Decay characterization of solutions to dissipative systems

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Pavia
March 2014

Navier-Stokes equations

Dissipation

We study dissipative fluid equations such as Navier-Stokes

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + \Delta u$$
$$\nabla \cdot u = 0$$
$$u_0(x) = u(x, 0).$$

u = velocity of homogeneous incompressible fluid in $\mathbb{R}^3$

- $u_0 \in L^2$, weak solutions exist (Leray 34).

Decay of solutions to the Navier-Stokes equations

- Suppose u regular, multiply NS-equation by u + integration $\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|u(x,t)|^2\,dx = -\int_{\mathbb{R}^3}|\nabla u(x,t)|^2\,dx < 0.$$

- *Original question (Leray, Kato):* How does the L^2-energy decay for weak solutions..
- Answer1: (Masuda '84): if $u_0 \in L^2$, $\Rightarrow$

$$\|u(t)\|_{L^2} \xrightarrow{t\to\infty} 0.$$

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More decay results

- (M.E. S. '85-86) If $u_0 \in L^1 \cap L^2, \Rightarrow$

$$\|u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}.$$

- (M.E. S '86) Decay without a rate : given $r, T, \epsilon > 0, \exists u_0,$ with $\|u_0\|_{L^2} = r$ satisfying

$$\frac{\|u(T)\|_{L^2}}{\|u_0\|_{L^2}} \geq 1 - \epsilon.$$

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Decay of solutions to the Navier-Stokes equations

- (Wiegner '87) If $\|e^{t\Delta}u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{2}\alpha_0}, \Rightarrow$

$$\|u(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}\min\{\alpha_0, \frac{5}{2}\}}.$$

- (M.E. Schonbek y Wiegner '96) If $\|e^{t\Delta}u_0\|_{L^2} \leq C(1+t)^{-\mu}$ + more hypothesis and $m \in \mathbb{N}$

$$\|D^m u(t)\|_{L^2} \leq C(1+t)^{-\left(\frac{1}{2}m+\mu\right)}.$$

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Ideas for the proof

- "Behavior of solutions for large time is determined by low frequencies of the solutions".
- Use a time depending filter to study the low frequencies.
- This is the Fourier Splitting method (S '80s).

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L^2 decay and frequencies at origin

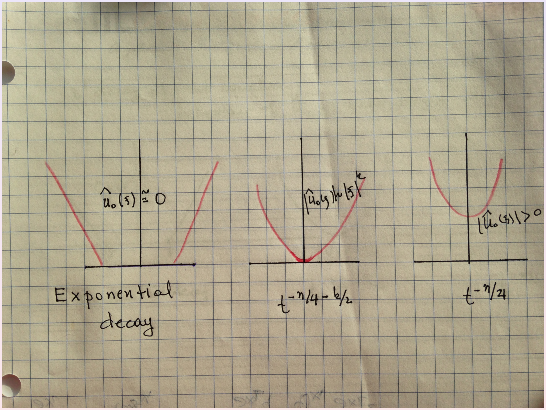

Figure: frequency behavior near origin

Heat equation

Heat equation in $\mathbb{R}^n$

$$u_t - \Delta u = 0$$
$$u_0(x) = u(x, 0),$$

well known solution

$$u(x,t) = G_t * u_0(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * u_0(x).$$

Heat equation: exponential decay

- Let $u_0 \in L^2(\mathbb{R}^n)$ and $\widehat{u_0}(\xi) = 0$, when $|\xi| < \delta$.
- $\Rightarrow$

$$\|\widehat{u}(t)\|_{L^2}^2 = \int_{|\xi|>\delta} e^{-8\pi|\xi|^2 t} |\widehat{u_0}(\xi)|^2 \, d\xi = C\, e^{-8\pi\delta^2 t}.$$

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Heat equation: slow decay

- Recall

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u(x,t)|^2\,dx = -\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\,dx < 0,$$

if $\|\nabla u(t)\|_{L^2} << 1 \;\Rightarrow$ decay rate becomes slower with smaller L^2 gradients.

Heat equation: slow decay

- Let $\mathcal{B} = \{v : \|v\|_2 = 1\}$.
- Let $u_0^\lambda(x) = \lambda^{\frac{n}{2}} e^{-\pi\frac{|\lambda x|^2}{2}}$, then $u_0^\lambda(x) \in \mathcal{B}$.
- Frequencies and L^2 norms of gradients

$$\widehat{u_0^\lambda}(\xi) = \left(\frac{2}{\lambda}\right)^{\frac{n}{2}} e^{-\frac{2\pi}{\lambda^2}|\xi|^2} \qquad \|\nabla u_0^\lambda\|_{L^2} = \pi\lambda\|\nabla u_0\|_{L^2}$$

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Heat equation slow decay

- For some fixed $t > 0$ decay for solutions with data $u_0^\lambda \in \mathcal{B}$ will not be uniform :

$$\frac{\|\widehat{u^\lambda}(t)\|_{L^2}^2}{\|\widehat{u_0^\lambda}\|_{L^2}^2} = \frac{1}{1 + 4\lambda^2 t} \xrightarrow{\lambda \to 0} 1.$$

- There exist solution to the heat equation with data in $L^2(\mathbb{R}^n)$ decaying arbitrarily slowly.

Theorem

Given $r, T, \varepsilon > 0 \Rightarrow \exists u_0$ with $\|u_0\|_{L^2} = r$ so that

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Heat equation: Fourier Splitting

- The behavior of large times is determined by the low frequencies of the solution.
- Filter: "$e^{t\Delta} \leq C$" yields "$t|\xi|^2 \leq C$" $\Rightarrow$

$$B(t) = \left\{ \xi : |\xi|^2 \leq \frac{C}{t+1} \right\}.$$

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Fourier Splitting : old idea (86)

- $\frac{d}{dt}\|u(t)\|_{L^2}^2 \leq -C\|\nabla u(t)\|_{L^2}^2$
- knowledge of bounds for $\widehat{u}_0(\xi), \ |\xi| \ll 1$

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\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= -2\|\nabla u(t)\|_{L^2}^2 \\
&= -2\int_{B(t)\cup B(t)^c} |\xi|^2|\widehat{u}(t)|^2\, d\xi \\
&\leq -2\int_{B(t)} |\xi|^2|\widehat{u}(t)|^2\, d\xi - \frac{C}{1+t}\int_{B(t)^c} |\widehat{u}(t)|^2\, d\xi \\
&\leq \frac{C}{1+t}\int_{B(t)} |\widehat{u}(t)|^2\, d\xi - \frac{C}{1+t}\int_{\mathbb{R}^n} |\widehat{u}(t)|^2\, d\xi.
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Fourier Splitting

- $$\frac{d}{dt}\left((t+1)^n \|u(t)\|_{L^2}^2\right) \le C(t+1)^{n-1} \int_{B(t)} |\widehat{u}(\xi,t)|^2 \, d\xi.$$
- If $u_0 \in L^1(\mathbb{R}^n)$, and $|\widehat{u_0}(\xi)| \le C$, for $|\xi| \ll 1 \Rightarrow$

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \le C(t+1)^{-\frac{n}{4}}.$$

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Fourier Splitting

- Fourier Splitting also works given info on FT near origin +

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^n)} \leq -C \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\widehat{u}(\xi,t)|^2 \, d\xi.$$

- Works for Parabolic conservation laws, Navier-Stokes, MHD, sistemas KdV-Burgers, dissipative QG, dissipative Camassa-Holm, $\cdots$.
- If solution stay in L^1 can use Cordoba-Cordoba methods. (generally not good for derivatives)

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Ingredients of ideas for Lower and Upper bounds of decay

- Knowledge of behavior of low frequencies of the solutions.
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Objetives

Objetive: give a good characterization of the (L^2, Sobolev) decay of solutions to dissipative equations

- **characterization of the initial datum,**
- **study of the linear part,**
- **study of the nonlinear part.**
- **study of difference of linear and nonlinear solutions**

Decay indicator

Definition (Bjorland - MES '09, Niche - MES '13)

Let $u_0 \in L^2(\mathbb{R}^n)$, $r \in \left(-\frac{n}{2}, \infty\right)$. The *decay indicator* of u_0 is defined by

$$P_r(u_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\widehat{u_0}(\xi)|^2 \, d\xi$$

where $B(\rho) = \{\xi : |\xi| \leq \rho\}$.

- The decay indicator compares $|\widehat{u_0}|$ with $f(\xi) = |\xi|^r$ at $\xi = 0$.

Decay character

Definition (Bjorland - MES '09, Niche - MES '13)

Let $u_0 \in L^2(\mathbb{R}^n)$. The decay character of u_0 is $r^* = r^*(u_0)$, is the unique $r \in \left(-\frac{n}{2}, \infty\right)$ so that $0 < P_r(u_0) < \infty$, if this number exists. If it does not exist then

$$r^*(u_0) = \begin{cases} -\frac{n}{2}, & \text{if } P_r(u_0) = \infty, \text{ for all } r \in \left(-\frac{n}{2}, \infty\right) \\ \infty, & \text{if } P_r(u_0) = 0, \text{ for all } r \in \left(-\frac{n}{2}, \infty\right). \end{cases}$$

s-decay indicator

Definition (CJN - MES '13)

Let $u_0 \in L^2(\mathbb{R}^n)$, $s > 0$, $r \in \left(-\frac{n}{2} + s, \infty\right)$. The *s-decay indicator* de $\Lambda^s u_0$ is defined as

$$P_r^s(u_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\xi|^{2s} |\widehat{u_0}(\xi)|^2 \, d\xi$$

where $B(\rho) = \{\xi : |\xi| \leq \rho\}$.

- The s-decay indicator compares $|\Lambda^s u_0|$ with $f(\xi) = |\xi|^r$ at $\xi = 0$.

s-decay character

Definition (CJN - MES '13)

The s-decay character of $\Lambda^s u_0$, is the unique $r_s^* = r_s^*(u_0)$, with $r \in \left(-\frac{n}{2} + s, \infty\right)$ tso that $0 < P_r^s(u_0) < \infty$, provided this number exists. If it does not exist the

$$r_s^*(u_0) = \begin{cases} \infty, & \text{if } P_r(u_0) = 0, \text{ for all } q \in \left(-\frac{n}{2} + s, \infty\right) \\ -\frac{n}{2} + s, & \text{if } P_r(u_0) = \infty, \text{ for all } q \in \left(-\frac{n}{2} + s, \infty\right). \end{cases}$$

Remark

If $u_0 \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $1 \leq p \leq 2$

- The definition of decay character $\Rightarrow r^*(u_0) = -n\left(1 - \frac{1}{p}\right)$.
- If $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow r^*(u_0) = 0$
- If $u_0 \in L^2(\mathbb{R}^n)$ but $u_0 \notin L^p(\mathbb{R}^n)$, for any $1 \leq p < 2 \quad \Rightarrow r^*(u_0) = -\frac{n}{2}$.

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Relation between the decay character and the s-decay character

Theorem (CJN - MES '13)

Let $u_0 \in H^s(\mathbb{R}^n)$, $s > 0$.

1. *If* $-\frac{n}{2} < r^*(u_0) < \infty$ *then* $-\frac{n}{2} + s < r^*_s(u_0) < \infty$ *and* $r^*_s(u_0) = s + r^*(u_0)$.
2. $r^*_s(u_0) = \infty$ *if an only if* $r^*(u_0) = \infty$.
3. $r^*(u_0) = -\frac{n}{2}$ *if and only if* $r^*_s(u_0) = r^*(u_0) + s = -\frac{n}{2} + s$.

How does the linear part need to behave ?

Recall Fourier Splitting works if we have info of frequencies near the origin and

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^n)} \leq -C \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\widehat{u}(\xi,t)|^2 \, d\xi.$$

Behavior of linear part?

Let $\mathcal{L} : X^n \to \left(L^2(\mathbb{R}^n)\right)^n$ be a pseudodifferential operator , on X a Hilbert space,so that:

- The symbol $\mathcal{M}(\xi)$ of $\mathcal{L}$ is such that

$$\mathcal{M}(\xi) = P^{-1}(\xi)D(\xi)P(\xi), \qquad \xi - a.e.$$

- $P(\xi) \in O(n)$ and $D(\xi) = -c_i|\xi|^{2\alpha}\delta_{ij}$, for $c_i > c > 0$ and $0 < \alpha \leq 1$. for constants $a, b > 0$, with $0 < \alpha \leq 1$.

$\Rightarrow$ linear equation

$$v_t = \mathcal{L}v. \tag{1}$$

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Behavior of linear part?

Given the linear equation

$$\partial_t v = \mathcal{L} v$$

haciendo el producto en L^2 con v y usando las propiedades de $\mathcal{L}$ llegamos a que podemos usar Fourier Splitting, i.e.

$$\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{R}^n)} \leq -C \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\widehat{v}(\xi,t)|^2 \, d\xi.$$

Example: Compressible approximation to Stokes

Example

Compressible approximation to the Stokes system in $\mathbb{R}^3$ (Temam)

$$u_t = \mathcal{L}u = \Delta u + \frac{1}{\epsilon}\nabla div\, u, \qquad \epsilon > 0. \tag{2}$$

Symbol $(\mathcal{M}(\xi))_{ij} = -|\xi|^2\delta_{ij} - \frac{1}{\epsilon}\xi_i\xi_j$, with
$D(\xi) = diag\left(-|\xi|^2, -|\xi|^2, -\left(1 + \frac{1}{\epsilon}\right)|\xi|^2\right)$ and

$$P(\xi) = \begin{pmatrix} \frac{-\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{-\xi_1\xi_3}{\sqrt{1-\xi_3^2}} & \xi_1 \\ \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{-\xi_2\xi_3}{\sqrt{1-\xi_3^2}} & \xi_2 \\ 0 & \frac{1-\xi_3^2}{\sqrt{1-\xi_3^2}} & \xi_3 \end{pmatrix},$$

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Example continuation

decay Kernel

$$\left(e^{t\mathcal{M}(\xi)}\right)_{ij} = e^{-t|\xi|^2}\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\left(e^{-t|\xi|^2} - e^{-\left(1+\frac{1}{\epsilon}\right)t|\xi|^2}\right), \qquad (3)$$

Results

Theorem (CJN - MES '13)

Let $v_0 \in L^2(\mathbb{R}^n)$ have decay character $r^*(v_0) = r^*$. Let $v(t)$ be a solution to the linear equation with data v_0. Then:

- *if $-\frac{n}{2} < r^* < \infty$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1(1+t)^{-\frac{1}{\alpha}\left(\frac{n}{2}+r^*\right)} \leq \|v(t)\|_{L^2}^2 \leq C_2(1+t)^{-\frac{1}{\alpha}\left(\frac{n}{2}+r^*\right)};$$

Theorem

- *if* $r^* = -\frac{n}{2}$, *there exists* $C > 0$ *such that*

$$\|v(t)\|_{L^2}^2 \geq C(1+t)^{-\epsilon}, \qquad \forall \epsilon > 0,$$

i.e. the decay of $\|v(t)\|_{L^2}^2$ *is slower than any uniform algebraic rate;*

- *if* $r^* = \infty$, *there exists* $C > 0$ *such that*

$$\|v(t)\|_{L^2}^2 \leq C(1+t)^{-m}, \qquad \forall m > 0,$$

i.e. the decay of $\|v(t)\|_{L^2}$ *is faster than any algebraic rate.*

Theorem (CJN - MES '13)

Let $v_0 \in H^s(\mathbb{R}^n), s > 0$ have decay character $r_s^* = r_s^*(v_0)$. Then

1. *if $-\frac{n}{2} \leq r^* < \infty$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1(1+t)^{-\frac{1}{\alpha}\left(\frac{n}{2}+r^*+s\right)} \leq \|v(t)\|_{\dot{H}^s}^2 \leq C_2(1+t)^{-\frac{1}{\alpha}\left(\frac{n}{2}+r^*+s\right)};$$

2. *if $r^* = \infty$, then*

$$\|v(t)\|_{\dot{H}^s}^2 \leq C(1+t)^{-r}, \qquad \forall r > 0,$$

i.e. the decay of $\|v(t)\|_{\dot{H}^s}$ is faster than any algebraic rate.

Quasi-Geostrophic equations

Dissipative Quasi-Geostrophic equations

$$\theta_t + u \cdot \nabla \theta + (-\Delta)^{\frac{\alpha}{2}} \theta = 0, \qquad 0 < \alpha \leq 2$$

where θ is the temperature of a fluid in $\mathbb{R}^2$, and

$$u = R^{\perp}\theta = (-R_2\theta, R_1\theta)$$

where R_i is the Riesz transform in x_i.

Results for QG

Theorem (CJN - MES '13)

Let $\theta_0 \in L^2(\mathbb{R}^2)$, *let* $r^* = r^*(\theta_0)$, $-1 < r^* < \infty$, *and* $0 < \alpha \leq 1$. *Let* θ *be a weak solution to QGE with data* θ_0. *Then:*

1. *If* $r^* \leq 1 - \alpha$, *then*

$$\|\theta(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{1}{\alpha}(1+r^*)};$$

2. *if* $r^* \geq 1 - \alpha$, *then*

$$\|\theta(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{1}{\alpha}(2-\alpha)}.$$

Results QG

Theorem (CJN - MES '13)

Let $\frac{1}{2} < \alpha \leq 1$, $\alpha \leq s$ and $\theta_0 \in H^s(\mathbb{R}^2)$. For $r^* = r^*(\theta_0)$ the solutions to QGE satisfy

1. *if $r^* \leq 1 - \alpha$, then*

$$\|\theta(t)\|^2_{\dot{H}^s} \leq C(t+1)^{-\frac{1}{\alpha}(s+1+r^*)};$$

2. *if $r^* \geq 1 - \alpha$, then*

$$\|\theta(t)\|^2_{\dot{H}^s} \leq C(t+1)^{-\frac{1}{\alpha}(s+2-\alpha)}.$$

Nonlinear minus Linear QG

Decay of the nonlinear part of QGE $w(t) = \theta(t) - \Theta(t)$.

Theorem (CJN - MES '13)

Let $0 < \alpha \leq 1$, $\theta_0 \in L^2(\mathbb{R}^2)$, $r^* = r^*(\theta_0)$. Then

1. *if $-1 < r^* \leq 1 - \alpha$ and $\frac{1+r^*}{\alpha} \geq 1$, we have*
$$\|\theta(t) - \Theta(t)\|^2_{L^2} \leq C(1+t)^{-\frac{1}{\alpha}\min\{2, 2-\alpha+r^*\}};$$
2. *if $-1 < r^* \leq 1 - \alpha$ and $\frac{1+r^*}{\alpha} < 1$, we have*
$$\|\theta(t) - \Theta(t)\|^2_{L^2} \leq C(1+t)^{-\frac{1}{\alpha}(2-\alpha+r^*)};$$
3. *if $r^* \geq 1 - \alpha$, then*
$$\|\theta(t) - \Theta(t)\|^2_{L^2} \leq C(1+t)^{-\frac{1}{\alpha}\min\{3-2\alpha, 2\}}.$$

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Lower Bounds QG

Theorem (CJN - MES '13)

Let $0 < \alpha \leq 1$, $\theta_0 \in L^2(\mathbb{R}^2)$, $r^* = r^*(\theta_0)$. *Then:*

1. *if* $r^* \geq 1 - \alpha$, *we have*

$$\|\theta(t)\|_{L^2}^2 \geq C(1+t)^{-\frac{1}{\alpha}(2-\alpha)};$$

2. *if* $r^* \leq 1 - \alpha$, *we have*

$$\|\theta(t)\|_{L^2}^2 \geq C(1+t)^{-\frac{1}{\alpha}(1+r^*)}.$$

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Approximation for compressible Navier-Stokes

- For Navier-Stokes, the presión p is a non local function of u, $\Rightarrow$ problems for numerical studies
- (Temam '68) Penalization method:

$$\nabla \cdot u = -\epsilon p$$

stabilize with a term of the form $\frac{1}{2}(div\, u)\, u$.

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Compressible Approximation

- We have

$$
\begin{aligned}
u_t^\epsilon + (u^\epsilon \cdot \nabla)\, u^\epsilon \quad &+ \quad \frac{1}{2}\,(div\, u^\epsilon)\ u^\epsilon = \Delta u^\epsilon + \frac{1}{\epsilon} \nabla div\, u^\epsilon, \\
u_0^\epsilon(x) \quad &= \quad u^\epsilon(x, 0).
\end{aligned}
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- Results: Temam '68, Brefort '88, Plecháč and Šverák '03, Rusin '12, + Numerical results.

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- Results: Temam '68, Brefort '88, Plecháč and Šverák '03, Rusin '12, + Numerical results.

Linear part, non linear term

- Linear part $\mathcal{L}$ works with our methods

$$u_t = \mathcal{L}u = \Delta u + \frac{1}{\epsilon} \nabla \cdot div\, u = 0,$$

$$(\mathcal{M}_\epsilon(\xi, t))_{kl} = e^{-t|\xi|^2} \delta_{kl} - \frac{\xi_k \xi_l}{|\xi|^2} \left(e^{-t|\xi|^2} - e^{-\left(1+\frac{1}{\epsilon}\right)t|\xi|^2} \right).$$

- Non linear part similar to NS

$$(u \cdot \nabla)\, u + \frac{1}{2} (div\, u)\, u = \nabla\, (u \otimes u) - \frac{1}{2} (div\, u)\, u.$$

- Compressible part no problem since

$$\int_{\mathbb{R}^3} u(u \cdot \nabla) u\, dx = -\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 div\, u\, dx.$$

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Results

Theorem (C.J.N. - M.E.S. '13)

Let $u_0^\epsilon \in L^2(\mathbb{R}^3)$, $r^* = r^*(u_0^\epsilon)$, $\epsilon > 0$. Then for a weak solution u^ϵ to the compressible approximation to NS we have that:

1. *if $-\frac{3}{2} < r^* \leq \frac{1}{4}$, then*

$$\|u^\epsilon(t)\|_{L^2}^2 \leq C(1+t)^{-\left(r^* + \frac{3}{2}\right)};$$

2. *if $r^* > \frac{1}{4}$, then*

$$\|u^\epsilon(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{7}{4}};$$

Theorem (C.J.N. - M.E.S. '13)

Let $u_0 \in H^r(\mathbb{R}^3)$, $r \geq 1$, $r^* = r^*(u_0)$. *Then, for* $1 \leq s \leq r$ *we have that*

$$\|u(t)\|_{\dot{H}^s} \leq C(1+t)^{-\frac{1}{2}\left(s+\min\{\frac{3}{2}, r^*+\frac{3}{2}\}\right)}.$$

Linear minus nonlinear

Theorem (C.J.N. - M.E.S. '13)

Let $u_0^\epsilon \in L^2(\mathbb{R}^3)$, $r^* = r^*(u_0)$. *Then for compressible NS*

$$\|u^\epsilon(t) - \bar{u}(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{7}{4}}.$$

lower bounds

Will follow by a triangle inequality and estimates of linear part and difference between linear and non linear.

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Will follow by a triangle inequality and estimates of linear part and difference between linear and non linear.

Commments

- Decay character classifies data for which decay is given by the linear part and those where the nonlinear part has a fundamental role.
- Gives a sharp division for upper and lower decay

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Thank you!