Equivalent to the classical Oldroyd-B model for viscoelastic fluids in the case of infinite Weissenberg number.
- From mathematical point of view:
  a coupling of a “parabolic” system for $\mathbf{u}$ with a “hyperbolic” system for F.

# Well-posedness results in the literature

Existence and uniqueness of local classical solutions and global classical solutions near-equilibrium.

(A) Incompressible case:

- 2D: Lin-Liu-Zhang 2005 CPAM, Lei-Zhou 2005 SIMA
- 3D: Chen-Zhang 2006 CPDE, Lei-Liu-Zhou 2008 ARMA
- Small strain/large rotation in 2D: Lei-Liu-Zhou 2007 CMS, Lei 2010 ARMA
- Critical space: Qian 2010 NA, Zhang-Fang 2012 SIMA
- Initial boundary value problem: Lin-Zhang 2008 CPAM

(B) Compressible case:

- Well-posedness: Hu-Wang 2010/2011/2012 JDE, Qian-Zhang 2010 ARMA, Qian 2011 JDE
- Decay estimate: Hu-Wu G.C. 2013 SIMA

# Basic energy law

- Total energy: kinetic + elastic

$$\mathcal{E}(t) = \frac{1}{2}\|\mathbf{u}(t)\|_{L^2}^2 + \frac{1}{2}\|\mathsf{F}\|_{L^2}^2.$$

- Basic energy law

$$\frac{d}{dt}\mathcal{E}(t) + \mu\|\nabla \mathbf{u}\|_{L^2}^2 = 0,$$

  $\Longrightarrow$ absence of damping mechanism in F.

- Partial dissipation structure is the **Main Difficulty** in the study of global existence of smooth solutions near equilibrium and its long-time behavior.

# Basic properties I

## Lemma (Incompressibility)

*Assume that* $\det \mathsf{F}_0 = 1$*, then*

$$\det \mathsf{F}(t,x) = 1, \quad \textit{for}\ t \geq 0,\ x \in \mathbb{R}^d,$$

*which is equivalent to*

$$\nabla \cdot \mathbf{u} = 0.$$

## Lemma (div-curl structure)

*Assume that* $\nabla \cdot \mathsf{F}_0^T = 0$*, then*

$$\nabla \cdot \mathsf{F}^T(t,x) = 0, \quad \textit{for}\ t \geq 0,\ x \in \mathbb{R}^d.$$

# Basic properties II

Let $\mathbb{I}$ be the $d \times d$ identity matrix. Introduce the strain tensor

$$\mathsf{E} = \mathsf{F} - \mathbb{I}.$$

## Lemma

*Assume that*

$$\nabla_m \mathsf{E}_{0ij} - \nabla_j \mathsf{E}_{0im} = \mathsf{E}_{0lj} \nabla_l \mathsf{E}_{0im} - \mathsf{E}_{0lm} \nabla_l \mathsf{E}_{0ij},$$

*then for* $t \geq 0$, $x \in \mathbb{R}^d$

$$\nabla_m \mathsf{E}_{ij}(t,x) - \nabla_j \mathsf{E}_{im}(t,x) = \mathsf{E}_{lj}(t,x) \nabla_l \mathsf{E}_{im}(t,x) - \mathsf{E}_{lm}(t,x) \nabla_l \mathsf{E}_{ij}(t,x).$$

$\Longrightarrow$ $\nabla \times \mathsf{E}$ is indeed a higher-order nonlinearity.

# Key to global existence near equilibrium

- $2D$: Lin-Liu-Zhang 2005 CPAM
  There exists a vector $\phi = (\phi_1, \phi_2)$ such that

$$\mathsf{F} = \nabla^\perp \phi = \begin{pmatrix} -\partial_2\phi_1 & -\partial_2\phi_2 \\ \partial_1\phi_1 & \partial_1\phi_2 \end{pmatrix}.$$

  Find damping effect for

$$\mathbf{w} = \mathbf{u} - \mu^{-1}(\phi(x) - x).$$

- $3D$: Lei-Liu-Zhou 2008 ARMA
  For the strain tensor $\mathsf{E} = \mathsf{F} - \mathbb{I}$, introducing

$$\mathbf{w} = -\Delta\mathbf{u} + \mu^{-1}\nabla \cdot \mathsf{E},$$

  which has a damping effect, then using the fact that $\nabla \times \mathsf{E}$ is indeed a higher-order nonlinear term.

# The Cauchy problem

Consider

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \nabla\cdot\mathsf{E} + \nabla\cdot\left(\mathsf{E}\mathsf{E}^T\right), \\ \nabla\cdot\mathbf{u} = 0, \\ \mathsf{E}_t + \mathbf{u}\cdot\nabla\mathsf{E} = \nabla\mathbf{u} + \nabla\mathbf{u}\mathsf{E}, \\ \mathbf{u}(t,x)|_{t=0} = \mathbf{u}_0(x), \quad \mathsf{E}(t,x)|_{t=0} = \mathsf{E}_0(x) = \mathsf{F}_0(x) - \mathbb{I}, \end{cases} \tag{2}$$

with following structural assumptions on the initial data:

(A1) $\det(\mathsf{E}_0 + \mathbb{I}) = 1, \quad \nabla\cdot\mathbf{u}_0 = 0,$

(A2) $\nabla\cdot\mathsf{E}_0^T = 0,$

(A3) $\nabla_m\mathsf{E}_{0ij} - \nabla_j\mathsf{E}_{0im} = \mathsf{E}_{0lj}\nabla_l\mathsf{E}_{0im} - \mathsf{E}_{0lm}\nabla_l\mathsf{E}_{0ij}.$

# Main result 1: Part I - Global existence

## Theorem

*Suppose that* $d = 3$ *and the initial data* $\mathbf{u}_0, \mathsf{E}_0 \in H^k(\mathbb{R}^3)$ $(k \geq 2$ *being an integer*) *fulfill the assumptions (A1)–(A3).*
*If the initial data satisfy*

$$\|\mathbf{u}_0\|_{H^2} + \|\mathsf{E}_0\|_{H^2} \leq \delta$$

*for certain sufficiently small* $\delta > 0$*, then the Cauchy problem* (2) *admits a unique global classical solution* $(\mathbf{u}, \mathsf{E})$ *such that*

$$\begin{cases} \partial_t^j \nabla^\alpha \mathbf{u} \in L^\infty(0,T; H^{k-2j-|\alpha|}(\mathbb{R}^3)) \cap L^2(0,T; H^{k-2j-|\alpha|+1}(\mathbb{R}^3)) \\ \partial_t^j \nabla^\alpha \mathsf{E} \in L^\infty(0,T; H^{k-2j-|\alpha|}(\mathbb{R}^3)) \end{cases}$$

*for all integer* $j$ *and multi-index* $\alpha$ *satisfying* $2j + |\alpha| \leq k$*. Moreover,*

$$\|\mathbf{u}(t)\|_{H^2} + \|\mathsf{E}(t)\|_{H^2} \leq 4\delta, \quad \forall\, t \geq 0, \qquad \int_0^{+\infty} \|\nabla\Delta\mathbf{u}(t)\|_{L^2}^2 dt \leq C,$$

*where the constant* $C$ *depends on* $\delta$*.*

# Main result 1: Part II - Optimal decay in $L^2$

## Theorem (Continued)

*If the initial data also satisfy* $\mathbf{u}_0, \mathsf{E}_0 \in L^1(\mathbb{R}^3)$*, then for the above unique global classical solution* $(\mathbf{u}, \mathsf{E})$ *to (2), the following decay estimates hold for all* $t \geq 0$*,*

$$\begin{aligned} \|\mathbf{u}(t)\|_{L^2} + \|\mathsf{E}(t)\|_{L^2} &\leq CM(1+t)^{-\frac{3}{4}}, \\ \|\nabla\mathbf{u}(t)\|_{H^1} + \|\nabla\mathsf{E}(t)\|_{H^1} &\leq CM(1+t)^{-\frac{5}{4}}, \end{aligned}$$

*where* $M = \|\mathbf{u}_0\|_{L^1\cap H^2} + \|\mathsf{E}_0\|_{L^1\cap H^2}$.
*Moreover, if the Fourier transforms of* $(\mathbf{u}_0, \mathbf{n}_0)$ *(where* $\mathbf{n}_0 = \Lambda^{-1}\nabla\cdot\mathsf{E}_0$*) also satisfy* $|\widehat{u_{i0}}| \geq c_0$*,* $|\widehat{n_{i0}}| \geq c_0$ *for* $0 \leq |\xi| << 1$*, where* $c_0 > 0$ *satisfies* $c_0 \sim O(\delta^\zeta)$ *with* $\zeta \in (0,1)$*, then there exists* $t_0 >> 1$ *such that*

$$\|\mathbf{u}(t)\|_{L^2} + \|\mathsf{E}(t)\|_{L^2} \geq C(1+t)^{-\frac{3}{4}}, \quad \forall\, t \geq t_0,$$

*i.e., the* $L^2$ *decay rate is optimal.*

# Idea of proof - decomposed system

- Let $\Lambda^s$ be the pseudo differential operator defined by

$$\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s \widehat{f}(\xi)), \quad s \in \mathbb{R}, \quad \text{e.g., } \Lambda^2 = -\Delta.$$

- Introduce new variables

$$\begin{cases} \mathbf{n} = \Lambda^{-1}(\nabla \cdot \mathsf{E}), \\ \Omega = \Lambda^{-1}(\nabla \mathbf{u} - \nabla^T \mathbf{u}), \\ \mathbb{E} = \mathsf{E}^T - \mathsf{E}. \end{cases}$$

- The "decomposed systems"

$$\begin{cases} \mathbf{u}_t - \mu \Delta \mathbf{u} - \Lambda \mathbf{n} = \mathbf{g}, \\ \mathbf{n}_t + \Lambda \mathbf{u} = \Lambda^{-1} \nabla \cdot \mathbf{h}, \end{cases} \tag{3}$$

$$\begin{cases} \Omega_t - \mu \Delta \Omega - \Lambda \mathbb{E} = \Lambda^{-1}(\nabla \mathbf{g} - \nabla^T \mathbf{g}) + \Lambda^{-1} \mathsf{S}, \\ \mathbb{E}_t + \Lambda \Omega = \mathbf{h}^T - \mathbf{h}, \end{cases} \tag{4}$$

# The decomposed system

- The nonlinearities $\mathbf{g}$ and $\mathbf{h}$ are given by

$$\mathbf{g} = -\mathbb{P}(\mathbf{u}\cdot\nabla\mathbf{u}) + \mathbb{P}\left(\nabla\cdot\left(\mathsf{E}\mathsf{E}^T\right)\right),$$
$$\mathbf{h} = -\mathbf{u}\cdot\nabla\mathsf{E} + \nabla\mathbf{u}\mathsf{E},$$

where

$$\mathbb{P} = \mathbb{I} - \Delta^{-1}(\nabla\otimes\nabla)$$

is the Leray projection operator.

- The higher-order anti-symmetric matrix S is given by

$$\mathsf{S}_{ij} = \nabla_k(\mathsf{E}_{lk}\nabla_l\mathsf{E}_{ij} - \mathsf{E}_{lj}\nabla_l E_{ik}) - \nabla_k(E_{lk}\nabla_l\mathsf{E}_{ji} - \mathsf{E}_{li}\nabla_l\mathsf{E}_{jk}).$$

# Some remarks

- Inspired by the decomposition in **Danchin 2000 Invent. Math.** for compressible Navier–Stokes equations.

  The linearized system for the density $\rho$ and the "compressible part" of the velocity $c = \Lambda^{-1}\nabla \cdot \mathbf{u}$ has a parabolic smoothing effect on $c$, and on $\rho$ in the low frequencies; a damping effect on $\rho$ in the high frequencies.

  $\Longrightarrow$ global existence of strong solutions near equilibrium (in critical spaces).

- The linearized systems (3)–(4) for $(\mathbf{u}, \mathbf{n})$ and $(\Omega, \mathbb{E})$ have a structure **similar** to the linearized system for $(\rho, c)$ of the compressible N–S equations.

$$u_i \sim c, \quad n_i \sim \rho.$$

- System (4) provides some **extra dissipation** on $\mathbb{E}$, which helps us to prove the existence of global smooth solutions near equilibrium.

# Useful estimates

## Lemma

*Let assumptions (A1)–(A3) be satisfied. The solution* E *to* (2) *satisfy:*

$$\begin{aligned} \|\mathsf{E}\|_{L^2} &\le C\left(\|\mathbf{n}\|_{L^2} + \|\mathsf{E}\|_{H^2}\|\mathsf{E}\|_{L^2}\right), \\ \|\nabla\mathsf{E}\|_{L^2} &\le C\left(\|\nabla\mathbf{n}\|_{L^2} + \|\mathsf{E}\|_{H^2}\|\nabla\mathsf{E}\|_{L^2}\right), \\ \|\Delta\mathsf{E}\|_{L^2} &\le C\left(\|\Delta\mathbb{E}\|_{L^2} + C\|\mathsf{E}\|_{H^2}\|\Delta\mathsf{E}\|_{L^2}\right). \end{aligned}$$

*where $C$ is a constant that does not depend on* E*.*

- If $\|\mathsf{E}\|_{H^2}$ is sufficiently small, then the norm of E can be controlled by the norms of $\mathbf{n}$ and $\mathbb{E}$.

# Global existence near equilibrium revisited

## Lemma

*Let* $(\mathbf{u}, \mathsf{E})$ *be a smooth solution to problem* (2)*, the following inequalities hold:*

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\|\Delta \mathbf{u}\|_{L^2}^2 + \|\Delta \mathsf{E}\|_{L^2}^2\right) + \mu\|\nabla\Delta \mathbf{u}\|_{L^2}^2 \\ \leq \quad & C\left(\|\mathbf{u}\|_{H^2} + \|\mathsf{E}\|_{H^2}\right)\left(\|\Delta \mathsf{E}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla\Delta \mathbf{u}\|_{L^2}^2\right), \end{aligned}$$

*and*

$$\begin{aligned} & \frac{d}{dt}(\Lambda\Omega, \Delta\mathbb{E}) + \frac{1}{2}\|\Delta\mathbb{E}\|_{L^2}^2 \\ \leq \quad & C\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla\Delta \mathbf{u}\|_{L^2}^2\right) + C\left(\|\mathbf{u}\|_{H^2}^2 + \|\mathsf{E}\|_{H^2}^2\right)\left(\|\Delta \mathbf{u}\|_{L^2}^2 + \|\Delta \mathsf{E}\|_{L^2}^2\right). \end{aligned}$$

*The constant* $C$ *is independent of the variables* $\mathbf{u}$*,* $\mathsf{E}$*,* $\Omega$ *and* $\mathbb{E}$*.*

# Global existence near equilibrium revisited

Denote

$$G(t) = \frac{1}{2}\left(\|\mathbf{u}\|_{L^2}^2 + \|\mathsf{E}\|_{L^2}^2 + \|\Delta\mathbf{u}\|_{L^2}^2 + \|\Delta\mathsf{E}\|_{L^2}^2\right) + \kappa(\Lambda\Omega, \Delta\mathbb{E}),$$

where $0 < \kappa << 1$. Then

$$G(t) \approx \|\mathbf{u}\|_{H^2}^2 + \|\mathsf{E}\|_{H^2}^2$$

and

$$\frac{d}{dt}G(t) + \left[C_1 - C_2\left(G(t) + G^{\frac{1}{2}}(t)\right)\right]\left(\|\Delta\mathsf{E}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\Delta\mathbf{u}\|_{L^2}^2\right) \le 0,$$

$C_1, C_2$ are independent of $\mathbf{u}$, E, $\Omega$ and $\mathbb{E}$.

- $G(0)$ is small $\Longrightarrow$ $G(t)$ is uniformly bounded for all $t \ge 0$.

# Long-time behavior

- Optimal decay rates for weak solutions of incompressible Navier–Stokes equations
  $\sim$ "the Fourier splitting method" due to Prof. M. Schonbek (series work in 1980's-1990's)
  - Application to the liquid crystal system (Dai-Qing-Schonbek 2012 CPDE):
    Decay for the director filed $\mathbf{d}$ + the Fourier splitting technique
    $\Longrightarrow$ decay for the velocity $\mathbf{u}$ in $L^2$.
- In the current case: lack of dissipation for F (or E) !
- Basic strategy:
  - Spectral analysis of the linearized problem of the decomposed system (3) for $(\mathbf{u}, \mathbf{n})$ in terms of the decomposition of wave modes at both lower and higher frequencies
  - Duhamel's principle + the fact of small global solution

$\Longrightarrow$ Decay of the nonlinear system.

# Decay of linear system

## Lemma

*Let* $(\mathbf{u}(t), \mathbf{n}(t))$ *be the solution to the linear problem*

$$\begin{cases} \mathbf{u}_t - \mu\Delta\mathbf{u} - \Lambda\mathbf{n} = 0, \\ \mathbf{n}_t + \Lambda\mathbf{u} = 0. \end{cases}$$

*with initial data* $(\mathbf{u}_0, \mathbf{n}_0) \in H^l(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$*.*
*(i) For* $0 \leq |\alpha| \leq l$ *and* $t \geq 0$*,*

$$\|(\partial_x^\alpha \mathbf{u}(t), \partial_x^\alpha \mathbf{n}(t))\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}} \left(\|(\mathbf{u}_0, \mathbf{n}_0)\|_{L^1} + \|\partial_x^\alpha (\mathbf{u}_0, \mathbf{n}_0)\|_{L^2}\right).$$

*(ii) Assume that* $\widehat{\mathbf{u}_0}$ *and* $\widehat{\mathbf{n}_0}$ *satisfy* $|\widehat{u_{i0}}| \geq c_0 > 0$*,* $|\widehat{n_{i0}}| \geq c_0$ *for* $0 \leq |\xi| << 1$ *with* $c_0$ *being a certain positive constant. Then for* $t$ *sufficiently large,*

$$Cc_0(1+t)^{-\frac{3}{4}} \leq \|(\mathbf{u}(t), \mathbf{n}(t))\|_{L^2} \leq C(1+t)^{-\frac{3}{4}},$$

*i.e., the* $L^2$*-decay rates are optimal.*

# Sketch of proof

- Linear operator

$$\mathcal{B} = \begin{pmatrix} \mu\Delta & \Lambda \\ -\Lambda & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{A}(\xi) = \begin{pmatrix} -\mu|\xi|^2 & |\xi| \\ -|\xi| & 0 \end{pmatrix}.$$

- Eigenvalues of $\mathcal{A}(\xi)$

$$\lambda_{\pm}(\xi) = \begin{cases} -\frac{\mu}{2}|\xi|^2 \pm \frac{\mathrm{i}}{2}\sqrt{-\mu^2|\xi|^4 + 4|\xi|^2}, & \text{if } |\xi| < \frac{2}{\mu}, \\ -\frac{\mu}{2}|\xi|^2 \pm \frac{1}{2}\sqrt{\mu^2|\xi|^4 - 4|\xi|^2}, & \text{if } |\xi| \geq \frac{2}{\mu}. \end{cases}$$

- Green's function for the semigroup

$$\widehat{G}(t,\xi) := e^{t\mathcal{A}(\xi)} = \begin{pmatrix} -\mu|\xi|^2\frac{(e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} - \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} & |\xi|\frac{(e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} \\ -|\xi|\frac{(e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} & -\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{pmatrix}$$

- Investigate the behavior of Green's function for both low frequency and high frequency.

# Decay of nonlinear problem 1

- Introduce

$$H(t) = \frac{1}{2}(\|\Delta \mathbf{u}\|_{L^2}^2 + \|\Delta \mathsf{E}\|_{L^2}^2) + \kappa(\Lambda\Omega, \Delta\mathbb{E}),$$

$$\widetilde{H}(t) = \sup_{0\le s\le t} (1+s)^{\frac{5}{2}} \Big( H(t) + \frac{1}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathsf{E}\|_{L^2}^2 \Big),$$

where $\kappa > 0$ is a sufficiently small constant such that

$$H(t) + \frac{1}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathsf{E}\|_{L^2}^2 \approx \|\nabla \mathbf{u}\|_{H^1}^2 + \|\nabla \mathsf{E}\|_{H^1}^2.$$

## Lemma

*For any $t \ge 0$, the global solution $(\mathbf{u}, \mathsf{E})$ (near equilibrium) satisfies*

$$\|\mathbf{u}(t)\|_{L^2} + \|\mathsf{E}(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}} \Big( \|\mathbf{u}_0\|_{L^1\cap L^2} + \|\mathsf{E}_0\|_{L^1\cap L^2} + \delta \widetilde{H}^{\frac{1}{2}}(t) \Big),$$

$$\|\nabla\mathbf{u}(t)\|_{L^2} + \|\nabla\mathsf{E}(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}} \Big( \|\mathbf{u}_0\|_{L^1\cap H^1} + \|\mathsf{E}_0\|_{L^1\cap H^1} + \delta \widetilde{H}^{\frac{1}{2}}(t) \Big).$$

# Sketch of proof

- Consider the nonlinear system for $\mathbf{W}_i = (u_i, n_i)^T$ $(i = 1, 2, 3)$

$$\begin{cases} \mathbf{W}_{it} = \mathcal{B}\mathbf{W}_i + \mathbf{f}_i, \\ \mathbf{W}_i(t,x)|_{t=0} = \mathbf{W}_{i0} := (u_{i0}, \Lambda^{-1}(\nabla \cdot \mathsf{E})_{i0})^T, \end{cases}$$

  where

$$\mathbf{f}_i = (g_i, \Lambda^{-1}(\nabla \cdot \mathbf{h})_i)^T.$$

- From the Duhamel's principle,

$$\mathbf{W}_i(t) = e^{t\mathcal{B}}\mathbf{W}_{i0} + \int_0^t e^{(t-s)\mathcal{B}}\mathbf{f}_i(s)ds.$$

- Using linear decay estimates and the elementary inequality

$$\int_0^t (1+t-s)^{-\gamma}(1+s)^{-\beta} ds \le C(1+t)^{-\gamma}, \quad \forall\, t \ge 0,$$

  for $\beta > 1$, $\beta \ge \gamma \ge 0$.

# Decay of nonlinear problem 2

If $\widetilde{H}(t)$ is uniformly bounded, then we can infer

## Lemma

*For $t \geq 0$, the small global solution $(\mathbf{u}, \mathsf{E})$ to system (2) satisfies*

$$\begin{aligned} \|\mathbf{u}(t)\|_{L^2} + \|\mathsf{E}(t)\|_{L^2} &\leq CM(1+t)^{-\frac{3}{4}}, \\ \|\nabla \mathbf{u}(t)\|_{H^1} + \|\nabla \mathsf{E}(t)\|_{H^1} &\leq CM(1+t)^{-\frac{5}{4}}, \end{aligned}$$

*where*

$$M = \|\mathbf{u}_0\|_{L^1 \cap H^2} + \|\mathsf{E}_0\|_{L^1 \cap H^2}.$$

*Moreover, we have the $L^p$-decay rate*

$$\|\mathbf{u}(t)\|_{L^p} + \|\mathsf{E}(t)\|_{L^p} \leq \begin{cases} CM(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, & p \in [2,6], \\ CM(1+t)^{-\frac{5}{4}}, & p \in [6,\infty], \end{cases}$$

# Sketch of proof

- From higher-order estimates and the fact of small global solution:

$$\frac{d}{dt}H(t) + \kappa_1 H(t) \le C\|\nabla \mathbf{u}\|_{L^2}^2.$$

- By Gronwall inequality and definition of the weighted function $\widetilde{H}(t)$,

$$\widetilde{H}(t) \quad \le \quad H(0)e^{-\kappa_1 t}(1+t)^{\frac{5}{2}} + C\Big(\|\mathbf{u}_0\|_{L^1\cap H^1}^2 + \|\mathsf{E}_0\|_{L^1\cap H^1}^2 + \delta^2\widetilde{H}(t)\Big).$$

- Since $\delta$ (the $H^2$-bound for global solution $(\mathbf{u}, \mathsf{E})$) is small,

$$\widetilde{H}(t) \le C\left(\|\mathbf{u}_0\|_{L^1\cap H^2}^2 + \|\mathsf{E}_0\|_{L^1\cap H^2}^2\right).$$

- $L^p$-decay rate follows from the interpolation inequality.

# Lower bound for $L^2$-decay rate

## Lemma

*Suppose the initial data* $(\mathbf{u}_0, \mathbf{n}_0)$ (*with* $\mathbf{n}_0 = \Lambda^{-1}\nabla \cdot \mathsf{E}_0$) *satisfy*

$$|\widehat{u_{i0}}| \geq c_0, \quad |\widehat{n_{i0}}| \geq c_0, \quad \textit{for } 0 \leq |\xi| << 1,$$

*where* $c_0 \sim O(\delta^\zeta)$ *with* $\zeta \in (0,1)$. *Then*

$$\|\mathbf{u}(t)\|_{L^2} + \|\mathsf{E}(t)\|_{L^2} \geq C(1+t)^{-\frac{3}{4}}, \quad \forall t \geq t_0 >> 1.$$

- By Duhamel's principle and previous decay estimates, for $\delta << 1$, it holds

$$\begin{aligned}
& \|(\mathbf{u}(t), \mathbf{n}(t))\|_{L^2} \\
\geq\ & C\delta^\eta (1+t)^{-\frac{3}{4}} - C\delta \widetilde{H}^{\frac{1}{2}}(t) \int_0^t (1+t-s)^{-\frac{3}{4}} (1+s)^{-\frac{5}{4}} ds \\
\geq\ & C(\delta^\eta - C\delta)(1+t)^{-\frac{3}{4}} \\
\geq\ & C(1+t)^{-\frac{3}{4}}.
\end{aligned}$$

# Weak solutions

## Definition

The pair $(\mathbf{u}, \mathsf{E})$ is a finite energy weak solution to the system (2) in $(0,T)\times\mathbb{R}^d$, if

$$\mathbf{u}\in L^\infty(0,T;L^2(\mathbb{R}^d)), \quad \nabla\mathbf{u}\in L^2(0,T;L^2(\mathbb{R}^d)), \quad \mathsf{E}=\mathsf{F}-\mathbb{I}\in L^\infty(0,T;L^2(\mathbb{R}^d)).$$

$(\mathbf{u}, \mathsf{E})$ satisfies the viscoelastic system (2) in the sense of distributions. Moreover, the following energy inequality holds for a.e. $t\in[0,T]$,

$$\mathcal{E}(t)+\mu\int_0^t\|\nabla\mathbf{u}(s)\|_{L^2}^2ds\leq\mathcal{E}(0),$$

where

$$\mathcal{E}(t)=\frac{1}{2}\left(\|\mathbf{u}(t)\|_{L^2}^2+\|\mathsf{E}(t)\|_{L^2}^2\right), \quad \mathcal{E}(0)=\frac{1}{2}\left(\|\mathbf{u}_0\|_{L^2}^2+\|\mathsf{E}_0\|_{L^2}^2\right).$$

# Main result 2: Weak-strong uniqueness

## Theorem

*Let $\mathbf{u}_0 \in H^k(\mathbb{R}^d)$ and $\mathsf{E}_0 = \mathsf{F}_0 - \mathbb{I} \in H^k(\mathbb{R}^d)$ $(k \geq 3)$, satisfying the assumptions (A1)–(A3).*
*Suppose that $(\hat{\mathbf{u}}, \hat{\mathsf{E}})$ is a weak solution of system (2) in $(0,T) \times \mathbb{R}^d$ and $(\mathbf{u}, \mathsf{E})$ is a strong solution emanating from the same initial data (e.g., the global classical solutions as before). Then we have $\hat{\mathbf{u}} \equiv \mathbf{u}$ and $\hat{\mathsf{E}} \equiv \mathsf{E}$ on the time interval of existence.*

- Existence of weak solutions to the viscoelastic system (2) for arbitrary initial data is **Open**.
- **Key difficulty**: to show the weak convergence of the nonlinear term $\mathsf{E}\mathsf{E}^T$ for suitable approximating solutions at least in the sense of distributions, which is NOT available from the basic energy law.
- In $2D$, Hu-Lin 2013, existence of weak solutions provided that

$$\mathbf{u}_0 \in L^p(\mathbb{R}^2), \quad p > 2,$$
$$\|\mathsf{E}_0\|_{L^\infty} + \int_{\mathbb{R}^2} (|\mathsf{E}_0|^2 + |\mathbf{u}_0|^2)(1 + |x|^2)dx << 1.$$

# Sketch of proof

- The weak solution $(\hat{\mathbf{u}}, \hat{\mathsf{E}})$ satisfies the *energy inequality* and the strong solution $(\mathbf{u}, \mathsf{E})$ satisfies the *energy equality*
- The strong solution $(\mathbf{u}, \mathsf{E})$ is regular enough to serve as the test functions in the weak formulation

$\Longrightarrow$ For the differences of solutions

$$\mathfrak{U} = \hat{\mathbf{u}} - \mathbf{u} \quad \text{and} \quad \mathfrak{E} = \hat{\mathsf{E}} - \mathsf{E},$$

it holds

$$\begin{aligned} & \|\mathfrak{U}\|_{L^2}^2 + \|\mathfrak{E}\|_{L^2}^2 + \mu \int_0^t \|\nabla \mathfrak{U}\|_{L^2}^2 d\tau \\ \leq \quad & \|\mathfrak{U}_0\|_{L^2}^2 + \|\mathfrak{E}_0\|_{L^2}^2 + C \int_0^t h(\tau) \left( \|\mathfrak{U}\|_{L^2}^2 + \|\mathfrak{E}\|_{L^2}^2 \right) d\tau, \end{aligned}$$

where

$$h(t) = \|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathsf{E}\|_{L^\infty} + \|\mathsf{E}\|_{L^\infty}^2.$$

- $h(t) \in L^1(0, T) \Longrightarrow$ conclusion by the Gronwall lemma.

# Open problems

- Decay estimate for incompressible Magnetohydrodynamic (MHD) system with zero magnetic diffusivity

$$
\begin{cases}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \mathbf{B}\cdot\nabla\mathbf{B}, \\
\mathbf{B}_t - \nabla\times(\mathbf{u}\times\mathbf{B}) = 0, \\
\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0.
\end{cases}
$$

Remark In 2D, there exists a *scalar* function $\phi$ such that $\mathbf{B} = \nabla^{\perp}\phi$, while for F, there exists a *vector* $\phi = (\phi_1, \phi_2)$ such that $\mathrm{F} = \nabla^{\perp}\phi$. For both cases, the system can be converted to

$$
\begin{cases}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = -\nabla\cdot(\nabla\phi\otimes\nabla\phi), \\
\phi_t + \mathbf{u}\cdot\nabla\phi = 0, \\
\nabla\cdot\mathbf{u} = 0.
\end{cases}
$$

- Global weak solutions with finite energy to viscoelastic system in 2D and 3D

# The End

Thank You !