Initial-Boundary Value Problem of a Coupled Navier-Stokes/Q-Tensor System

Yuning Liu

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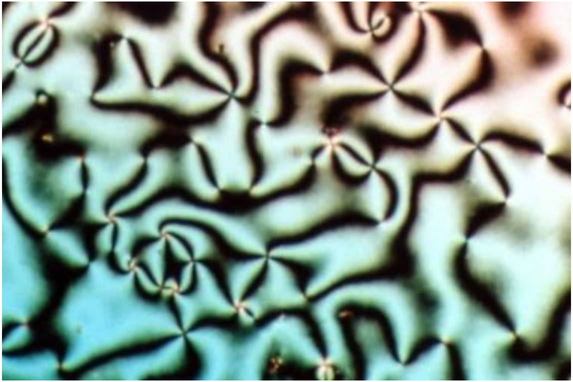
in collaboration with
Helmut Abels and Georg Dolzmann

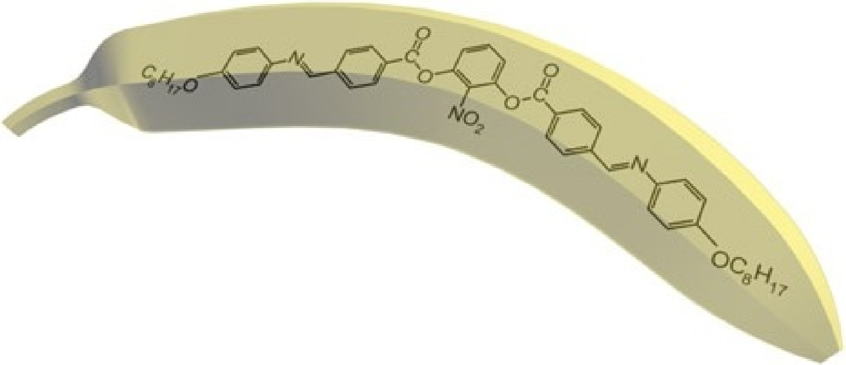
Table of contents

Schlieren Texture

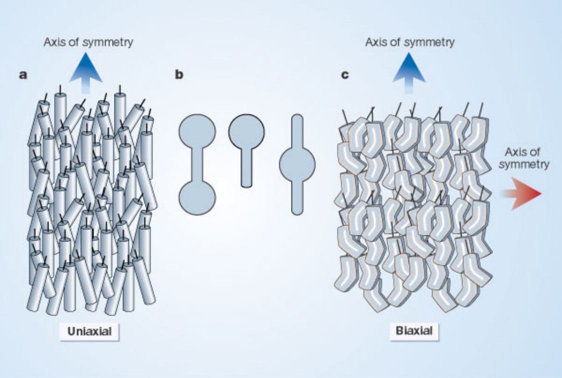


schlieren texture

The ‘Banana’-Shaped Molecule



Uniaxial and Biaxial



- In the uniaxial phase, the rod-like molecule self-align to have long-range directional order with their long axis roughly parallel.
- In the biaxial phase, in addition to orient their long axis, they also orient along a secondary axis.

Continumm Theories: 'Vector Models'

- Ossen-Frank theory: for any unit vector field $n : \Omega \to \mathbb{S}^2$, the free energy is given by

$$\mathcal{I}_{OF}[n] = \int_\Omega K_1(\operatorname{div} n)^2 + K_2|n \cdot (\nabla \times n)|^2 + K_3|n \times (\nabla \times n)|^2 + \int_\Omega (K_2 + K_4)\left(\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2\right),$$

- Ericksen's theory: for $s : \Omega \to [-\frac{1}{2}, 1]$ and $n : \Omega \to \mathbb{S}^2$, the free energy is defined by

$$\mathcal{I}_e(s, n) := \int_\Omega k|\nabla s|^2 + s^2|\nabla n|^2 + \Psi(s)$$

where $\Psi(s)$ is a potential function: positive C^2-function defined on $(-\frac{1}{2}, 1)$ and $\lim_{s\to 1}\Psi(s) = \lim_{s\to -\frac{1}{2}}\Psi(s) = +\infty$.

F-H.Lin, Proceeding ICM, 1990, Tokyo.

Landau-De Gennes Theory of Nematic Liquid Crystals

In the Landau-De Gennes framework, the state of a nematic liquid crystal is modeled by a symmetric, traceless 3×3 matrix (denoted by $\mathbb{S}_0$), known as the Q-tensor. A nematic liquid crystal is said to be:

- **Isotropic** when $Q = 0$.
- **uniaxial** when Q has two equal non-zero eigenvalues:

$$Q = s\left(n \otimes n - \frac{1}{3} I_d\right); \quad s \in \mathbb{R}\backslash\{0\}, \quad n \in \mathbb{S}^2.$$

- **biaxial** when Q has three distinct eigenvalues:

$$Q = s\left(n \otimes n - \frac{1}{3} I_d\right) + r\left(m \otimes m - \frac{1}{3} I_d\right); \quad s,\ r \in \mathbb{R}\backslash\{0\}, \quad n,\ m \in \mathbb{S}^2.$$

A. Majumdar and A. Zarnescu'2010, J.M. Ball and A. Zarnescu, 2011.

Free Energy of Nematic Liquid Crystals

Free energy of nematic liquid crystals:

$$\mathcal{F}[Q] := \int_U \frac{L}{2} \underbrace{|\nabla Q|^2(x)}_{\text{elastic energy density}} + \underbrace{f_B(Q(x))}_{\text{bulk energy density}} \, \mathrm{d}x.$$

Non-singular potential f_B:

$$f_B(Q) := \frac{1}{\epsilon}\left(a\,\mathrm{tr}(Q^2) - b\,\mathrm{tr}(Q^3) + c\left(\mathrm{tr}(Q^2)\right)^2\right).$$

The elastic constant is typically very small compared with the coefficients in the bulk energy.

The boundary condition:

- Strong anchoring: $Q|_{\partial U} = s_+\left(n \otimes n - \frac{1}{3} I_d\right), \quad n \in C^\infty(\partial U; \mathbb{S}^2)$.
- Neumann: $\partial_n Q|_{\partial U} = 0$.

De Gennes, P.G. '1974, A. Majumdar and A. Zarnescu'2010.

The Dynamical System

The system we shall study was proposed by Beris, A.N., Edwards, B.J.:

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla p &= \nabla \cdot \big(\tau(Q) + \sigma(Q, H(Q))\big), \\ \nabla \cdot u &= 0, \\ Q_t + u \cdot \nabla Q &= S(\nabla u, Q) + H(Q), \end{cases}$$

with the following boundary conditions

$$\begin{cases} u &= 0, & (t, x) \in (0, T) \times \partial U, \\ Q &= Q_D(x), & (t, x) \in (0, T) \times \partial U. \end{cases}$$

We only consider time-independent boundary conditions.

The Tensors

$$H_{\alpha\beta} = L\Delta Q_{\alpha\beta} + \frac{1}{\epsilon}\left(-aQ_{\alpha\beta} + b\left[Q_{\alpha\gamma}Q_{\gamma\beta} - \frac{\delta_{\alpha\beta}}{d}\operatorname{tr} Q^2\right] - cQ_{\alpha\beta}\operatorname{tr} Q^2\right),$$

$$\tau_{\alpha\beta} = -L\left(Q_{\gamma\delta,\beta}Q_{\gamma\delta,\alpha} + \frac{\delta_{\alpha\beta}}{d}\operatorname{tr} Q^2\right),$$

$$-\xi\left(Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{d}\right)H_{\gamma\beta} - \xi H_{\alpha\gamma}\left(Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{d}\right) + 2\xi\left(Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{d}\right)Q_{\gamma\delta}H_{\gamma\delta},$$

$$\sigma_{\alpha\beta} = Q_{\alpha\gamma}H_{\gamma\beta} - H_{\alpha\gamma}Q_{\gamma\beta} = Q_{\alpha\gamma}\Delta Q_{\gamma\beta} - \Delta Q_{\alpha\gamma}Q_{\gamma\beta},$$

$$(Du)_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}),$$

$$\Omega_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha}),$$

$$S(\nabla u, Q) = (\xi Du + \Omega)(Q + \frac{1}{d}I_d) + (Q + \frac{1}{d}I_d)(\xi Du - \Omega) - 2\xi(Q + \frac{1}{d}I_d)\operatorname{tr}(Q\nabla u).$$

Relation to Ericksen–Leslie theory

In a recent work by Wang, Zhang and Zhang, preprint, the rigious derivation from Q-tensor system to Ericksen–Leslie system is accomplished by using the **Hilbert expansion**.

Formally, if we assume

$$u = u_0 + \sum_{i \geqslant 1} \epsilon u_i, \; Q = Q_0 + \sum_{i \geqslant 1} \epsilon Q_i$$

and plugin them in the Q-tensor system, by matching the order of ϵ:

- The order $O(\epsilon^{-1})$ system implies that Q_0 is uniaxial:

$$\frac{\delta f_B(Q)}{\delta Q}(Q_0) \equiv 0 \Leftrightarrow Q_0 = s_0(d_0 \otimes d_0 - \frac{1}{3} I_d) \tag{1}$$

  where $|d_0| = 1$ and $s_0 = 0$ or it is the solution of $2cs_0^2 - bs_0 - 3a = 0$. The assertion (1) is due to A.Majumdar, 2008..
- The order $O(1)$ system implies that (u_0, d_0) satisfies the Ericksen–Leslie system.
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The case when $\xi = 0$

Without loss of generality, we shall focus on the case when $L = a = b = c = \epsilon = 1$ and $\xi = 0$:

$$u_t + u \cdot \nabla u - \Delta u + \nabla p = \nabla \cdot \big(\overbrace{\tau(Q) + \sigma(Q, H(Q))}^{\text{additional stress tensor}}\big),$$
$$\nabla \cdot u = 0,$$
$$Q_t + u \cdot \nabla Q = S(\nabla u, Q) + H(Q).$$

$$\begin{cases} H(Q) &= \Delta Q - Q + \left[QQ - \frac{1}{d}\mathbb{I}\,\mathrm{tr}\,Q^2\right] - Q\,\mathrm{tr}\,Q^2, \\ \tau(Q) &= -\nabla Q \cdot \nabla Q - \frac{1}{d}\mathbb{I}\,\mathrm{tr}\,Q^2, \\ \sigma(Q, H(Q)) &= Q \cdot \Delta Q - \Delta Q \cdot Q, \\ 2S(\nabla u, Q) &= (\nabla u - (\nabla u)^T)Q - Q(\nabla u - (\nabla u)^T). \end{cases}$$

Known Results

Here we list part of the analytic result for this system:

1. M. Paicu and A. Zarnescu, 2011, The system on $\mathbb{R}^d$, $|\xi| \ll 1$
2. M. Wilkinson: singular potential, periodic boundary condition: $\mathbb{T}^d$.
3. E. Feireisl, E.Rocca, G.Schimperna, A.Zarnescu: singular potential and non-isothermal, periodic boundary condition: $\mathbb{T}^d$.
4. H.Abels, G.Dolzmann and Y.Liu, F. Guillén-González and M. A. Rodríguez-Bellido: initial-boundary-value problems

Table of contents

Energy Dissipation Law

The total energy of the system is composed of two parts:

$$E(t) := \frac{1}{2}\int_U |u(t,x)|^2 \mathrm{d}x + \mathcal{F}\left[Q(t)\right].$$

Testing $(u, -H(Q))$ and integrating over U gives

$$\frac{d}{dt}E(t) + \int_U |Du|^2 \,\mathrm{d}x + \int_U |H(Q)|^2 \,\mathrm{d}x$$

$$= \underbrace{\int_U \nabla\cdot\sigma(Q,H(Q))\cdot u - S(\nabla u, Q)H(Q)}_{\text{algebra structure: } \sigma(Q,H)\nabla u = S(\nabla u, Q)H} + \underbrace{\int_U \nabla\cdot\tau(Q)\cdot u + u\cdot\nabla Q H(Q)}_{\text{vanishes by divergence-free condition}}.$$

Integrating over $[0,t]$ gives

$$E(t) + \int_{U_t} |Du(\tau,x)|^2 + |H(\tau,x)|^2 \mathrm{d}x\mathrm{d}\tau = E(0), \quad t\in[0,T]$$

Global Weak Solutions

Theorem

For any $u_0 \in L^2_\sigma(U)$ and $Q_0 \in H^1(U)$, there exists

$$u \in L^\infty(0, T; L^2_\sigma(U)) \cap L^2(0, T; H^1_{0,\sigma}(U)),$$
$$Q \in L^\infty(0, T; H^1(U)) \cap L^2(0, T; H^1(U) \cap H^2(U))$$

satisfying the initial-boundary value problem **in the sense of distribution** and the energy dissipation law holds.

Similar strategy to **Lin and Liu, 1995, CPAM**:

1. finite dimensional approximation which keeps the energy dissipation law.
2. Aubin-Lions compactness.

The time derivative of (u, Q) are very 'bad':

$$u_t \in L^2(0, T; H^{-2}(\Omega)), \quad Q \in L^2(0, T; H^{-1}(\Omega)).$$

Table of contents

Main Result

Theorem (Differentiability in time)

For any $u_0 \in H^1_{0,\sigma}(U;\mathbb{R}^3)$ and $Q_0 \in H^2(U;\mathbb{S}_0)$ satisfying the 'compatibility condition', there exists $T > 0$ such that there is a unique solution

$$u \in H^2(0,T;V'(U)) \cap H^1(0,T;H^1_{0,\sigma}(U)),$$
$$Q \in H^2(0,T;L^2(U)) \cap H^1(0,T;H^2(U))$$

to the non-homogeneous initial-boundary value problem (in the sense of distribution).

Here and V' is the dual space of $H^1_{0,\sigma}(U)$. Compared with the existence of global weak solutions

$$u \in L^\infty(0,T;L^2_\sigma(U)) \cap L^2(0,T;H^1_{0,\sigma}(U)),$$
$$Q \in L^\infty(0,T;H^1(U)) \cap L^2(0,T;H^1(U) \cap H^2(U)),$$

the regularity in time increases.

Linearization around an initial data

Let Q_0 be the initial data satisfying some compatibility condition which shall be seen later. We then linearize the nonlinear system around Q_0 by introducing the following linear operators:

$$\mathcal{S}(Q_0)\begin{pmatrix} u \\ Q \end{pmatrix} = \begin{pmatrix} P_\sigma \Delta u + P_\sigma \nabla \cdot (Q_0 \Delta Q - \Delta Q Q_0) \\ \Delta Q + \frac{1}{2}(\nabla u - (\nabla u)^T) Q_0 - \frac{1}{2} Q_0 (\nabla u - (\nabla u)^T) \end{pmatrix},$$

$$\mathcal{L}(Q_0)\begin{pmatrix} u \\ Q \end{pmatrix} = \begin{pmatrix} u_t \\ Q_t \end{pmatrix} - \mathcal{S}(Q_0)\begin{pmatrix} u \\ Q \end{pmatrix}.$$

$P_\sigma : H^{-1}(U)^3 \to V'$ is the generalized Leray-Projector:
$P_\sigma f := f|_{V(U)}, \quad \forall f \in H^{-1}(U)^3$.

Nonlinear Terms

Now we define the nonlinear operator $\mathcal{N}$ by

$$\mathcal{N}(Q_0)\begin{pmatrix} u \\ Q \end{pmatrix} = \begin{pmatrix} P_\sigma\left[\nabla\cdot(\tau(Q)+\sigma(Q-Q_0,\Delta Q)-u\otimes u)\right] \\ -u\cdot\nabla Q - S(\nabla u, Q-Q_0) - L(Q) \end{pmatrix}.$$

Then we can write the nonlinear system by

$$\begin{pmatrix} u_t \\ Q_t \end{pmatrix} = \mathcal{S}(Q_0)\begin{pmatrix} u \\ Q \end{pmatrix} + \mathcal{N}(Q_0)\begin{pmatrix} u \\ Q \end{pmatrix}$$

Initial Data

Compatibility conditions:

$$\mathcal{S}(Q_0)\begin{pmatrix} u_0 \\ Q_0 \end{pmatrix} + \mathcal{N}(Q_0)\begin{pmatrix} u_0 \\ Q_0 \end{pmatrix} := \mathcal{E}\begin{pmatrix} u_0 \\ Q_0 \end{pmatrix}$$

The functional space for the initial date (u_0, Q_0):

$$Z := \left\{ (u_0, Q_0) \in H^1_{0,\sigma} \times H^2 : \mathcal{E}\begin{pmatrix} u_0 \\ Q_0 \end{pmatrix} \in L^2_\sigma \times H^1_0, Q_0|_{\partial U} = Q_D(x) \right\}$$

Note that the phase space defined above is non-empty. For instance, we can choose $u_0 \in H^2(U) \cap H^1_{0,\sigma}(U)$ and then choose $Q_0 \in H^3(U)$ by solving the following elliptic system on U along with boundary condition $Q_0|_{\partial U} = Q_D(x)$:

$$u_0 \cdot \nabla Q_0 + S(\nabla u_0, Q_0) + \underbrace{\Delta Q_0 + L(Q_0)}_{H(Q_0)} = h, \ \forall h \in H^1_0(\Omega; \mathbb{S}_0).$$

Functional Spaces

Lemma 1.

$\mathcal{L}(Q_0) : X_0 \to Y_0$ *is a bounded linear operator*

$$
\begin{aligned}
X_u &= H^2(0,T;V'(U)) \cap H^1(0,T;V(U)),\\
X_Q &= H^2(0,T;L^2(U)) \cap H^1(0,T;H^2(U)),\\
X_0 &= \left\{(u,Q) \in X_u \times X_Q : \; Q|_{\partial U} = 0, (u,Q)\,|_{t=0} = (0,0)\right\},
\end{aligned}
$$

$$
\begin{aligned}
Y_u &= H^1(0,T;V'(U)), \quad Y_Q = H^1(0,T;L^2(U;\mathbb{S}_0)),\\
Y_0 &= \left\{(f,g) \in Y_u \times Y_Q : \begin{pmatrix} f \\ g \end{pmatrix}\Bigg|_{t=0} \in L^2_\sigma(U) \times H^1_0(U;\mathbb{S}_0)\right\},
\end{aligned}
$$

and Y_0 is equipped with the norm
$\|(f,g)\|_{Y_0} := \|(f,g)\|_{Y_u \times Y_Q} + \|(f,g)|_{t=0}\|_{L^2_\sigma(U)\times H^1(U)}$.

The invertibility of the linearized operator

The following result establishes the solvability of the operator equation:

$$\mathcal{L}(Q_0)\begin{pmatrix} u_h \\ Q_h \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Lemma 2.

For every $(f, g) \in Y_0$, the above operator equation has a unique solution $(u_h, Q_h) \in X_0$ satisfying

$$\|(u_h, Q_h)\|_{X_0} \leqslant C_{\mathcal{L}}(Q_0)\|(f, g)\|_{Y_0}.$$

Nonlinear terms I

Seeking the solution amounts to looking for $(u, Q) = (u_h, Q_h) + (u_0, Q_0)$ satisfying:

$$\mathcal{L}(Q_0)\begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix} = \mathcal{N}(Q_0)\begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix}, \quad \begin{pmatrix} u_h \\ Q_h \end{pmatrix} \in X_0$$

or equivalently ($\mathcal{L}$ is linear):

$$\mathcal{L}(Q_0)\begin{pmatrix} u_h \\ Q_h \end{pmatrix} = \mathcal{N}(Q_0)\begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix} - \mathcal{L}(Q_0)\begin{pmatrix} u_0 \\ Q_0 \end{pmatrix} \in Y_0, \quad \begin{pmatrix} u_h \\ Q_h \end{pmatrix} \in X_0$$

we define a new nonlinear operator by

$$\mathcal{N}_0(Q_0)\begin{pmatrix} u_h \\ Q_h \end{pmatrix} = \mathcal{N}(Q_0)\begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix} - \mathcal{L}(Q_0)\begin{pmatrix} u_0 \\ Q_0 \end{pmatrix} \in Y_0.$$

Nonlinear Terms II

Lemma 3.

The operator $\mathcal{N}_0(Q_0)$ maps X_0 to Y_0:

$$\|\mathcal{N}_0(Q_0)(u_1, Q_1) - \mathcal{N}_0(Q_0)(u_2, Q_2)\|_{Y_0} \leqslant C_{\mathcal{N}}(T, R, u_0, Q_0)\|(u_1 - u_2, Q_1 - Q_2)\|_{X_0}$$

for all $(u_i, Q_i) \in \mathcal{B}_R := \{(u, Q) \in X_0;\ \|(u, Q)\|_{X_0} \leqslant R\}$, $i = 1, 2$. Moreover, for each $R > 0$, $\lim_{T \to 0} C_{\mathcal{N}}(T, R, u_0, Q_0) = 0$.

Proof of Main Theorem:
Show that the nonlinear mapping has a fixed point:

$$\mathcal{T}(Q_0) := \mathcal{L}^{-1}\mathcal{N}_0 : X_0 \to X_0$$

For any $(u_i, Q_i) \in \mathcal{B}_R$:

$$\begin{aligned}
&\left\|\mathcal{L}^{-1}(Q_0)\mathcal{N}_0(Q_0)\begin{pmatrix} u_1 \\ Q_1 \end{pmatrix} - \mathcal{L}^{-1}(Q_0)\mathcal{N}_0(Q_0)\begin{pmatrix} u_2 \\ Q_2 \end{pmatrix}\right\|_{X_0} \\
\leqslant & C_{\mathcal{L}^{-1}}(Q_0)\|\mathcal{N}_0(Q_0)(u_1, Q_1) - \mathcal{N}_0(Q_0)(u_2, Q_2)\|_{Y_0} \\
\leqslant & C_{\mathcal{L}^{-1}}(Q_0)C_{\mathcal{N}}(T, R, u_0, Q_0)\|(u_1 - u_2, Q_1 - Q_2)\|_{X_0}
\end{aligned}$$

Table of contents

The Linearized Operator: I

To prove the invertibility of the linearized operator amounts to establishing the wellposedness of the following system:

$$\begin{cases} u_t - \Delta u + \nabla p - \nabla \cdot \sigma(Q_0, \Delta Q) = f, \\ Q_t - \Delta Q - S(\nabla u, Q_0) = g, \\ u = 0, \ x \in \partial U, \\ Q = 0, \ x \in \partial U, \\ (u, Q)|_{t=0} = (0, 0) \end{cases} \tag{2}$$

with $(f, g) \in Y_0$ and $(u, Q) \in X_0$.

Two Observations

- Testing the equations by u and ΔQ respectively will give an energy dissipation law, which implies the global weak solutions.
- Differentiating (2) in time and testing by $(u_t, \Delta Q_t)$ will give the a priori estimate for the time derivatives.

The Linearized Operator: II

The equations for (u_t, Q_t) is the following:

$$\begin{cases} (u_t)_t - \Delta u_t + \nabla p_t - \nabla \cdot \sigma(Q_0, \Delta Q_t) = f_t, \\ (Q_t)_t - \Delta Q_t - S(\nabla u_t, Q_0) = g_t, \\ u_t = 0, \ x \in \partial U, \\ Q_t = 0, \ x \in \partial U. \end{cases}$$

The initial date for the above system is given by the linearized system at $t = 0$:

$$\begin{cases} u_t - \overbrace{P_\sigma \left[\Delta u + \nabla p - \nabla \cdot \sigma(Q_0, \Delta Q)\right]}^{\text{vanishes at } t=0} = P_\sigma f, \\ Q_t \underbrace{- \Delta Q - S(\nabla u, Q_0)}_{\text{vanishes at } t=0} = g, \end{cases}$$

The Linearized Operator: III

The a priori estimate for (u_t, Q_t) are:

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(U)} + \int_U |Du_t|^2 + \langle \sigma\left(Q_0, \Delta Q_t\right), \nabla u_t\rangle = \langle f_t, \partial_t u\rangle ,$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla Q_t\|^2_{L^2(U)} + \int_U |\Delta Q_t|^2 + \langle S(\nabla u_t, Q_0), \Delta Q_t\rangle = \int_U g_t \Delta Q_t,$$

Adding up the above estimates and estimating the lower-order terms yields:

$$\sup_{t\in[0,T]}\left(\|u_t\|^2_{L^2(U)} + \|\nabla Q_t\|^2_{L^2(U)}\right) + \|\nabla u_t\|^2_{L^2(U_T)} + \|\Delta Q_t\|^2_{L^2(U_T)}$$

$$\leqslant C(\xi, Q_0)\left(\|f_t\|^2_{L^2(H^{-1})} + \|g_t\|^2_{L^2(L^2)}\right) + \underbrace{\|u_t(0)\|^2_{L^2(U)} + \|\nabla Q_t(0)\|^2_{L^2(U)}}_{\mathcal{R}}$$

The Linearized Operator: IV

Since the initial date is zero, we can estimate $\mathcal{R}$ by

$$\mathcal{R} \lesssim \|(u_t, Q_t)|_{t=0}\|_{L^2_\sigma(U)\times H^1(U)} \lesssim \underbrace{\|(f,g)|_{t=0}\|_{L^2_\sigma(U)\times H^1(U)}}_{\text{compatibility conditions}} \leqslant \|(f,g)\|_{Y_0}$$

As a result:

$$\sup_{t\in[0,T]} \left(\|u_t\|^2_{L^2(U)} + \|\nabla Q_t\|^2_{L^2(U)}\right) + \|\nabla u_t\|^2_{L^2(U_T)} + \|\Delta Q_t\|^2_{L^2(U_T)}$$
$$\leqslant C(\xi, Q_0)\|(f,g)\|_{Y_0}.$$

Abstract approach: J.Wloka, Partial differential equations, 1987.

Table of contents

From Dynamical System to the Stationary System

Our previous time-regularity motivates us to consider, for almost every $t \in [0, T]$, the regularity problem of the following nonlinear elliptic system on Ω

$$\left\{ \begin{aligned} u \cdot \nabla u - \Delta u + \nabla p &= \nabla \cdot \big(\tau(Q) + \sigma(Q, H(Q)) \big) + f, \\ \nabla \cdot u &= 0, \\ u \cdot \nabla Q &= S(\nabla u, Q) + H(Q) + g, \end{aligned} \right. \tag{3}$$

with the following boundary conditions

$$\left\{ \begin{array}{lll} u &= 0, & x \in \partial U, \\ Q &= Q_D(x), & x \in \partial U. \end{array} \right. \tag{4}$$

where $(u, Q) \in H^1_{0,\sigma}(\Omega) \times H^2(\Omega; \mathbb{S}_0)$ is a ‘ weak’ solution to (3) and (4) for almost every $t \in [0, T]$ and $(f, g) \in H^1_\sigma(\Omega) \times H^2(\Omega; \mathbb{S}_0)$.

System in Two Dimension: I

In 2-d, we still assume that $Q : \Omega \subset \mathbb{R}^2 \to \mathbb{S}_0$ and $u : \Omega \to \mathbb{R}^2$:

$$\begin{cases} u \cdot \nabla u - \Delta u + \nabla p & = \nabla \cdot \left(\tau(\widetilde{Q}) + \sigma(\widetilde{Q}, H(\widetilde{Q})) \right) + f, \\ \nabla \cdot u & = 0, \\ u \cdot \nabla Q & = S(\widehat{\nabla u}, \hat{Q}) + H(Q) + g, \end{cases} \tag{5}$$

where $\hat{u} = (u_1, u_2, 0)$ and for any $d \times d$-matrix M ($d \geqslant 2$), we define

$$\hat{M} = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

In this case, we can see from the last equations of (5) that, the equations for Q_{13}, Q_{23}, Q_{33} are essentially decoupled and can be solved independently. Consequently, the L^p-estimate for linear elliptic equations implies that

$$Q_{13}, Q_{23}, Q_{33} \in W^{3,\frac{3}{2}}(\Omega).$$

System in Two Dimension: II

It remains to solve the following system:

$$\begin{cases} u \cdot \nabla u - \Delta u + \nabla p &= \nabla \cdot \sigma(\widetilde{Q}, H(\widetilde{Q})) + f + \nabla \cdot \tau(\widetilde{Q}), \\ \nabla \cdot u &= 0, \\ u \cdot \nabla \widetilde{Q} &= S(\nabla u, \widetilde{Q}) + \Delta \widetilde{Q} + L(Q) + \widetilde{g}, \end{cases}$$

In this case, we have the following cancellation law

$$S(\nabla u, \widetilde{Q_1}) : \widetilde{Q_2} + \sigma(\widetilde{Q_1}, \widetilde{Q_2}) : \nabla u = 0$$

As a result, the time-regularity results that we obtained previously are also valid for the 2-d system.

Elimination of the Highest Order Terms

Now we perform the following elimination process:

$$u \cdot \nabla u - \Delta u + \nabla p = \nabla \cdot \tau(\widetilde{Q}) + \nabla \cdot \sigma(\widetilde{Q}, \underbrace{u \cdot \nabla \widetilde{Q} - S(\nabla u, \widetilde{Q}) - \widetilde{g}}_{=H(\widetilde{Q})}) + f,$$

Since σ-tensor is bilinear

$$\begin{aligned} &\sigma(\widetilde{Q}, u \cdot \nabla \widetilde{Q} - S(\nabla u, \widetilde{Q}) - \widetilde{g}) \\ = &-\sigma(\widetilde{Q}, S(\nabla u, \widetilde{Q})) + \underbrace{\sigma(\widetilde{Q}, u \cdot \nabla \widetilde{Q}) - \sigma(\widetilde{Q}, \widetilde{g})}_{\text{lower order terms}} \end{aligned}$$

A New System

The above step leads to the following system about u:

$$\operatorname{div}\sigma(\widetilde{Q}, S(\nabla u, \widetilde{Q})) - \Delta u + \nabla p = \widetilde{f} \in L^{\frac{3}{2}}(\Omega). \tag{6}$$

Direct computation shows that

$$\sigma(\widetilde{Q}, S(\nabla u, \widetilde{Q})) = \omega \left(2Q_{12}^2 + \frac{1}{2}(Q_{22} - Q_{11})^2\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where ω denotes the vorticity (scalar) $\omega := u_{1,2} - u_{2,1}$. Consequently, we can write (6) by

$$\begin{pmatrix} -\Delta - \Theta\partial_2^2 & \Theta\partial_1\partial_2 & \partial_1 \\ \Theta\partial_1\partial_2 & -\Theta\partial_1^2 - \Delta & \partial_2 \\ -\partial_1 & -\partial_2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ p \end{pmatrix} = \begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \\ 0 \end{pmatrix}$$

where $\Theta(x,t) = 2Q_{12}^2 + \frac{1}{2}(Q_{22} - Q_{11})^2$, which is non-negative.

Strong-Elliptic System

One can easily obtain the symbol (of its leading 2×2 sub-matrix) and calculate its determinant :

$$\det \begin{pmatrix} |\xi|^2 + q\xi_2^2 & -q\xi_1\xi_2 \\ -q\xi_1\xi_2 & q\xi_1^2 + |\xi|^2 \end{pmatrix} = (q+1)|\xi|^4$$

and this shows that the above operator is strong-elliptic.

Questions: Does the weak solution $u \in H^1_{0,\sigma}(\Omega)$ of the 2-d stationary Beris-Edwards system belongs to $W^{2,\frac{3}{2}}(\Omega)$ such that one can get $Q \in W^{3,\frac{3}{2}}(\Omega)$ through a ‘bootstrap’ arguments ?

Proof of $W^{2,\frac{3}{2}}$-Regularity for u

I shall end my talk by proving the $W^{2,\frac{3}{2}}$ regularity of the following system

$$\begin{pmatrix} -\Delta - \Theta\partial_2^2 & \Theta\partial_1\partial_2 & \partial_1 \\ \Theta\partial_1\partial_2 & -\Theta\partial_1^2 - \Delta & \partial_2 \\ -\partial_1 & -\partial_2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ p \end{pmatrix} = \begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \\ 0 \end{pmatrix}$$

We shall mollify Θ and $\widetilde{f}$ in the above system by the standard mollifier operator and denote the corresponding solutions by (u^ϵ, p^ϵ):

$$\begin{pmatrix} -\Delta - \Theta_\epsilon\partial_2^2 & \Theta_\epsilon\partial_1\partial_2 & \partial_1 \\ \Theta_\epsilon\partial_1\partial_2 & -\Theta_\epsilon\partial_1^2 - \Delta & \partial_2 \\ -\partial_1 & -\partial_2 & 0 \end{pmatrix} \begin{pmatrix} u_1^\epsilon \\ u_2^\epsilon \\ p \end{pmatrix} = \begin{pmatrix} \widetilde{f}_1^\epsilon \\ \widetilde{f}_2^\epsilon \\ 0 \end{pmatrix} \tag{7}$$

1. System (7) has a solution $(u^\epsilon, p^\epsilon) \in H^2(\Omega) \times \dot{H}^1(\Omega)$. See 'Constantin and Foias, Navier-Stokes Equations, Chapter 3'.
2. System (7) is elliptic in the sense of Agmon, Douglis and Nirenberg:

$$\|u^\epsilon\|_{W^{2,p}(\Omega)} + \|\nabla p^\epsilon\|_{L^p(\Omega)} \leqslant C\|\widetilde{f}^\epsilon\|_{L^p(\Omega)} \leqslant C\|\widetilde{f}\|_{L^p(\Omega)}$$

   where $p = \frac{3}{2}$ and C depends on $\|\Theta_\epsilon\|_{C(\overline{\Omega})}$ and thus on $\Theta \in H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ but not ϵ.
3. Pass to the limit $\epsilon \to 0$ and use weak compactness.

Future Works

Future Works

- What is the situation for $\xi \neq 0$?
- What about $d = 3$?

Thank you for your attention!