# Uniform attractors for a phase transition model coupling momentum balance and phase dynamics 

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#### Abstract

In this paper we are concerned with the uniform attractor for a nonautonomous dynamical system related to the Frémond thermo-mechanical model of shape memory alloys. The dynamical system consists in a diffusive equation for the phase proportions coupled with the hyperbolic momentum balance equation, in the case when a damping term is considered in the latter and the temperature field is prescribed. We prove that the solution to the related initial-boundary value problem yields a semiprocess which is continuous on the proper phase space and satisfies a dissipativity property. Then we show the existence of a unique compact and connected uniform attractor for the system.


## 1 Introduction

Let us fix a bounded and regular subset $\Omega$ of $\mathbb{R}^{3}$ and consider the following system (VSMA) of partial differential equations and relations in terms of the unknown functions $\chi_{1}, \chi_{2}$

Acknowledgemnents. The authors would like to point out the financial support from the MIURCOFIN 2006 research program on "Free boundary problems, phase transitions and models of hysteresis". The work also benefited from a partial support of the IMATI of CNR in Pavia, Italy.
and $\boldsymbol{u}$, in the space-time domain $Q=\Omega \times(0,+\infty)$

$$
\begin{gather*}
k\binom{\chi_{1}}{\chi_{2}}_{t}-\eta\binom{\Delta \chi_{1}}{\Delta \chi_{2}}+\binom{\frac{\ell}{\vartheta^{*}}\left(\vartheta-\vartheta^{*}\right)}{\alpha(\vartheta) \operatorname{div} \boldsymbol{u}}+\partial I_{\mathcal{K}}\left(\chi_{1}, \chi_{2}\right) \ni\binom{0}{0},  \tag{1.1}\\
\boldsymbol{u}_{t t}+c \boldsymbol{u}_{t}-\operatorname{div}\left((-\nu \Delta(\operatorname{div} \boldsymbol{u})+\lambda \operatorname{div} \boldsymbol{u}) \mathbb{I}+2 \mu \varepsilon(\boldsymbol{u})+\alpha(\vartheta) \chi_{2} \mathbb{I}\right)=\boldsymbol{G} ;  \tag{1.2}\\
\chi_{1}(\cdot, 0)=\chi_{1}^{0}, \quad \chi_{2}(\cdot, 0)=\chi_{2}^{0}, \quad \boldsymbol{u}(\cdot, 0)=\boldsymbol{u}_{0}, \quad \boldsymbol{u}_{t}(\cdot, 0)=\boldsymbol{v}_{0} \quad \text { in } \Omega,  \tag{1.3}\\
\partial_{n} \chi_{j}=0 \quad \text { on } \partial \Omega \times(0,+\infty), \quad j=1,2,  \tag{1.4}\\
\boldsymbol{u}=\mathbf{0} \quad \text { on } \partial \Omega \times(0,+\infty),  \tag{1.5}\\
\partial_{n}(\nu \operatorname{div} \boldsymbol{u})=0 \quad \text { on } \partial \Omega \times(0,+\infty) . \tag{1.6}
\end{gather*}
$$

Here, $k, \eta, \ell, \vartheta^{*}, c, \lambda, \mu$ are strictly positive parameters, while the coefficient $\nu$ is allowed to be greater than or equal to 0 . Note that $\boldsymbol{u}=\left(u_{1}, u_{2}, u_{3}\right) \in \mathbb{R}^{3}, \varepsilon(\boldsymbol{u})$ is the tensor given by

$$
\varepsilon(\boldsymbol{u}):=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial x_{j}}+\frac{\partial u_{j}}{\partial x_{i}}\right), \quad i, j=1,2,3,
$$

and $\mathbb{I}$ denotes the identity matrix in $\mathbb{R}^{3}$. Moreover, $\vartheta, \alpha(\vartheta)$ and $\boldsymbol{G}$ are given functions with some properties to be specified later, and $\mathcal{K}$ is the triangle

$$
\begin{equation*}
\mathcal{K}:=\left\{\left(\gamma_{1}, \gamma_{2}\right) \in \mathbb{R}^{2} \text { such that }\left|\gamma_{2}\right| \leq \gamma_{1} \leq 1\right\} \tag{1.7}
\end{equation*}
$$

The symbol $I_{\mathcal{K}}$ in (1.1) denotes the indicator function of the convex set $\mathcal{K}$, namely $I_{\mathcal{K}}(v)=$ 0 if $v$ belongs to $\mathcal{K}$ and $I_{\mathcal{K}}(v)=+\infty$ elsewhere, while $\partial I_{\mathcal{K}}: \mathbb{R}^{2} \rightarrow 2^{\mathbb{R}^{2}}$ stands for its subdifferential, namely $y \in \partial I_{\mathcal{K}}(x)$ iff $x \in \mathcal{K}$ and $(y, x-w) \geq 0 \forall w \in \mathcal{K}$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^{2}$. In particular, $\partial I_{\mathcal{K}}$ turns out to be a multivalued maximal monotone operator (see, e.g., [5, p. 25]). Finally, $\partial_{n}$ denotes the outward normal derivative to the boundary $\partial \Omega$.

The system (VSMA) arises in the study of the behaviour of a viscoelastic shape memory body subjected to mechanical deformations when the temperature field is prescribed. A shape memory material is a metallic alloy which exhibits this peculiar and surprising behaviour: it could be permanently deformed (avoiding fractures) up to $8 \%$ of its strain and subsequently forced to recover its original shape just by thermal means. This unusual property is used nowadays in a variety of engineering applications. In particular, the field of applications of shape memory technologies ranges from bio-engineering, to structuresengineering and aerospace sciences (see [8]). The shape memory phenomenon has been interpreted as the effect of a thermo-elastic solid-solid phase transition between two different crystallographic configurations, the austenite, which is stable at higher temperatures and variants of martensite, stable at lower temperatures (see [1] and [15]). Here, we are interested in the macroscopic modelling proposed by Frémond in [15]. In this connection, $\vartheta$ has to be regarded as the absolute temperature (assumed to be known) of the shape memory sample, while $\boldsymbol{u}$ stands for the vector of small displacements. Hence, the 2-tensor $\varepsilon(\boldsymbol{u})$ represents the linearized strain tensor. Finally, $\chi_{1}, \chi_{2}$ are quantities related to the pointwise proportions of the phases. In particular, if $\beta_{1}, \beta_{2}, \beta_{3}$ denote the local proportions of the two martensitic variants and of the austenite, respectively, we point out that the following conditions have to be fulfilled

$$
\begin{equation*}
0 \leq \beta_{i} \leq 1 \text { for } i=1,2,3, \text { and } \beta_{1}+\beta_{2}+\beta_{3}=1 \tag{1.8}
\end{equation*}
$$

eqn:sposta

We stress that the latter condition forces the phases to attain only meaningful values, that is we are requiring that neither voids nor overlapping zones appear between the phases. Then, by simply eliminating $\beta_{3}$ in the equation above and letting $\chi_{1}=\beta_{1}+\beta_{2}$ and $\chi_{2}=\beta_{1}-\beta_{2}$, it turns out that (1.8) reduces to (see (1.7))

$$
\begin{equation*}
\left(\chi_{1}, \chi_{2}\right) \in \mathcal{K} . \tag{1.9}
\end{equation*}
$$

In particular, the set $\left\{\chi_{1}=1\right\}$ corresponds to the situation in which no austenite is present, and the set $\left\{\chi_{1}=\chi_{2}\right\}$ (resp. $\left\{\chi_{1}=-\chi_{2}\right\}$ ) corresponds to the region where only the first (resp. second) variant of martensite is present.

Finally, $\boldsymbol{G}$ stands for the density of the body forces while $\alpha$ is a rather smooth function related to the thermal expansion of the system (see [15] for further details): in fact, let us refer to [15] for the physical derivation of the model and the related comments. However, we point out that although the full Frémond's model rules the evolution of an unknown temperature field as well, our setting in which $\vartheta$ is a given datum is physically justified and interesting for applications. We recall that also the positive damping $c \boldsymbol{u}_{t}$ (it is not present in the original Frémond's model) in (1.2) has a physical motivation. In particular, this element can be understood as a friction term, hence it serves as a dissipation mechanism.

The mathematical analysis of Frémond's model was initiated in [11] and then extended in various directions. In particular, the reader is referred to [3] and [4] (and references therein) for an updated and minute presentation of the analytical results concerning Frémond's model. We note however, that [11], [3], [4] and most of quoted references deal with the quasistationary situation, in which the macroscopic accelerations are not taken into account. On the other hand, the case in which the macroscopic accelerations are retained in the momentum balance has been studied in [9] and more recently in [23].

The problem of the long-time behaviour of the solutions to Frémond's model has been first considered in [13]. In this paper, the authors investigate the problem of the convergence to the steady state solutions for the full model (but without the inertial term in the momentum balance equation) in the one-dimensional situation. The structure of the $\omega$-limit set has been further analysed in [27]. There, still in the 1-D case it is shown how, in a prescribed temperature range, the set of solutions to the stationary problem contains elements that present a deeply structured alternance of martensitic variants. This fact is in complete agreement with experimental evidence. The study of the asymptotic stability from the point of view of the global attractors has been tackled in [14] in the one-dimensional setting and then extended to the three-dimensional situation in [10]. The analysis of [14] relies on the crucial observation that in the absence of inertial terms the momentum balance equation (i.e., our (1.2) without the part $\boldsymbol{u}_{t t}+c \boldsymbol{u}_{t}$ ) along with the boundary conditions (1.5) and (1.6) allows to completely determine the displacement $\boldsymbol{u}$ in terms of data of the problem and others unknowns. Thus, the original system for three unknowns $\left(\chi_{1}, \chi_{2}\right), \boldsymbol{u}, \vartheta$ reduces to a system for the two unknowns $\left(\chi_{1}, \chi_{2}\right), \vartheta$ in which $\boldsymbol{u}$ (which is now a function of $\left.\left(\chi_{1}, \chi_{2}\right), \vartheta\right)$ plays the role of a driving force depending on time. Consequently, the system is intrinsically nonautonomous. The long-time dynamics of the related dynamical system has been characterized with the aid of the study of a proper limiting autonomous system. In [10] the same type of result has been extended to the three dimensional situation.

In this paper, we aim to analyse the long-time behaviour in terms of the global attractor for (1.1)-(1.2). Our situation differs form the one studied in [14] and [10] since
we retain the macroscopic acceleration in the momentum balance equation (1.2), thus obtaining a hyperbolic equation for the vector $\boldsymbol{u}$. Moreover, we assume that the evolution of the temperature field $\vartheta$ is known. On this regard, the function $\vartheta$ becomes a forcing term depending on time. In particular, our system (VSMA) is nonautonomous. The problem of existence, uniqueness and continuous dependence of solutions to the full system (1.1)-(1.6) has been investigated in [23] when $c=0$ in (1.2). The presence of this damping term is however mandatory from the long-time behaviour point of view: in fact, it provides some dissipation to the system. The existence and uniqueness analysis of [23] refers to the situation in which $c=0$, but let us note that all the results established in [23] extend to the case $c \neq 0$. In this paper, we rely on the concept of uniform attractors to handle the fact that the system is nonautonomous (see Subsection 2.2). In particular, we will prove that the solution operator to (the proper weak formulation of) (1.1)-(1.6) is a semiprocess which is continuous on the proper phase space (see Theorem 2.8). Then, we show the dissipativity of the system in Theorem 3.1 and finally the existence of a unique compact and connected uniform attractor in Theorem 3.5. The crucial step in proving the existence of the uniform attractor relies in the proof of some form of compactness for the solution operator. The simplest and, by the way, the strongest form of compactness one could expect is that the solution operator itself becomes a compact operator after some finite time. Unfortunately, this form of compactness is not usually available for hyperbolic equations (as our (1.2)). Thus, we rely here on the concept of uniform asymptotic compactness, which is well known for autonomous systems and also for nonautonomuos systems (see [21]). In particular, we prove the asymptotic compactness for our system (VSMA) by using the so called energy method introduced by J.M. Ball in [2] and then extended to nonautonomous systems in [21]. It is worthwhile noting that there exist at least another method to prove the asymptotic compactness of the system. As in the autonomous case, one can try to decompose the solution operator into two parts: a (uniformly) compact part and a part which decays to zero as the time goes to infinity (see, e.g., [CHE COSA CITARE QUA?]). This method could be succesfully applied to our system (1.1)-(1.6), under some extra regularity assumptions for the forcing functions $\vartheta$ and $\boldsymbol{G}$ than the one we use to prove the mere existence of solutions and the continuity of the semiprocess (cf. Theorems 2.6 and ... CHE COSA CITARE QUA?). In this concern, we can say that the energy method, since it essentially relies on the standard energy estimate for hyperbolic problems, is optimal with respect to the regularity of the data.

This is the plan of the paper. In Section 2 we introduce the weak formulation of (1.1)(1.6). Moreover, we summarize some preliminary machinery on the long-time behaviour of nonautonomous dynamical system. Section 2.3 contains the results on the well-posedness of the weak formulation of (1.1)-(1.6). Finally, in Section 3 we prove the existence of the uniform attractor for (VSMA).

## 2 Mathematical Setting and Preliminaries

### 2.1 Function spaces and weak formulation

We first introduce some notation. We set

$$
\begin{gathered}
H:=L^{2}(\Omega), \quad V:=H^{1}(\Omega) \\
\boldsymbol{H}:=\left(L^{2}(\Omega)\right)^{3}, \quad \boldsymbol{V}:=\left\{\boldsymbol{v} \in\left(H_{0}^{1}(\Omega)\right)^{3}: \nu \operatorname{div} \boldsymbol{v} \in V\right\},
\end{gathered}
$$

where the coefficient $\nu$ in the definition of $\boldsymbol{V}$ allows to consider at the same time both the $\nu=0$ case and the $\nu>0$ situation. As usual, we identify $H$ and $\boldsymbol{H}$ with their respective dual spaces $H^{\prime}$ and $\boldsymbol{H}^{\prime}$, so that $V \subset H \subset V^{\prime}$ and $\boldsymbol{V} \subset \boldsymbol{H} \subset \boldsymbol{V}^{\prime}$ may be regarded as classical Hilbert triplets. The spaces $H, V, \boldsymbol{H}$ will be endowed with usual norms, while for $\boldsymbol{V}$ we prescribe the equivalent norm

$$
\begin{equation*}
\|\boldsymbol{v}\|_{\boldsymbol{V}}:=\left\{\sum_{i=1}^{3}\left\|\nabla v_{i}\right\|_{\boldsymbol{H}}^{2}+\nu\|\nabla(\operatorname{div} \boldsymbol{v})\|_{\boldsymbol{H}}^{2}\right\}^{1 / 2} \tag{2.1}
\end{equation*}
$$

In the sequel, we denote by $(\cdot, \cdot)_{\Omega}$ the scalar product in $H$ or in $\boldsymbol{H}$, by $\langle\cdot, \cdot\rangle$ the duality pairing between $\boldsymbol{V}^{\prime}$ and $\boldsymbol{V}$ or between $V^{\prime}$ and $V$. The symbol $\|\cdot\|_{E}$ will indicate the norm in the generic normed vector space $E$. In addition, we introduce the continuous and symmetric bilinear form $a(\cdot, \cdot)$ defined for all $\boldsymbol{v}_{1}, \boldsymbol{v}_{2}$ in $\boldsymbol{V}$ by

$$
\begin{gather*}
a\left(\boldsymbol{v}_{1}, \boldsymbol{v}_{2}\right):=\nu \int_{\Omega} \nabla\left(\operatorname{div} \boldsymbol{v}_{1}\right) \cdot \nabla\left(\operatorname{div} \boldsymbol{v}_{2}\right)+\lambda \int_{\Omega} \operatorname{div} \boldsymbol{v}_{1} \operatorname{div} \boldsymbol{v}_{2} \\
+2 \mu \sum_{i, j=1}^{3} \int_{\Omega} \varepsilon_{i j}\left(\boldsymbol{v}_{1}\right) \varepsilon_{i j} .\left(\boldsymbol{v}_{2}\right) \tag{2.2}
\end{gather*}
$$

Recalling the well-known Korn inequality, the following property holds

$$
\begin{equation*}
a(\boldsymbol{v}, \boldsymbol{v}) \geq c_{a}\|\boldsymbol{v}\|_{\boldsymbol{V}}^{2} \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{2.3}
\end{equation*}
$$

Since the special triangular form of $\mathcal{K}$ in (1.7) is not needed in our analysis, we let $\mathcal{K}$ stand for any bounded, convex and closed subset of $\mathbb{R}^{2}$ such that $(0,0) \in \mathcal{K}$. Consequently, we denote by $K:=\left\{\left(\gamma_{1}, \gamma_{2}\right) \in H \times H\right.$ a.e. in $\left.\Omega\right\}$ the realization of $\mathcal{K}$ in $H \times H$, which is clearly bounded, convex and closed. In particular, it is straightforward to find a positive constant $c_{K}$ such that

$$
\begin{equation*}
\left\{\left|\gamma_{1}\right|^{2}+\left|\gamma_{2}\right|^{2}\right\}^{1 / 2} \leq c_{K} \tag{2.4}
\end{equation*}
$$

for all $\left(\gamma_{1}, \gamma_{2}\right) \in K$ and almost everywhere in $\Omega$. The symbol $I_{K}$ will clearly indicate the indicator function of $K$, while $\partial I_{K}$ stands for its subdifferential, which is now a maximal monotone operator in $L^{2}(\Omega) \times L^{2}(\Omega)$.

In order to describe the asymptotic behaviour of solutions we need to introduce the Banach space of $L_{\mathrm{loc}}^{p}$-translation bounded functions with values in a Banach space $B$. More precisely, for $p \geq 1$ we set

$$
\begin{equation*}
L_{\mathrm{loc}}^{p}(0,+\infty ; B):=\left\{v:(0,+\infty) \rightarrow B \text { measurable, } v \in L^{p}(0, T ; B) \forall T>0\right\} \tag{2.5}
\end{equation*}
$$

(pay attention, this is not the standard position) and consequently define

$$
\begin{equation*}
\mathcal{T}_{p}(B):=\left\{v \in L_{\mathrm{loc}}^{p}(0,+\infty ; B):\|v\|_{\mathcal{T}_{p}(B)}=\sup _{t \geq 0} \int_{t}^{t+1}\|v(s)\|_{B}^{p} d s<+\infty\right\} \tag{2.6}
\end{equation*}
$$

We prescribe the following assumptions on data

$$
\begin{gather*}
\left(\chi_{1}^{0}, \chi_{2}^{0}\right) \in K \cap(V \times V)  \tag{2.7}\\
\boldsymbol{u}_{0} \in \boldsymbol{V}, \quad \boldsymbol{v}_{0} \in \boldsymbol{H}  \tag{2.8}\\
\alpha \in W^{1, \infty}(\mathbb{R}),  \tag{2.9}\\
\boldsymbol{G} \in L_{\mathrm{loc}}^{2}(0,+\infty ; \boldsymbol{H}),  \tag{2.10}\\
\vartheta \in L_{\mathrm{loc}}^{2}(0,+\infty ; V) . \tag{2.11}
\end{gather*}
$$

VANNO BENE LE ULTIME DUE IPOTESI? NON TI SERVE CHE I DUE DATI SIANO IN $\mathcal{T}_{2}(\cdot)$ VISTO CHE HAI INTRODOTTO QUESTI SPAZI? Assumptions (2.7) and (2.8) suggest to investigate the long-time behaviour of solutions in the complete phase space $\mathcal{X}:=\left(K \cap V^{2}\right) \times \boldsymbol{V} \times \boldsymbol{H}$, with the metric

$$
\begin{align*}
& d_{\mathcal{X}}\left(\left(\left(\xi_{1}, \xi_{2}\right), \boldsymbol{\xi}_{3}, \boldsymbol{\xi}_{4}\right),\left(\left(\zeta_{1}, \zeta_{2}\right), \boldsymbol{\zeta}_{3}, \boldsymbol{\zeta}_{4}\right)\right) \\
& :=\sum_{j=1}^{2}\left\|\xi_{j}-\zeta_{j}\right\|_{V}+\left\|\boldsymbol{\xi}_{3}-\boldsymbol{\zeta}_{3}\right\|_{\boldsymbol{V}}+\left\|\boldsymbol{\xi}_{4}-\boldsymbol{\zeta}_{4}\right\|_{\boldsymbol{H}} \tag{2.12}
\end{align*}
$$

We consider now the weak formulation of (1.1)-(1.6). For our convenience, we also introduce the space

$$
\begin{equation*}
W:=\left\{v \in H^{2}(\Omega): \partial_{n} v=0 \text { in } \partial \Omega\right\}, \tag{2.13}
\end{equation*}
$$

h2neumann
which takes into account Neumann homogeneous boundary conditions.
problema Problem 2.1. Under the assumptions (2.7)-(2.11), find $\chi_{1}, \chi_{2}, h_{1}, h_{2}$, $\boldsymbol{u}$ satisfying

$$
\begin{gather*}
\chi_{1}, \chi_{2} \in H^{1}(0, T ; H) \cap C^{0}([0, T] ; V) \cap L^{2}(0, T ; W),  \tag{2.14}\\
\quad h_{1}, h_{2} \in L^{2}(0, T ; H),  \tag{2.15}\\
\boldsymbol{u} \in H^{2}\left(0, T ; \boldsymbol{V}^{\prime}\right) \cap C^{1}([0, T] ; \boldsymbol{H}) \cap C^{0}([0, T] ; \boldsymbol{V}) \tag{2.16}
\end{gather*}
$$

for all $T>0$,

$$
\begin{equation*}
\left(\chi_{1}(t), \chi_{2}(t)\right) \in K \tag{2.17}
\end{equation*}
$$

for every $t \geq 0$, solving almost everywhere in the time interval $(0,+\infty)$

$$
\begin{gather*}
k\binom{\chi_{1 t}}{\chi_{2 t}}-\eta\binom{\Delta \chi_{1}}{\Delta \chi_{2}}+\binom{\frac{\ell}{\vartheta^{*}}\left(\vartheta-\vartheta^{*}\right)}{\alpha(\vartheta) \operatorname{div} \boldsymbol{u}}+\binom{h_{1}}{h_{2}}=\binom{0}{0}, \text { a.e. in } \Omega,  \tag{2.18}\\
\binom{h_{1}}{h_{2}} \in \partial I_{K}\left(\chi_{1}, \chi_{2}\right), \text { a.e. in } \Omega,  \tag{2.19}\\
\left\langle\boldsymbol{u}_{t t}, \boldsymbol{v}\right\rangle+c\left\langle\boldsymbol{u}_{t}, \boldsymbol{v}\right\rangle+a(\boldsymbol{u}, \boldsymbol{v})+\left(\alpha(\vartheta) \chi_{2}, \operatorname{div} \boldsymbol{v}\right)_{\Omega}=\langle\boldsymbol{G}, \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.20}
\end{gather*}
$$

and such that

$$
\begin{array}{ll}
\boldsymbol{u}(0)=\boldsymbol{u}_{0} \text { in } \boldsymbol{V}, & \boldsymbol{u}_{t}(0)=\boldsymbol{v}_{0} \text { in } \boldsymbol{H}, \\
\chi_{1}(0)=\chi_{1}^{0} \text { in } H, & \chi_{2}(0)=\chi_{2}^{0} \text { in } H . \tag{2.21}
\end{array}
$$

### 2.2 Long-time behaviour of nonautonomous evolution systems: the abstract setting

In this subsection, we present some known results on the long-time behaviour of nonautonomous systems especially in connection with the construction of the so-called uniform attractor. The reader is referred to the seminal references $[25,26,17,6]$ for the related proofs and further remarks.

The basic concept in studying the long-time behaviour of a nonautonomous system is the notion of semiprocess. Let $\mathcal{X}$ and $\Sigma$ be two complete metric spaces. We say that the set $\left\{U_{\sigma}(t, \tau)\right\}_{t \geq \tau \geq 0, \sigma \in \Sigma}$ is a family of semiprocesses in $\mathcal{X}$ if the following properties are satisfied

$$
\begin{align*}
& U_{\sigma}(t, \tau): \mathcal{X} \rightarrow \mathcal{X} \text { for any } t \geq \tau \geq 0  \tag{2.22}\\
& U_{\sigma}(\tau, \tau) \text { is the identity map on } \mathcal{X} \text { for any } \tau \geq 0  \tag{2.23}\\
& U_{\sigma}(t, s) U_{\sigma}(s, \tau)=U_{\sigma}(t, \tau) \text { for any } t \geq s \geq \tau \geq 0 \tag{2.24}
\end{align*}
$$

for each $\sigma \in \Sigma$. Such a $\Sigma$ is the so-called symbol space. As we shall see in the concrete case of system (2.18)-(2.20), the symbol space $\Sigma$ will be a space of time dependent functions which collects all the forcing terms that depend on time. Let $\left\{T_{h}\right\}_{h \geq 0}$ be a semigroup of translations in $\Sigma$, that is $\left(T_{h}(\sigma)\right)(t):=\sigma(t+h)$, and assume the following translation invariance condition

$$
\begin{equation*}
U_{T_{s}(\sigma)}(t, \tau)=U_{\sigma}(t+s, \tau+s), \quad \forall \sigma \in \Sigma, \quad \forall t \geq \tau \geq 0, \quad \forall s \geq 0 \tag{2.25}
\end{equation*}
$$

The parameter $\sigma$ is then termed the time symbol of the semiprocess $U_{\sigma}(t, \tau)$. The class of translation compact forcing functions will be of interest for us: we say that $\sigma$ is translation compact if

$$
\begin{equation*}
\text { the hull } \mathcal{H}(\sigma):=\left[T_{h}(\sigma), h \in[0,+\infty)\right]_{\Sigma} \text { is compact in } \Sigma, \tag{2.26}
\end{equation*}
$$

where $[\cdot]_{\Sigma}$ denotes the closure in $\Sigma$. (HO CORRETTO METTENDO $[0,+\infty)$ CON 0 INCLUSO QUI SOPRA: PREGO ANTONIO DI CONTROLLARE) The class of translation compact functions is quite large. A useful criterion to check if a given function $\sigma \in L_{\mathrm{loc}}^{p}(0,+\infty ; B)$, with $B$ a Banach space, is translation-compact is as follows (see [7, Proposition V.3.3]).

Proposition 2.2. A function $\sigma$ is translation-compact in $L_{\mathrm{loc}}^{p}(0,+\infty ; B)$ if and only if

1. for any $h \geq 0$ the set $\left\{\int_{t}^{t+h} \sigma(s) d s: t \geq 0\right\}$ is precompact in $B$;
2. there exists a function $\lambda$, with $\lambda(s) \searrow 0$ as $s \searrow 0$, such that

$$
\begin{equation*}
\int_{t}^{t+1}\|\sigma(s+h)-\sigma(s)\|_{B}^{p} d s \leq \lambda(h) \tag{2.27}
\end{equation*}
$$

for all $t \geq 0$ and $h \geq 0$.
The family of semiprocesses is said to be $\mathcal{X} \times \Sigma$-continuous if, for any $t, \tau$ with $t \geq \tau \geq 0$, the map $(v, \sigma) \mapsto U_{\sigma}(t, \tau) v$ is continuous from $\mathcal{X} \times \Sigma$ to $\mathcal{X}$.

Now, let us recall the notions of absorbing set and attractor for the family of semiprocesses $\left\{U_{\sigma}(t, \tau)\right\}_{t>\tau>0, \sigma \in \Sigma}$. We say that $\mathcal{B} \subset \mathcal{X}$ is a uniformly absorbing set if for any $\tau \geq 0$ and any $B \subset \mathcal{X}$ bounded, there exists a time $T=T(\tau, B) \geq \tau$ such that $U_{\sigma}(t, \tau) B \subset \mathcal{B}$ for any $t \geq T$ and for all $\sigma \in \Sigma$. Then, we say that $\mathcal{K} \subset \mathcal{X}$ is uniformly attracting for $\left\{U_{\sigma}(t, \tau)\right\}$ if

$$
\begin{equation*}
\lim _{t /+\infty} \sup _{\sigma \in \Sigma} \operatorname{dist}_{\mathcal{X}}\left(U_{\sigma}(t, \tau) B, \mathcal{K}\right)=0, \quad \forall \tau \geq 0, \forall B \subset \mathcal{X} \text { bounded } \tag{2.28}
\end{equation*}
$$

where

$$
\operatorname{dist}_{\mathcal{X}}(A, B):=\sup _{a \in A} \inf _{b \in B} d_{\mathcal{X}}(a, b)
$$

denotes the semidistance of two sets $A, B \subset \mathcal{X}$. Finally, we say that $\mathcal{A}$ is the uniform attractor for the family $\left\{U_{\sigma}(t, \tau)\right\}$ if it is at the same time uniformly attracting and contained in every closed uniformly attracting set (minimality property). Then, it is unique by construction.

Now, we quote a general abstract criterion providing sufficient conditions for the existence of the uniform attractor (see [21, Theorem 2.3]).
attractor Theorem 2.3. Let $\left\{U_{f}(t, \tau)\right\}_{t \geq r \geq 0}$ be a continuous family of semiprocess in $\mathcal{X}$ with $f \in$ $\mathcal{H}(\sigma)$ and $\sigma$ translation compact. Let us assume that

1. $\left\{U_{f}(t, \tau)\right\}$ possesses a bounded uniformly absorbing set $\mathcal{B}$ (dissipativity);
2. $\left\{U_{f}(t, \tau)\right\}$ is uniformly asymptotically compact, i.e.

$$
\left\{\begin{array}{l}
\left\{z_{0 n}\right\}_{n \in \mathbb{N}} \text { bounded in } \mathcal{X}  \tag{2.29}\\
\left\{f_{n}\right\}_{n \in \mathbb{N}} \subset \mathcal{H}(\sigma) \\
t_{n} \nearrow+\infty
\end{array} \Longrightarrow\left\{U_{f_{n}}\left(t_{n}, 0\right) z_{0 n}\right\}_{n \in \mathbb{N}} \quad \text { precompact in } \mathcal{X}\right.
$$

Then, $\left\{U_{f}(t, \tau)\right\}$ possesses a compact uniform attractor.
Note that the above definition of uniform asymptotic compactness (taken from [21]) is different from the one given by Haraux [17]. More precisely, in [17] a semiprocess is said to be uniformly asymptotically compact if it possesses a compact uniformly attracting set in the sense of (2.28). However, it is not difficult to prove that if a semiprocess is uniformly asymptotically compact in the sense of (2.29) and possesses a bounded uniformly absorbing set, then it is uniformly asymptotically compact in the sense of Haraux. LE NOTE CHE ANTONIO AVEVA SCRITTO A MARGINE DELLA PRECEDENTE VERSIONE VANNO INCORPORATE QUA? Furthermore, we can note that the notion of uniform asymptotic compactness introduced in [21] and used in this paper, is also completely in agreement with the corresponding definition for semigroups (see [29]) and seems easier to be verified using the energy method.

Starting from the semiprocess $U_{f}(t, \tau), f \in \mathcal{H}(\sigma)$ we can define a semigroup $\boldsymbol{S}_{t}$ acting on the extended phase space $\mathcal{X} \times \mathcal{H}(\sigma)$ as follows

$$
\begin{equation*}
\boldsymbol{S}_{t}\left(z_{0}, f\right):=\left(U_{f}(t, 0) z_{0}, T_{t}(f)\right), \quad \boldsymbol{S}_{t}: \mathcal{X} \times \mathcal{H}(\sigma) \rightarrow \mathcal{X} \times \mathcal{H}(\sigma) \tag{2.30}
\end{equation*}
$$

This construction is well known (see, e.g., [6]). It is also well known that the uniform attractor $\mathcal{A}$ could be equivalently defined in terms of the global attractor $\boldsymbol{A} \subset \mathcal{X} \times \mathcal{H}(\sigma)$ of the semigroup $\boldsymbol{S}_{t}$, that is

$$
\begin{equation*}
\mathcal{A}=\Pi_{1} \boldsymbol{A}, \quad \text { where } \Pi_{1} \text { is the projection on the first component. } \tag{2.31}
\end{equation*}
$$

This construction will help us in proving the connectedness of the uniform attractor for our system (VSMA).

We conclude this subsection by recalling two technical results which will be useful in the course of our argumentation. we start with the so-called Uniform Gronwall Lemma (for the proof see [29, Lemma III.1.1]).

Lemma 2.4. Let $y, a, b \in L_{\mathrm{loc}}^{1}(0,+\infty)$ be three non negative functions such that $y^{\prime} \in$ $L_{\mathrm{loc}}^{1}(0,+\infty)$ and

$$
y^{\prime}(t) \leq a(t) y(t)+b(t) \quad \text { for a.e. } t>0,
$$

and let $a_{1}, a_{2}, a_{3}$ be three non negative constants such that

$$
\|a\|_{\mathcal{T}_{1}} \leq a_{1}, \quad\|b\|_{\mathcal{T}_{1}} \leq a_{2}, \quad\|y\|_{\mathcal{T}_{1}} \leq a_{3}
$$

Then, we have

$$
y(t+1) \leq\left(a_{2}+a_{3}\right) \exp ^{a_{1}} \quad \text { for all } t>0
$$

Next, we prepare a Gronwall-type lemma prompted by [22].
nwall-diff Lemma 2.5. Let $\varphi, m_{1}$ and $m_{2}$ be three non-negative locally summable functions on $[\tau,+\infty)$ which satisfy, for some $\varepsilon>0$, the differential inequality

$$
\begin{equation*}
\frac{d}{d t} \varphi^{2}(t)+\varepsilon \varphi^{2}(t) \leq m_{1}(t) \varphi(t)+m_{2}(t) \quad \text { for a.e. } t \in[\tau,+\infty) \tag{2.32}
\end{equation*}
$$

Then it results that

$$
\begin{equation*}
\varphi^{2}(t) \leq 2 \varphi^{2}(\tau) e^{-\varepsilon(t-\tau)}+\left(\int_{\tau}^{t} m_{1}(s) e^{-\varepsilon(t-s) / 2} d s\right)^{2}+2 \int_{\tau}^{t} m_{2}(s) e^{-\varepsilon(t-s)} d s \tag{2.33}
\end{equation*}
$$

gronwall-d
gronwall-d
for any $t \in[\tau,+\infty)$. Moreover, the inequality

$$
\begin{equation*}
\int_{\tau}^{t} m(s) e^{-\varepsilon(t-s)} d s \leq \frac{1}{1-e^{-\varepsilon}} \sup _{r \geq \tau} \int_{r}^{r+1} m(s) d s \tag{2.34}
\end{equation*}
$$

holds for every non-negative locally summable function $m$ on $[\tau,+\infty)$ and every $\varepsilon>0$.
Proof. Observe that (2.32) entails

$$
\frac{d}{d t}\left(e^{\varepsilon(t-\tau)} \varphi^{2}(t)\right) \leq m_{1}(t) e^{\varepsilon(t-\tau)} \varphi(t)+m_{2}(t) e^{\varepsilon(t-\tau)}
$$

Therefore, setting $\psi(t)=e^{\varepsilon(t-\tau) / 2} \varphi(t)$ and integrating, it turns out that

$$
\frac{1}{2} \psi^{2}(t) \leq \frac{1}{2} \psi^{2}(\tau)+\frac{1}{2} \int_{\tau}^{t} m_{2}(s) e^{\varepsilon(s-\tau)} d s+\int_{\tau}^{t} \frac{m_{1}(s)}{2} e^{\varepsilon(s-\tau) / 2} \psi(s) d s
$$

for any $t \geq \tau$. Now, by applying, e.g., [5, Lemme A.5, p. 157] we infer that

$$
\psi(t) \leq\left(\psi^{2}(\tau)+\int_{\tau}^{t} m_{2}(s) e^{\varepsilon(s-\tau)} d s\right)^{1 / 2}+\int_{\tau}^{t} \frac{m_{1}(s)}{2} e^{\varepsilon(s-\tau) / 2} d s
$$

whence (2.33) follows easily. Let us now check (2.34). Denoting by $j$ the integer part of $(t-\tau)$, we have

$$
\begin{aligned}
& \int_{\tau}^{t} m(s) e^{-\varepsilon(t-s)} d s \leq \int_{0}^{t-\tau} m(t-s) e^{-\varepsilon s} d s \\
& \leq \sum_{n=0}^{j-1}\left(e^{-\varepsilon n} \int_{n}^{n+1} m(t-s) d s\right)+e^{-\varepsilon j} \int_{\tau}^{\tau+1} m(s) d s \\
& \leq \sup _{r \geq \tau} \int_{r}^{r+1} m(s) d s \sum_{n=0}^{j} e^{-\varepsilon n} \leq \frac{1}{1-e^{-\varepsilon}} \sup _{r \geq \tau} \int_{r}^{r+1} m(s) d s
\end{aligned}
$$

and this concludes the proof.

### 2.3 Well-posedness

The well-posedness of (2.18)-(2.20) has been proved in [23, Theorems 2.2 and 2.4] for the case $c=0$ in (2.20). The argument of the proof relies on a time discretization - a priori estimates - passage to the limit procedure. Moreover, [24] contains the error estimates for the time discretization scheme approximating (2.18)-(2.20). The situation in which $c>0$ can be analysed similarly, by simply adapting the proofs of [23] and [24]. The following statement holds.
esun Theorem 2.6 (Well-posedness). Under the assumptions (2.7)-(2.11), there exists a quintuple $\left(\chi_{1}, \chi_{2}, h_{1}, h_{2}, \boldsymbol{u}\right)$ solving Problem 2.1. Moreover, if

$$
\begin{equation*}
\vartheta \in L_{\mathrm{loc}}^{2}\left(0,+\infty ; W^{1,3}(\Omega)\right), \tag{2.35}
\end{equation*}
$$

then the solution is unique and depends continuously on data. Namely, letting $\mathcal{F}_{i}=$ $\left\{\boldsymbol{u}_{i}^{0}, \boldsymbol{w}_{i}^{0},\left(\chi_{1 i}^{0}, \chi_{2 i}^{0}\right), \vartheta_{i}, \boldsymbol{G}_{i}\right\}, i=1,2$, represent two families of data that satisfy (2.7)(2.10) and (2.35), and denoting by $\left(\boldsymbol{u}_{1}, \chi_{11}, \chi_{21}\right),\left(\boldsymbol{u}_{2}, \chi_{12}, \chi_{22}\right)$ the corresponding solution components, then there is a positive constant $\boldsymbol{\Lambda}$, which depends only on the quantities $k, \eta, \ell, \vartheta^{*}, c_{\kappa}, c, c_{a}, T, \Omega,\|\alpha\|_{W^{1, \infty}(\mathbb{R})}$, and $\max _{i=1,2}\left\{\left\|\nabla \vartheta_{i}\right\|_{L^{2}\left(0, T ;\left(L^{3}(\Omega)\right)^{3}\right)}\right\}$, such that

$$
\begin{align*}
& \left\|\boldsymbol{u}_{1}-\boldsymbol{u}_{2}\right\|_{C^{1}([0, T] ; \boldsymbol{H}) \cap C^{0}([0, T] ; \boldsymbol{V})}^{2}+\sum_{j=1}^{2}\left\|\chi_{j 1}-\chi_{j 2}\right\|_{C^{0}([0, T] ; H) \cap L^{2}(0, T ; V)}^{2} \\
& \leq \boldsymbol{\Lambda}\left(\left\|\boldsymbol{w}_{01}-\boldsymbol{w}_{02}\right\|_{\boldsymbol{H}}^{2}+\left\|\boldsymbol{u}_{01}-\boldsymbol{u}_{02}\right\|_{\mathbf{V}}^{2}+\sum_{j=1}^{2}\left\|\chi_{j 1}^{0}-\chi_{j 2}^{0}\right\|_{H}^{2}\right. \\
& \left.+\left\|\boldsymbol{G}_{1}-\boldsymbol{G}_{2}\right\|_{L^{2}(0, T ; \boldsymbol{H})}^{2}+\left\|\vartheta_{1}-\vartheta_{2}\right\|_{L^{2}(0, T ; H)}^{2}+\left\|\alpha\left(\vartheta_{1}\right)-\alpha\left(\vartheta_{2}\right)\right\|_{L^{2}(0, T ; V)}^{2}\right) \tag{2.36}
\end{align*}
$$

Finally, if ( $\boldsymbol{u}, \chi_{1}, \chi_{2}$ ) yields a solution to (2.18)-(2.20) and we set

$$
\begin{equation*}
\mathcal{E}\left(\boldsymbol{u}_{t}, \boldsymbol{u}\right)(t):=\frac{1}{2}\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}+\frac{c}{2}\left\langle\boldsymbol{u}_{t}, \boldsymbol{u}\right\rangle+\frac{1}{2} a(\boldsymbol{u}(t), \boldsymbol{u}(t)), \tag{2.37}
\end{equation*}
$$

## the following identity

$$
\begin{gather*}
\mathcal{E}\left(\boldsymbol{u}_{t}, \boldsymbol{u}\right)(M)=e^{-c M} \mathcal{E}\left(\boldsymbol{u}_{t}, \boldsymbol{u}\right)(0)+\int_{0}^{M} e^{c(t-M)}\left(\boldsymbol{G}(t), \boldsymbol{u}_{t}(t)+\frac{c}{2} \boldsymbol{u}(t)\right)_{\Omega} d t \\
+\int_{0}^{M} e^{c(t-M)}\left(\nabla\left(\alpha(\vartheta(t)) \chi_{2}(t)\right), \boldsymbol{u}_{t}(t)+\frac{c}{2} \boldsymbol{u}(t)\right)_{\Omega} d t \tag{2.38}
\end{gather*}
$$

is satisfied for all $M>0$.
Proof. The existence, uniqueness and continuous dependence result is essentially proved in [23]. Here, we give a proof of the energy equality (2.38) that will be of fundamental importance in proving the uniform asymptotic compactness of the system. To show (2.38) we rely on an approximation argument similar to the one devised in [12, Appendix]. If $\boldsymbol{u} \in H_{\mathrm{loc}}^{2}\left(0,+\infty, \boldsymbol{V}^{\prime}\right) \cap C^{1}([0,+\infty), \boldsymbol{H}) \cap C^{0}([0,+\infty) ; \boldsymbol{V})$ solve the hyperbolic equation (2.20), for any $\varepsilon>0$ we let $\boldsymbol{u}^{\varepsilon}$ be the unique solution of

$$
\begin{equation*}
\left\langle\boldsymbol{u}^{\varepsilon}, \boldsymbol{v}\right\rangle+\varepsilon^{2} a\left(\boldsymbol{u}^{\varepsilon}, \boldsymbol{v}\right)=\langle\boldsymbol{u}, \boldsymbol{v}\rangle, \text { a.e. in }(0,+\infty), \forall \boldsymbol{v} \in \boldsymbol{V} \tag{2.39}
\end{equation*}
$$

The behaviour of $\boldsymbol{u}^{\varepsilon}$ as $\varepsilon \searrow 0$ is well known (cf., e.g., [19]). In particular, for all $T \in$ $[0,+\infty)$ we have that

$$
\begin{gather*}
\boldsymbol{u}^{\varepsilon}(T) \rightarrow \boldsymbol{u}(T) \text { in } \boldsymbol{V},  \tag{2.40}\\
\boldsymbol{u}_{t}^{\varepsilon}(T) \rightarrow \boldsymbol{u}_{t}(T) \text { in } \boldsymbol{H},  \tag{2.41}\\
\varepsilon \boldsymbol{u}_{t}^{\varepsilon}(T) \rightarrow \mathbf{0} \text { in } \boldsymbol{V},  \tag{2.42}\\
\boldsymbol{u}_{t t}^{\varepsilon} \rightarrow \boldsymbol{u}_{t t} \text { in } L^{2}\left(0, T ; \boldsymbol{V}^{\prime}\right) \tag{2.43}
\end{gather*}
$$

pert-conv1
pert-conv2
pert-conv3
pert-conv4
as well as other convergences that can be inferred from the regularity of $\boldsymbol{u}$ and (2.39). Now, putting $\boldsymbol{v}=\boldsymbol{u}_{t}^{\varepsilon}+(c / 2) \boldsymbol{u}^{\varepsilon}$ in (2.20) and using also a Green formula, we obtain

$$
\begin{align*}
& \frac{d}{d t} \mathcal{E}\left(\boldsymbol{u}_{t}^{\varepsilon}, \boldsymbol{u}^{\varepsilon}\right)(t)+\mathcal{E}\left(\boldsymbol{u}_{t}^{\varepsilon}, \boldsymbol{u}^{\varepsilon}\right)(t)+\left\langle\boldsymbol{u}_{t t}(t)-\boldsymbol{u}_{t t}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right\rangle+c\left\langle\boldsymbol{u}_{t}(t)-\boldsymbol{u}_{t}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right\rangle \\
&+\left\langle\boldsymbol{u}_{t t}(t)-\boldsymbol{u}_{t t}^{\varepsilon}(t), \frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right\rangle+c\left\langle\boldsymbol{u}_{t}(t)-\boldsymbol{u}_{t}^{\varepsilon}(t), \frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right\rangle \\
&+a\left(\boldsymbol{u}(t)-\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right)+\frac{c}{2} a\left(\boldsymbol{u}(t)-\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{u}^{\varepsilon}(t)\right) \\
&=\left(\boldsymbol{G}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)+\frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right)_{\Omega}+\left(\nabla\left(\alpha(\vartheta(t)) \chi_{2}(t)\right), \boldsymbol{u}_{t}^{\varepsilon}(t)+\frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right)_{\Omega} \tag{2.44}
\end{align*}
$$

for a.e. $t \in(0,+\infty)$. Next, we multipy (2.44) by $e^{c(t-M)}$, with $M>0$, and integrate between 0 and $M$. We infer that

$$
\begin{gather*}
\mathcal{E}\left(\boldsymbol{u}_{t}^{\varepsilon}, \boldsymbol{u}^{\varepsilon}\right)(M)=e^{-c M} \mathcal{E}\left(\boldsymbol{u}_{t}^{\varepsilon}, \boldsymbol{u}^{\varepsilon}\right)(0)+\int_{0}^{M} e^{c(t-M)}\left(\boldsymbol{G}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)+\frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right)_{\Omega} d t \\
\int_{0}^{M} e^{c(t-M)}\left(\nabla\left(\alpha(\vartheta(t)) \chi_{2}(t)\right), \boldsymbol{u}_{t}^{\varepsilon}(t)+\frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right)_{\Omega} d t-\sum_{i=1}^{6} J_{i}^{\varepsilon}(M), \tag{2.45}
\end{gather*}
$$

where the residual terms $\left\{J_{i}^{\varepsilon}(M)\right\}_{i=1}^{6}$ are

$$
\begin{align*}
J_{1}^{\varepsilon}(M) & =\int_{0}^{M} e^{c(t-M)}\left\langle\boldsymbol{u}_{t t}(t)-\boldsymbol{u}_{t t}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right\rangle d t  \tag{2.46}\\
J_{2}^{\varepsilon}(M) & =\int_{0}^{M} c e^{c(t-M)}\left\langle\boldsymbol{u}_{t}(t)-\boldsymbol{u}_{t}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right\rangle d t  \tag{2.47}\\
J_{3}^{\varepsilon}(M) & =\int_{0}^{M} e^{c(t-M)}\left\langle\boldsymbol{u}_{t t}(t)-\boldsymbol{u}_{t t}^{\varepsilon}(t), \frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right\rangle d t  \tag{2.48}\\
J_{4}^{\varepsilon}(M) & =\int_{0}^{M} c e^{c(t-M)}\left\langle\boldsymbol{u}_{t}(t)-\boldsymbol{u}_{t}^{\varepsilon}(t), \frac{c}{2} \boldsymbol{u}^{\varepsilon}(t)\right\rangle d t  \tag{2.49}\\
J_{5}^{\varepsilon}(M) & =\int_{0}^{M} e^{c(t-M)} a\left(\boldsymbol{u}(t)-\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right) d t  \tag{2.50}\\
J_{6}^{\varepsilon}(M) & =\int_{0}^{M} \frac{c}{2} e^{c(t-M)} a\left(\boldsymbol{u}(t)-\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{u}^{\varepsilon}(t)\right) d t \tag{2.51}
\end{align*}
$$

The goal is plainly to prove that any of these $J_{i}^{\varepsilon}(M)$-terms tends to 0 as $\varepsilon \searrow 0$. Indeed, from (2.40)- (2.43) and the regularity of $\boldsymbol{u}$ it is clear that the left hand side and the first three terms in the right hand side of (2.45) converge to their respective limits in (2.38). On the other hand, the convergence to 0 of the residual terms $\left\{J_{i}^{\varepsilon}(M)\right\}_{i=1}^{6}$ can be shown using the methods employed in [12, Appendix], to which we refer for getting the right hints on how to manage things. Just for helping the reader a bit, let us develop the computation for

$$
\begin{gathered}
J_{1}^{\varepsilon}(M)=\int_{0}^{M} e^{c(t-M)}\left\langle\boldsymbol{u}_{t t}(t)-\boldsymbol{u}_{t t}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right\rangle d t=\int_{0}^{M} e^{c(t-M)} \varepsilon^{2} a\left(\boldsymbol{u}_{t t}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right) d t \\
=\frac{1}{2} a\left(\varepsilon \boldsymbol{u}_{t}^{\varepsilon}(M), \varepsilon \boldsymbol{u}_{t}^{\varepsilon}(M)\right)-\frac{1}{2} e^{-c M} a\left(\varepsilon \boldsymbol{u}_{t}^{\varepsilon}(0), \varepsilon \boldsymbol{u}_{t}^{\varepsilon}(0)\right)-\int_{0}^{M} \frac{c \varepsilon^{2}}{2} e^{c(t-M)} a\left(\boldsymbol{u}_{t}^{\varepsilon}(t), \boldsymbol{u}_{t}^{\varepsilon}(t)\right) d t \\
=\frac{1}{2} a\left(\varepsilon \boldsymbol{u}_{t}^{\varepsilon}(M), \varepsilon \boldsymbol{u}_{t}^{\varepsilon}(M)\right)-\frac{1}{2} e^{-c M} a\left(\varepsilon \boldsymbol{u}_{t}^{\varepsilon}(0), \varepsilon \boldsymbol{u}_{t}^{\varepsilon}(0)\right)-\frac{1}{2} J_{2}^{\varepsilon}(M)
\end{gathered}
$$

and note that the last line tends to 0 as $\varepsilon \searrow 0$ because of (2.41)-(2.42).
Definition 2.7. For $t \geq \tau \geq 0$ we denote by $U_{\sigma}(t, \tau) z_{0}$ the triplet

$$
\left(\left(\chi_{1}, \chi_{2}\right)(t), \boldsymbol{u}(t), \boldsymbol{u}_{t}(t)\right)
$$

related to the solution of Problem 2.1 but precisely

- satisfying (2.17) for every $t \geq \tau$;
- solving (2.18)-(2.20) almost everywhere in $(\tau,+\infty)$, for some selection pair $\left(h_{1}, h_{2}\right)$, with source term

$$
\sigma=(\boldsymbol{G}, \vartheta) \in L_{\mathrm{loc}}^{2}(0, \infty ; \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(0, \infty ; W^{1,3}(\Omega)\right) ;
$$

- assuming the initial value

$$
z_{0}=\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right) \in\left(K \cap V^{2}\right) \times \boldsymbol{V} \times \boldsymbol{H}
$$

at time $\tau$, that is $U_{\sigma}(\tau, \tau) z_{0}=z_{0}$.

As an immediate consequence of Theorem 2.6, we have that the collection of solving operators $\left\{U_{\sigma}(t, \tau)\right\}$ yields a family of semiprocesses in $\mathcal{X}$ (see (2.12)) with (time) symbol space

$$
\begin{equation*}
\Sigma=L_{\mathrm{loc}}^{2}(0, \infty ; \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(0, \infty ; W^{1,3}(\Omega)\right) \tag{2.52}
\end{equation*}
$$

In fact, it satisfies (2.22)-(2.24). Moreover, the solution operator $U_{\sigma}(t, \tau)$ enjoys also the translation invariance condition (2.25). Also, note that the space $\Sigma$, equipped with the topology of the local $L^{2}$-convergence in all intervals $(0, T), T>0$ (cf. (2.5)), is metrizable and the corresponding metric space is complete (for more details on such spaces see, e.g., [7, Chapter V]).

Observe however that we cannot immediately conclude from (2.36) that the the mapping $\left(z_{0}, \sigma\right) \mapsto U_{\sigma}(t, \tau) z_{0}$ is continuous from $\mathcal{X} \times \Sigma$ to $\mathcal{X}$. Indeed, the metric on $\mathcal{X}$ (cf. (2.12)) involves the gradients of the phase variable ( $\chi_{1}, \chi_{2}$ ) and the continuous dependence estimate (2.36) entails no pointwise control in time for the gradients of $\left(\chi_{1}, \chi_{2}\right)$. Nonetheless, in the next theorem we will actually check such continuity property.

Theorem 2.8. Assume (2.7)-(2.10) and (2.35). Then the family of semiprocesses

$$
\left\{U_{\sigma}(t, \tau) z_{0}\right\}, \quad \text { with } z_{0}=\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right) \text { and } \sigma=(\boldsymbol{G}, \vartheta),
$$

is $\mathcal{X} \times \Sigma$-continuous.
Proof. Assume that $\left(\left(\chi_{1 n}^{0}, \chi_{2 n}^{0}\right), \boldsymbol{u}_{0 n}, \boldsymbol{v}_{0 n}\right) \subset \mathcal{X}$ is a sequence of initial data converging to $\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right) \in \mathcal{X}$ in the metric (2.12) and take a sequence $\left(\boldsymbol{G}_{n}, \vartheta_{n}\right) \in \Sigma$ converging to $(\boldsymbol{G}, \vartheta)$ in $\Sigma$. Moreover, let us fix $t$ and denote by

$$
\left(\left(\chi_{1 n}(t), \chi_{2 n}(t)\right), \boldsymbol{u}_{n}(t), \boldsymbol{u}_{n t}(t)\right) \quad\left(\operatorname{resp} . \quad\left(\left(\chi_{1}(t), \chi_{2}(t)\right), \boldsymbol{u}(t), \boldsymbol{u}_{t}(t)\right)\right)
$$

the solution to (2.18)-(2.20) at time $t$ starting from

$$
\left(\left(\chi_{1 n}^{0}, \chi_{2 n}^{0}\right), \boldsymbol{u}_{0 n}, \boldsymbol{v}_{0 n}\right) \quad\left(\operatorname{resp} .\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right)\right)
$$

at time $\tau$ and with forcing terms $\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)$ (resp. $(\boldsymbol{G}, \vartheta)$ ). The existence and the uniqueness for both $\left(\left(\chi_{1 n}(t), \chi_{2 n}(t)\right), \boldsymbol{u}_{n}(t), \boldsymbol{u}_{n t}(t)\right)$ and $\left(\left(\chi_{1}(t), \chi_{2}(t)\right), \boldsymbol{u}(t), \boldsymbol{u}_{t}(t)\right)$ are obviously guaranteed by Theorem 2.6. Moreover, thanks to (2.36), we have that, for any $t \geq \tau \geq 0$,

$$
\begin{align*}
\boldsymbol{u}_{n} & \rightarrow \boldsymbol{u} \text { in } C^{0}([\tau, t] ; \boldsymbol{V}),  \tag{2.53}\\
\boldsymbol{u}_{n t} & \rightarrow \boldsymbol{u}_{t} \text { in } C^{0}([\tau, t] ; \boldsymbol{H}),  \tag{2.54}\\
\chi_{j n} & \rightarrow \chi_{j} \text { in } C^{0}([\tau, t] ; H), \tag{2.55}
\end{align*}
$$

since, in particular, it turns out that (see [20] or, e.g., [18, Théorème 16.7]) $\alpha\left(\vartheta_{n}\right) \rightarrow \alpha(\vartheta)$ in $L^{2}(\tau, t ; V)$ for any $t \geq \tau \geq 0$. Thus, in view of (2.12) we only have to prove that

$$
\begin{equation*}
\sum_{j=1}^{2}\left\|\nabla\left(\chi_{j n}(t)-\chi_{j}(t)\right)\right\|_{\boldsymbol{H}}^{2} \rightarrow 0 \quad \text { for all } t>0 \tag{2.56}
\end{equation*}
$$

Standard energy estimates (see [23] for details) entail the boundedness of the sequences

$$
\begin{gathered}
\left\{\chi_{1 n}\right\},\left\{\chi_{2 n}\right\} \text { in } H^{1}(0, T ; H) \cap L^{\infty}(0, T ; V) \cap L^{2}(0, T ; W), \\
\left\{\boldsymbol{u}_{n}\right\} \text { in } H^{2}\left(0, T ; \boldsymbol{V}^{\prime}\right) \cap L^{\infty}(0, T ; \boldsymbol{V}) \text {, and }\left\{\boldsymbol{u}_{n t}\right\} \text { in } H^{1}\left(0, T ; \boldsymbol{V}^{\prime}\right) \cap L^{\infty}(0, T ; \boldsymbol{H})
\end{gathered}
$$

(cf. the norms in (2.14) and (2.16)) uniformly with respect to $n$. Thus, by well-known compactness arguments, the related weak or weak star convergences to $\chi_{1}, \chi_{2}, \boldsymbol{u}, \boldsymbol{u}_{t}$ hold. Note that the whole sequences converge since the limits are perfectly identified and, in particular, let us point out the following convergence

$$
\begin{equation*}
\chi_{j n} \rightharpoonup \chi_{j} \text { in } H^{1}(\tau, t ; H) \text { for all } t \geq \tau \geq 0 \text { and for } j=1,2 . \tag{2.57}
\end{equation*}
$$

Now, in order to show (2.56), we exploit the following semicontinuity comparison argument. Formally test (2.20) at level $n$ by the vector of components $\chi_{1 n t}, \chi_{2 n t}$ and then integrate over $(\tau, t)$, with $t \geq \tau, \tau \geq 0$. We get

$$
\begin{align*}
& \frac{\eta}{2} \sum_{j=1}^{2}\left\|\nabla \chi_{j n}(t)\right\|_{\boldsymbol{H}}^{2}+I_{K}\left(\chi_{1 n}(t), \chi_{2 n}(t)\right) \\
& =\frac{\eta}{2} \sum_{j=1}^{2}\left\|\nabla \chi_{j n}^{0}\right\|_{\boldsymbol{H}}^{2}+I_{K}\left(\chi_{1 n}^{0}, \chi_{2 n}^{0}\right)-\sum_{j=1}^{2} \int_{\tau}^{t}\left\|\partial_{t} \chi_{j n}(s)\right\|_{H}^{2} d s \\
& -\int_{\tau}^{t} \frac{\ell}{\vartheta^{*}}\left(\vartheta_{n}-\vartheta^{*}, \partial_{t} \chi_{1 n}\right)_{\Omega}(s) d s-\int_{\tau}^{t}\left(\alpha\left(\vartheta_{n}\right) \operatorname{div} \boldsymbol{u}_{n}, \partial_{t} \chi_{2 n}\right)_{\Omega}(s) d s . \tag{2.58}
\end{align*}
$$

Observe that the same identity follows rigorously from [5, Théorème 3.6, pp. 72-73]. Taking the limsup as $n \nearrow+\infty$ of both sides of (2.58), our aim is clearly to verify that the terms on the right hand side actually pass to the limit. This is the case. In fact, thanks to (3.15) and the lower semicontinuity of norms with respect to weak convergence, from (2.53) and the convergence of $\vartheta_{n}$ to $\vartheta$ il follows that

$$
\begin{align*}
& \limsup _{n \nearrow+\infty} \int_{\tau}^{t}-\left(\left(\vartheta_{n}-\vartheta^{*}, \partial_{t} \chi_{1 n}\right)_{\Omega}(s)+\left(\alpha\left(\vartheta_{n}\right) \operatorname{div} \boldsymbol{u}_{n}, \partial_{t} \chi_{2 n}\right)_{\Omega}(s)+\sum_{j=1}^{2}\left\|\partial_{t} \chi_{j n}(s)\right\|_{H}^{2}\right) d s \\
& \leq \int_{\tau}^{t}-\left(\left(\vartheta-\vartheta^{*}, \partial_{t} \chi_{1}\right)_{\Omega}(s)+\left(\alpha(\vartheta) \operatorname{div} \boldsymbol{u}, \partial_{t} \chi_{2}\right)_{\Omega}(s)+\sum_{j=1}^{2}\left\|\partial_{t} \chi_{j}(s)\right\|_{H}^{2}\right) d s . \tag{2.59}
\end{align*}
$$

Then, by recovering the identity analogous to (2.58) for the limiting pair ( $\chi_{1}, \chi_{2}$ ), a comparison with (2.58) yields

$$
\begin{gather*}
\limsup _{n \nearrow+\infty}\left(\frac{\eta}{2} \sum_{j=1}^{2}\left\|\nabla \chi_{j n}(t)\right\|_{\boldsymbol{H}}^{2}+I_{K}\left(\chi_{1 n}(t), \chi_{2 n}(t)\right)\right) \\
\quad \leq \frac{\eta}{2} \sum_{j=1}^{2}\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}+I_{K}\left(\chi_{1}(t), \chi_{2}(t)\right) \tag{2.60}
\end{gather*}
$$

for all $t \geq 0$. Since the functions $t \mapsto \frac{\eta}{2} \sum_{j=1}^{2}\left\|\nabla \chi_{j n}(t)\right\|_{\boldsymbol{H}}^{2}+I_{K}\left(\chi_{1 n}(t), \chi_{2 n}(t)\right)$ and $t \mapsto$ $\frac{\eta}{2} \sum_{j=1}^{2}\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}+I_{K}\left(\chi_{1}(t), \chi_{2}(t)\right)$ are absolutely continuous [5, Théorème 3.6, pp. 72-73] and (2.17) holds, (2.60) reduces to

$$
\begin{equation*}
\limsup _{n \nearrow+\infty} \sum_{j=1}^{2}\left\|\nabla \chi_{j n}(t)\right\|_{\boldsymbol{H}}^{2} \leq \sum_{j=1}^{2}\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2} . \tag{2.61}
\end{equation*}
$$

The converse liminf inequality clearly follows from the weak lower semicontinuity of norms and from the fact that $\chi_{j n}(t) \rightarrow \chi_{j}(t), j=1,2$, strongly in $H$ and weakly in $V$ for all $t \geq 0$, due to (2.55) and the boundedness of $\left\{\chi_{j n}\right\}, j=1,2$, in $L^{\infty}(0, T ; V)$ for all $T>0$. Thus, we have that $\left\|\nabla \chi_{j n}(t)\right\|_{\boldsymbol{H}}^{2} \rightarrow\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}$ for $j=1,2$ and any $t \geq 0$. This convergence combined with the abovementioned weak convergence plainly leads to the strong convergence of $\chi_{j n}(t)$ to $\chi_{j}(t)$ in $V$ for $j=1,2$ and any $t \geq 0$. Then, recalling again (2.53)-(2.54), it turns out that the theorem is completely proved.

## 3 Uniform Attractor for (2.18)-(2.20)

In this section, we prove that the system (2.18)-(2.20) possesses the compact uniform at$\operatorname{tractor} \mathcal{A}$. We advise the reader that in the sequel we will make often use of some formal estimates, which can be rigorously justified by adopting some, by now standard, approximation argument. The occurrence of these formal estimates will be however marked to the reader. To simplify the notation, from now on we denote by $C$ (or $C_{i}, i=1,2, \ldots$ ) some possibly different constants depending on the data of the problem. Moreover, we let $c=1$ in (2.20).

We start with the proof of the dissipativity of the system and state the following result.
Theorem 3.1 (Uniformly absorbing set). Under the same conditions as in Definition 2.7, let the triplet

$$
\left(\boldsymbol{G}, \vartheta, \vartheta_{t}\right) \text { lie in a bounded subset } F \text { of } \mathcal{T}_{2}(\boldsymbol{H}) \times \mathcal{T}_{2}(H) \times \mathcal{T}_{2}(H) .
$$

Then, there exists a constant $D>0$ depending on the quantity

$$
M_{F}:=\sup _{\left(\boldsymbol{G}, \vartheta, \vartheta_{t}\right) \in F}\left(\|\boldsymbol{G}\|_{\mathcal{T}_{2}(\boldsymbol{H})}^{2}+\|\vartheta\|_{\mathcal{T}_{2}(H)}^{2}+\left\|\vartheta_{t}\right\|_{\mathcal{T}_{2}(H)}^{2}\right)
$$

such that the $\mathcal{X}$-ball with radius $D$ turns out to be a uniform absorbing set for the family $\left\{U_{(\boldsymbol{G}, \vartheta)}(t, \tau), \quad\left(\boldsymbol{G}, \vartheta, \vartheta_{t}\right) \in F\right\}$. Moreover, for any $R>0$ there is a contant $C$, which depends on $M_{F}$ as well, such that for any $\tau \geq 0$ there holds

$$
\begin{equation*}
\sup _{t \geq \tau} \int_{t}^{t+1} \sum_{j=1}^{2}\left\|\chi_{j t}(s)\right\|_{H}^{2} d s \leq C \tag{3.1}
\end{equation*}
$$

whenever $d_{\mathcal{X}}\left(\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right), 0\right) \leq R$ and $\left(\chi_{1}(t), \chi_{2}(t)\right)$ stands the first component of $U_{(\boldsymbol{G}, \vartheta)}(t, \tau)\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right)$.
Proof. The notation of Definition 2.7 being in force, let us test (2.18) by $\binom{\chi_{1 t}+\chi_{1}}{\chi_{2 t}+\chi_{2}}$. We obtain

$$
\begin{gathered}
\sum_{j=1}^{2}\left(k\left\|\chi_{j t}(t)\right\|_{H}^{2}+\eta\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}\right)+I_{K}\left(\chi_{1}(t), \chi_{2}(t)\right) \\
+\frac{d}{d t}\left(\sum_{j=1}^{2}\left(\frac{k}{2}\left\|\chi_{j}(t)\right\|_{H}^{2}+\frac{\eta}{2}\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}\right)+I_{K}\left(\chi_{1}(t), \chi_{2}(t)\right)\right) \\
\leq I_{K}(0,0)+\frac{l}{\vartheta^{*}}\left|\left(\vartheta-\vartheta^{*}, \chi_{1 t}+\chi_{1}\right)_{\Omega}(t)\right|-\left(\alpha(\vartheta) \operatorname{div} \boldsymbol{u}, \chi_{2 t}+\chi_{2}\right)_{\Omega}(t)
\end{gathered}
$$

for a.e. $t \in(\tau,+\infty)$. Using (2.17), (2.4), and the elementary Young inequality, we infer that

$$
\begin{gather*}
\sum_{j=1}^{2}\left(\frac{k}{2}\left\|\chi_{j t}(t)\right\|_{H}^{2}+\eta\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}\right)+\frac{d}{d t} \sum_{j=1}^{2}\left(\frac{k}{2}\left\|\chi_{j}(t)\right\|_{H}^{2}+\frac{\eta}{2}\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}\right) \\
\leq C\left(1+\|\vartheta(t)\|_{H}^{2}\right)-\left(\alpha(\vartheta) \operatorname{div} \boldsymbol{u}, \chi_{2 t}+\chi_{2}\right)_{\Omega}(t) \tag{3.2}
\end{gather*}
$$

for a.e. $t \in(\tau,+\infty)$. Now, we take $\boldsymbol{v}=\boldsymbol{u}_{t}+\delta \boldsymbol{u}$ as test function in (2.20), with $\delta \in[0,1 / 2]$ to be chosen later. Note that this procedure is formal since $\boldsymbol{u}_{t} \notin \boldsymbol{V}$, however one can argue rigorously as in Theorem 2.6. Anyway, by the computation we are led to the equality

$$
\begin{gather*}
\frac{d}{d t}\left(\frac{1}{2}\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}+\frac{1}{2} a(\boldsymbol{u}(t), \boldsymbol{u}(t))+\delta\left\langle\boldsymbol{u}_{t}(t), \boldsymbol{u}(t)\right\rangle+\left(\alpha(\vartheta) \chi_{2}, \operatorname{div} \boldsymbol{u}\right)_{\Omega}(t)\right) \\
+(1-\delta)\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}+\delta a(\boldsymbol{u}(t), \boldsymbol{u}(t))+\delta^{2}\left\langle\boldsymbol{u}_{t}(t), \boldsymbol{u}(t)\right\rangle+\delta\left(\alpha(\vartheta) \chi_{2}, \operatorname{div} \boldsymbol{u}\right)_{\Omega} \\
=\left\langle\boldsymbol{G}, \boldsymbol{u}_{t}+\delta \boldsymbol{u}\right\rangle(t)+\left(\delta^{2}-\delta\right)\left\langle\boldsymbol{u}_{t}(t), \boldsymbol{u}(t)\right\rangle \\
+\left(\alpha^{\prime}(\vartheta) \vartheta_{t} \chi_{2}, \operatorname{div} \boldsymbol{u}\right)_{\Omega}(t)+\left(\alpha(\vartheta) \chi_{2 t}, \operatorname{div} \boldsymbol{u}\right)_{\Omega}(t) . \tag{3.3}
\end{gather*}
$$

At this point, in view of (2.3) and (2.9) we deduce that

$$
\begin{gathered}
\left\langle\boldsymbol{G}, \boldsymbol{u}_{t}+\delta \boldsymbol{u}\right\rangle(t) \leq C\|\boldsymbol{G}(t)\|_{\boldsymbol{H}}^{2}+\frac{1}{8}\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}+\frac{\delta}{8} a(\boldsymbol{u}(t), \boldsymbol{u}(t)), \\
\left(\delta^{2}-\delta\right)\left\langle\boldsymbol{u}_{t}(t), \boldsymbol{u}(t)\right\rangle \leq C \delta\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}+\frac{\delta}{8} a(\boldsymbol{u}(t), \boldsymbol{u}(t)) \\
\left(\alpha^{\prime}(\vartheta) \vartheta_{t} \chi_{2}, \operatorname{div} \boldsymbol{u}\right)_{\Omega}(t) \leq \frac{C}{\delta}\left\|\vartheta_{t}(t)\right\|_{H}^{2}+\frac{\delta}{8} a(\boldsymbol{u}(t), \boldsymbol{u}(t)) .
\end{gathered}
$$

Hence, by introducing the function

$$
\begin{equation*}
\Psi(t):=\frac{1}{2}\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}+\frac{1}{2} a(\boldsymbol{u}(t), \boldsymbol{u}(t))+\delta\left\langle\boldsymbol{u}_{t}(t), \boldsymbol{u}(t)\right\rangle+\left(\alpha(\vartheta) \chi_{2}, \operatorname{div} \boldsymbol{u}\right)_{\Omega}(t) \tag{3.4}
\end{equation*}
$$

provided $\delta$ is sufficiently small we find out that

$$
\begin{equation*}
\frac{d}{d t} \Psi(t)+\delta \Psi(t) \leq C\|\boldsymbol{G}(t)\|_{\boldsymbol{H}}^{2}+\frac{C}{\delta}\left\|\vartheta_{t}(t)\right\|_{H}^{2}+\left(\alpha(\vartheta) \chi_{2 t}, \operatorname{div} \boldsymbol{u}\right)_{\Omega}(t) \tag{3.5}
\end{equation*}
$$

On the other hand, with the help of (2.3), (2.4) and (2.9) we can determine two positive constants $C_{1}, C_{2}$ such that

$$
\begin{equation*}
\Psi(t)+C_{1} \geq C_{2}\left(\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}+\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^{2}\right) \tag{3.6}
\end{equation*}
$$

again for $\delta$ small enough. Now, we set

$$
\begin{equation*}
\Phi(t):=\sum_{j=1}^{2}\left(\frac{k}{2}\left\|\chi_{j}(t)\right\|_{H}^{2}+\frac{\eta}{2}\left\|\nabla \chi_{j}(t)\right\|_{\boldsymbol{H}}^{2}\right)+\Psi(t)+C_{1} \tag{3.7}
\end{equation*}
$$

and sum (3.2) and (3.5) noting that two terms cancel out. Then we obtain

$$
\begin{array}{r}
\frac{k}{2} \sum_{j=1}^{2}\left\|\chi_{j t}(t)\right\|_{H}^{2}+\frac{d}{d t} \Phi(t)+\delta \Phi(t) \leq \frac{k}{2} \delta \sum_{j=1}^{2}\left\|\chi_{j}(t)\right\|_{H}^{2}+C-\left(\alpha(\vartheta) \operatorname{div} \boldsymbol{u}, \chi_{2}\right)_{\Omega}(t) \\
+C\|\vartheta(t)\|_{H}^{2}+C\|\boldsymbol{G}(t)\|_{\boldsymbol{H}}^{2}+\frac{C}{\delta}\left\|\vartheta_{t}(t)\right\|_{H}^{2}
\end{array}
$$

for a.e. $t \in(\tau,+\infty)$. Using again (2.4), (2.9) and the Young inequality, eventually we derive

$$
\begin{equation*}
\frac{k}{2} \sum_{j=1}^{2}\left\|\chi_{j t}(t)\right\|_{H}^{2}+\frac{d}{d t} \Phi(t)+\frac{\delta}{2} \Phi(t) \leq C\left(1+\|\vartheta(t)\|_{H}^{2}+\left\|\vartheta_{t}(t)\right\|_{H}^{2}+\|\boldsymbol{G}(t)\|_{\boldsymbol{H}}^{2}\right) \tag{3.8}
\end{equation*}
$$

in which $\delta$ is finally fixed and the constant $C$ on the right hand side depends also on $\delta$. Thus, Lemma 2.5 applies with $\varphi(t)=\sqrt{\Phi(t)}, \varepsilon=\delta / 2, m_{1}(t)=0$, and obvious position for $m_{2}(t)$. Then, we obtain the fundamental inequality

$$
\begin{equation*}
\Phi(t) \leq 2 \Phi(\tau) e^{-\delta(t-\tau) / 2}+C_{3}\left(1+\|\vartheta\|_{\mathcal{T}_{2}(H)}^{2}+\left\|\vartheta_{t}\right\|_{\mathcal{T}_{2}(H)}^{2}+\|\boldsymbol{G}\|_{\mathcal{T}_{2}(\boldsymbol{H})}^{2}\right) \tag{3.9}
\end{equation*}
$$

for every $t \geq \tau, \tau \geq 0$, and for some positive constant $C_{3}$. Then, referring to our statement and fixing a constant $C_{4}>C_{3}\left(1+M_{F}\right)$, it results that we can always determine a proper time $T$ depending on $\tau$ and on the radius of the $\mathcal{X}$-ball in which $\left(\left(\chi_{1}, \chi_{2}\right)(\tau), \boldsymbol{u}(\tau), \boldsymbol{u}_{t}(\tau)\right)$ lives such that $2 \Phi(\tau) e^{-\delta(T-\tau) / 2} \leq C_{4}-C_{3}\left(1+M_{F}\right)$ and consequently $\Phi(t) \leq C_{4}$ for all $t \geq T$. Thus, in view of (3.6)-(3.7), it is straightforward to find the desired radius $D$. Finally, to obtain (3.1), we only need to integrate (3.8) between $t$ and $t+1$ and use (3.9).

The following lemmata will be crucial in the course of our investigation. In fact, in the first one we will prove a weak continuity result for the solution semiprocess, while in the second one we will show a smoothing property in finite times for the ( $\chi_{1}, \chi_{2}$ )-component of the solution operator $U_{(G, \vartheta)}(t, \tau)$.
weakcont Lemma 3.2. Let $\sigma_{n}=\left(\boldsymbol{G}_{n}, \vartheta_{n}\right) \rightarrow \sigma:=(\boldsymbol{G}, \vartheta)$ in $L_{\mathrm{loc}}^{2}(0,+\infty ; \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(0, \infty ; W^{1,3}(\Omega)\right)$ and let $z_{0 n}=\left(\left(\chi_{1 n}^{0}, \chi_{2 n}^{0}\right), \boldsymbol{u}_{0 n}, \boldsymbol{v}_{0 n}\right)$ specify a sequence in $\mathcal{X}$ that weakly converges to $z_{0}:=$ $\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right)$ in $V^{2} \times \boldsymbol{V} \times \boldsymbol{H}$ as $n \nearrow \infty$. Then

$$
\begin{equation*}
U_{\sigma_{n}}(t, \tau) z_{0 n} \rightharpoonup U_{\sigma}(t, \tau) z_{0} \quad \text { weakly in } V^{2} \times \boldsymbol{V} \times \boldsymbol{H} \quad \text { for all } t \geq \tau \geq 0 \tag{3.10}
\end{equation*}
$$

Proof. As in the proof of Theorem 2.8, we agree to set

$$
\begin{equation*}
\left(\left(\chi_{1 n}(t), \chi_{2 n}(t)\right), \boldsymbol{u}_{n}(t), \boldsymbol{u}_{n t}(t)\right)=U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}(t, \tau) z_{0 n} \tag{3.11}
\end{equation*}
$$

Now, we apply standard estimates, that is, work on (2.18) as in (2.58) and formally test (2.20) by $\boldsymbol{u}_{n t}$. Note that

$$
-\int_{\tau}^{t}\left(\alpha\left(\vartheta_{n}\right) \chi_{2 n}, \operatorname{div} \boldsymbol{u}_{n t}\right)_{\Omega}(s) d s=\int_{\tau}^{t}\left(\chi_{2 n} \alpha^{\prime}\left(\vartheta_{n}\right) \nabla \vartheta_{n}+\alpha\left(\vartheta_{n}\right) \nabla \chi_{2 n}, \boldsymbol{u}_{n t}\right)_{\Omega}(s) d s
$$

where the product $\alpha^{\prime}\left(\vartheta_{n}\right) \nabla \vartheta_{n}$ has to be understood properly. Then, summing the resulting inequalities and so on, with the help of (2.3), (2.4), (2.9) and the Gronwall lemma it is not difficult to obtain the following bound

$$
\begin{equation*}
\sum_{j=1}^{2}\left\|\chi_{j n}\right\|_{H^{1}(\tau, t ; H) \cap L^{\infty}(\tau, t ; V)}+\left\|\boldsymbol{u}_{n t}\right\|_{L^{\infty}(\tau, t ; \boldsymbol{H})}+\left\|\boldsymbol{u}_{n}\right\|_{L^{\infty}(\tau, t ; \boldsymbol{V})} \leq C \tag{3.12}
\end{equation*}
$$

where the constant $C$ depends on data and on $t$, but is independent of $n$ due to the convergences $z_{0 n} \rightharpoonup z_{0}$ and $\left(\boldsymbol{G}_{n}, \vartheta_{n}\right) \rightarrow(\boldsymbol{G}, \vartheta)$ in the related spaces. Consequently, a
formal test of (2.18) by the vector of components $h_{1 n}, h_{2 n}$ and a subsequent comparison argument along with well-known regularity results yield (cf. (2.14))

$$
\begin{equation*}
\sum_{j=1}^{2}\left(\left\|h_{j n}\right\|_{L^{2}(\tau, t ; H)}+\left\|\chi_{j n}\right\|_{L^{2}(\tau, t ; W)}\right) \leq C \tag{3.13}
\end{equation*}
$$

while a comparison of terms in (2.20) lead to the estimate

$$
\begin{equation*}
\left\|\boldsymbol{u}_{n t t}\right\|_{L^{2}\left(\tau, t ; \boldsymbol{V}^{\prime}\right)} \leq C \tag{3.14}
\end{equation*}
$$

where, again, $C$ is independent of $n$. Thus, we have (up to a subsequence not relabeled)

$$
\begin{gather*}
\chi_{j n} \stackrel{*}{\rightharpoonup} \chi_{j} \text { in } H^{1}(\tau, t ; H) \cap L^{\infty}(\tau, t ; V) \cap L^{2}(\tau, t ; W), j=1,2,  \tag{3.15}\\
h_{j n} \rightharpoonup h_{j} \text { in } L^{2}(\tau, t ; H), j=1,2,  \tag{3.16}\\
\boldsymbol{u}_{n} \stackrel{*}{\rightharpoonup} \boldsymbol{u} \text { in } L^{\infty}(\tau, t ; \boldsymbol{V}),  \tag{3.17}\\
\boldsymbol{u}_{n t} \stackrel{*}{\rightharpoonup} \boldsymbol{u}_{t} \text { in } L^{\infty}(\tau, t ; \boldsymbol{H}),  \tag{3.18}\\
\boldsymbol{u}_{n t t} \rightharpoonup \boldsymbol{u}_{t t} \text { in } L^{2}\left(\tau, t ; \boldsymbol{V}^{\prime}\right) \tag{3.19}
\end{gather*}
$$

for any $t \geq \tau \geq 0$ and for some limit functions $\chi_{1}, \chi_{2}, h_{1}, h_{2}, \boldsymbol{u}$. Note that known compactness results (cf., e.g., [28, Corollary 4, p. 84]) imply

$$
\begin{equation*}
\chi_{j n} \rightarrow \chi_{j} \text { in } C^{0}([\tau, t] ; H) \cap L^{2}(\tau, t ; V), \quad j=1,2 . \tag{3.20}
\end{equation*}
$$

Convergences (3.15)-(3.20) are enough to conclude that the limit triplet ( $\chi_{1}, \chi_{2}, \boldsymbol{u}$ ) yields indeed the unique solution to (2.18)-(2.20) starting from $z_{0}$ and with forcing terms $(\boldsymbol{G}, \vartheta)$. In fact, the only delicate point consists in showing the identification

$$
\left(h_{1}, h_{2}\right) \in \partial I_{K}\left(\chi_{1}, \chi_{2}\right) \text { in } \Omega \times(\tau, t) .
$$

However, this is a direct consequence of the strong-weak closure of the maximal monotone operators (see, e.g., [5, pp. 24-27]) and of the convergences (3.16) and $\chi_{j} \rightarrow \chi_{j}$ in $L^{2}(\tau, t ; H)$. Therefore, we have that

$$
\begin{equation*}
\left.\left(\left(\chi_{1}(t), \chi_{2}(t)\right), \boldsymbol{u}(t), \boldsymbol{u}_{t}(t)\right)\right)=U_{(\boldsymbol{G}, \vartheta)}(t, \tau) z_{0} . \tag{3.21}
\end{equation*}
$$

Notice that, as the limiting solution is unique, the convergences in (3.15)-(3.19) hold for the whole sequence of $n$ and not only for a proper subsequence. It remains to to show (3.10). Actually, by (3.15)-(3.19) and the generalized Ascoli theorem we also infer (see again [28, Corollary 4, p. 84])

$$
\begin{equation*}
\boldsymbol{u}_{n} \rightarrow \boldsymbol{u} \text { in } C^{0}([\tau, t] ; \boldsymbol{H}), \quad \boldsymbol{u}_{n t} \rightarrow \boldsymbol{u}_{t} \text { in } C^{0}\left([\tau, t] ; \boldsymbol{V}^{\prime}\right) \tag{3.22}
\end{equation*}
$$

for all $t \geq \tau \geq 0$. At this point, it is not difficult to obtain (3.10) from (3.20), (3.22) and the uniform estimates in (3.12).
eff-rego Lemma 3.3. Under the assumption of Theorem 3.1, let moreover

$$
\left(\boldsymbol{G}, \vartheta, \vartheta_{t}\right) \in \mathcal{T}_{2}(\boldsymbol{H}) \times \mathcal{T}_{2}\left(W^{1,3}(\Omega)\right) \times \mathcal{T}_{2}\left(L^{3}(\Omega)\right)
$$

Then, for any $R>0$ there exists a constant $\Lambda$ such that for any $\tau \geq 0$ there holds

$$
\begin{equation*}
\sup _{t \geq \tau+1} \sum_{j=1}^{2}\left(\left\|\chi_{j t}(t)\right\|_{H}^{2}+\left\|\chi_{j}(t)\right\|_{W}^{2}\right) \leq \Lambda \tag{3.23}
\end{equation*}
$$

whenever $d_{\mathcal{X}}\left(\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right), 0\right) \leq R$ and $\|\boldsymbol{G}\|_{\mathcal{T}_{2}(\boldsymbol{H})}+\|\vartheta\|_{\mathcal{T}_{2}\left(W^{1,3}(\Omega)\right)}+\left\|\vartheta_{t}\right\|_{\mathcal{T}_{2}\left(L^{3}(\Omega)\right)} \leq R$, $\left(\chi_{1}(t), \chi_{2}(t)\right)$ denoting the first component of $U_{(\boldsymbol{G}, \vartheta)}(t, \tau)\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right)$.

Proof. Adopting the usual notation as in Definition 2.7, take $R>0$ such that

$$
d_{\mathcal{X}}\left(\left(\left(\chi_{1}^{0}, \chi_{2}^{0}\right), \boldsymbol{u}_{0}, \boldsymbol{v}_{0}\right), 0\right) \leq R \quad \text { and } \quad\|\boldsymbol{G}\|_{\mathcal{T}_{2}(\boldsymbol{H})}+\|\vartheta\|_{\mathcal{T}_{2}\left(W^{1,3}(\Omega)\right)}+\left\|\vartheta_{t}\right\|_{\tau_{2}\left(L^{3}(\Omega)\right)} \leq R .
$$

Let us differentiate (2.18) with respect to time and then test by the vector of components $\chi_{1 t}, \chi_{2 t}$. This procedure is only formal. However, it might be made rigorous by working at a regularized level and then passing to the limit. Since this technique is quite standard, we prefer to avoid all such details and proceed in a formal way. Thus, by the monotonicity of $\partial I_{K}$ we get

$$
\begin{align*}
& \sum_{j=1}^{2}\left(\frac{k}{2} \frac{d}{d t}\left\|\chi_{j t}(t)\right\|_{H}^{2}+\eta\left\|\nabla \chi_{j t}(t)\right\|_{\boldsymbol{H}}^{2}\right) \\
& \leq-\frac{\ell}{\vartheta^{*}}\left(\vartheta_{t}, \chi_{1 t}\right)_{\Omega}(t)-\left\langle\operatorname{div} \boldsymbol{u}_{t}, \alpha(\vartheta) \chi_{2 t}\right\rangle(t)-\left(\operatorname{div} \boldsymbol{u}, \alpha^{\prime}(\vartheta) \vartheta_{t} \chi_{2 t}\right)_{\Omega}(t) \tag{3.24}
\end{align*}
$$

Now, we sum $\sum_{j=1}^{2} \eta\left\|\chi_{j t}(t)\right\|_{H}^{2}$ to both sides of (3.24) in order to get the full $V^{2}$-norm in the left hand side of (3.24). Subsequently, taking advantage of (2.9), the Hölder inequality, the continuous embedding $H^{1}(\Omega) \subset L^{6}(\Omega)$ and the Young inequality in the form $a b \leq \frac{1}{2 \varepsilon} a^{2}+\frac{\varepsilon}{2} b^{2}$ for all $\varepsilon>0$ and $a, b \in \mathbb{R}$, we get

$$
\begin{gathered}
-\frac{\ell}{\vartheta^{*}}\left(\vartheta_{t}, \chi_{1 t}\right)_{\Omega}(t) \leq C\left(\left\|\vartheta_{t}(t)\right\|_{H}^{2}+\left\|\chi_{1 t}(t)\right\|_{H}^{2}\right) \\
-\left\langle\operatorname{div} \boldsymbol{u}_{t}, \alpha(\vartheta) \chi_{2 t}\right\rangle(t) \leq C\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}\left(\left\|\chi_{2 t}(t)\right\|_{V}+\|\nabla \vartheta(t)\|_{L^{3}(\Omega)^{3}}\left\|\chi_{2 t}(t)\right\|_{L^{6}(\Omega)}\right) \\
\leq \frac{\eta}{4}\left\|\chi_{2 t}(t)\right\|_{V}^{2}+C\left\|\boldsymbol{u}_{t}(t)\right\|_{\boldsymbol{H}}^{2}\left(1+\|\vartheta(t)\|_{W^{1,3}(\Omega)}^{2}\right) \\
-\left(\operatorname{div} \boldsymbol{u}, \alpha^{\prime}(\vartheta) \vartheta_{t} \chi_{2 t}\right)_{\Omega}(t) \leq C\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}\left\|\vartheta_{t}(t)\right\|_{L^{3}(\Omega)}\left\|\chi_{2 t}(t)\right\|_{L^{6}(\Omega)} \\
\leq \frac{\eta}{4}\left\|\chi_{2 t}(t)\right\|_{V}^{2}+C\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^{2}\left\|\vartheta_{t}(t)\right\|_{L^{3}(\Omega)}^{2} .
\end{gathered}
$$

Recalling Theorem 3.1 (cf., in particular, (3.6)-(3.7) and (3.9)), it turns out that

$$
\begin{align*}
& \sum_{j=1}^{2}\left(\frac{d}{d t}\left\|\chi_{j t}(t)\right\|_{H}^{2}+\left\|\chi_{j t}(t)\right\|_{V}^{2}\right) \\
& \leq C\left(1+\|\vartheta(t)\|_{W^{1,3}(\Omega)}^{2}+\left\|\vartheta_{t}(t)\right\|_{L^{3}(\Omega)}^{2}+\sum_{j=1}^{2}\left\|\chi_{j t}(t)\right\|_{H}^{2}\right) \tag{3.25}
\end{align*}
$$

for some constant $C$ depending especially on $R$. Thus, recalling (3.1) we are in the position to apply the Uniform Gronwall Lemma 2.4 to obtain the uniform (in time) bound for $\left\|\chi_{j t}(t)\right\|_{H}^{2}, j=1,2$. Next, by Theorem 3.1 and a simple comparison argument in (2.18) we infer

$$
\sup _{\tau \geq 0} \sup _{t \geq \tau+1} \sum_{j=1}^{2}\left\|-\Delta \chi_{j}(t)+h_{j}(t)\right\|_{H}^{2} \leq C .
$$

Then, the monotonicity of $\partial I_{K}$ and standard elliptic regularity results allow us to conclude that(cf. also (2.14))

$$
\begin{equation*}
\sum_{j=1}^{2}\left(\left\|\chi_{j}(t)\right\|_{W}^{2}+\left\|h_{j}(t)\right\|_{H}^{2}\right) \leq \sum_{j=1}^{2}\left\|-\Delta \chi_{j}(t)+h_{j}(t)\right\|_{H}^{2} \leq C \tag{3.26}
\end{equation*}
$$

for all $\tau \geq 0$ and $t \geq \tau+1$, whence (3.23) is completely proved.
We now prove the uniform asymptotic compactness of the system. To this end, we fix a translation compact function $\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$ in $L_{\mathrm{loc}}^{2}(0,+\infty, \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(0, \infty ; W^{1,3}(\Omega)\right)$. Then, we allow $(\boldsymbol{G}, \vartheta)$ to vary in $\mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$. First of all, note that (see, e.g., [7, Proposition V.3.4])

$$
\begin{equation*}
\|(\boldsymbol{G}, \vartheta)\|_{\mathcal{T}_{2}\left(\boldsymbol{H} \times W^{1,3}(\Omega)\right)} \leq\left\|\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)\right\|_{\mathcal{T}_{2}\left(\boldsymbol{H} \times W^{1,3}(\Omega)\right)}<+\infty \tag{3.27}
\end{equation*}
$$

for any $(\boldsymbol{G}, \vartheta) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$.
Theorem 3.4 (Uniform asymptotic compactness). Within the framework of Definition 2.7, assume in addition that $\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right) \in L_{\mathrm{loc}}^{2}(0,+\infty, \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(0, \infty ; W^{1,3}(\Omega)\right)$ is translation compact and

$$
\begin{equation*}
\text { there exists } R>0 \text { such that }\left\|\left(\widehat{\sigma}_{2}\right)_{t}\right\|_{\left.\mathcal{T}_{2}\left(L^{3}(\Omega)\right)\right)} \leq R . \tag{3.28}
\end{equation*}
$$

Then, the family $\left\{U_{(\boldsymbol{G}, \vartheta)}(t, \tau), \quad(\boldsymbol{G}, \vartheta) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{\mathbf{1}}, \widehat{\sigma}_{2}\right)\right\}$ is uniformly asymptotically compact in $\mathcal{X}$.

Proof. Recalling Definition 2.7, we let

$$
\begin{equation*}
\left(\left(\chi_{1 n}(t), \chi_{2 n}(t)\right), \boldsymbol{u}_{n}(t), \boldsymbol{u}_{n t}(t)\right):=U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}(t, 0) z_{0 n} \tag{3.29}
\end{equation*}
$$

denote the solutions emanating from the $\mathcal{X}$-bounded sequence $z_{0 n}=\left(\left(\chi_{1 n}^{0}, \chi_{2 n}^{0}\right), \boldsymbol{u}_{0 n}, \boldsymbol{v}_{0 n}\right)$ at initial time 0 , with forcing terms $\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)$ in $\mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$. Moreover, take an arbitrary time sequence $t_{n} \nearrow+\infty$. Owing to Theorem 3.1, it turns out that (3.27)-(3.28) and the boundedness of $\left\{z_{0 n}\right\}$ in $\left(\mathcal{X}, d_{\mathcal{X}}\right)$ entail the following uniform estimate

$$
\begin{equation*}
\sum_{j=1}^{2}\left\|\chi_{j n}(t)\right\|_{V}+\left\|\boldsymbol{u}_{n}(t)\right\|_{\boldsymbol{V}}+\left\|\boldsymbol{u}_{n t}(t)\right\|_{\boldsymbol{H}} \leq C \quad \forall t \geq 0 \tag{3.30}
\end{equation*}
$$

Thus, (3.30) holds in particular for $t=t_{n}$ and, up to the extraction of a subsequence of $n$, it results that

$$
\begin{align*}
& U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}(t, 0) z_{0 n}=\left(\left(\chi_{1 n}\left(t_{n}\right), \chi_{2 n}\left(t_{n}\right)\right), \boldsymbol{u}_{n}\left(t_{n}\right), \boldsymbol{u}_{n t}\left(t_{n}\right)\right) \\
& \quad \rightharpoonup\left(\left(\chi_{1 \infty}, \chi_{2 \infty}\right), \boldsymbol{u}_{\infty}, \boldsymbol{v}_{\infty}\right)=: z_{\infty} \text { in } V^{2} \times \boldsymbol{V} \times \boldsymbol{H} . \tag{3.31}
\end{align*}
$$

On the other hand, since $\mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{\mathbf{1}}, \widehat{\sigma}_{2}\right)$ is compact, in view of [7, Proposition V.3.4] we have that

$$
\begin{equation*}
T_{t_{n}} \boldsymbol{G}_{n} \rightarrow \boldsymbol{G}_{\infty} \text { in } L_{\mathrm{loc}}^{2}(0,+\infty ; \boldsymbol{H}), \quad T_{t_{n}} \vartheta_{n} \rightarrow \vartheta_{\infty} \text { in } L_{\mathrm{loc}}^{2}\left(0,+\infty ; W^{1,3}(\Omega)\right) \tag{3.32}
\end{equation*}
$$

still up to a subsequence, for some pair $\left(\boldsymbol{G}_{\infty}, \vartheta_{\infty}\right) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$. Moreover, possibly by a diagonal procedure one can select another subsequence of $n$ such that for all $M \in \mathbb{N}$ there holds

$$
\begin{gather*}
U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}\left(t_{n}-M, 0\right) z_{0 n}\left(\text { extended with } z_{0 n} \text { value for } t_{n} \leq M\right) \\
\text { weakly converges in } V^{2} \times \boldsymbol{V} \times \boldsymbol{H} \\
\text { to some element } z_{M}:=\left(\left(\chi_{1 M}, \chi_{2 M}\right), \boldsymbol{u}_{M}, \boldsymbol{v}_{M}\right) \tag{3.33}
\end{gather*}
$$

as well as

$$
\begin{equation*}
T_{t_{n}-M} \boldsymbol{G}_{n} \rightarrow \boldsymbol{G}_{M} \text { in } L_{\mathrm{loc}}^{2}(0,+\infty ; \boldsymbol{H}), \quad T_{t_{n}-M} \vartheta_{n} \rightarrow \vartheta_{M} \text { in } L_{\mathrm{loc}}^{2}\left(0,+\infty ; W^{1,3}(\Omega)\right), \tag{3.34}
\end{equation*}
$$

with the limits $\left(\boldsymbol{G}_{M}, \vartheta_{M}\right) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{\mathbf{1}}, \widehat{\sigma}_{2}\right)$. Incidentally, note that

$$
\boldsymbol{G}_{M}(s):=\boldsymbol{G}_{\infty}(s-M), \quad \vartheta_{M}(s):=\vartheta_{\infty}(s-M) \text { for } s>M
$$

Then, from the translation invariance condition (2.25) we have that

$$
\begin{gather*}
\quad\left(\left(\chi_{1 n}, \chi_{2 n}\right), \boldsymbol{u}_{n}, \boldsymbol{u}_{n t}\right)\left(t_{n}\right)=U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}\left(t_{n}, 0\right) z_{0 n} \\
=U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}\left(t_{n}-M+M, t_{n}-M\right) U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}\left(t_{n}-M, 0\right) z_{0 n} \\
=U_{T_{t_{n}-M}\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}(M, 0) U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}\left(t_{n}-M, 0\right) z_{0 n} \\
=U_{T_{t_{n}-M}\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}(M, 0)\left(\left(\chi_{1 n}, \chi_{2 n}\right), \boldsymbol{u}_{n}, \boldsymbol{u}_{n t}\right)\left(t_{n}-M\right) \tag{3.35}
\end{gather*}
$$

if $t_{n} \geq M$. Thus, by the weak continuity property stated in Lemma 3.2 and by (3.31), (3.33) we deduce that

$$
\begin{equation*}
U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}\left(t_{n}, 0\right) z_{0 n} \rightharpoonup U_{\left(\boldsymbol{G}_{M}, \vartheta_{M}\right)}(M, 0) z_{M} \equiv z_{\infty} \text { in } V^{2} \times \boldsymbol{V} \times \boldsymbol{H} . \tag{3.36}
\end{equation*}
$$

Let us now recall (2.29) and observe that, in order to complete the proof, we should check that actually the strong convergence holds as well in (3.36). Let us analyse separately the components of $U_{\left(\boldsymbol{G}_{n}, \vartheta_{n}\right)}\left(t_{n}, 0\right) z_{0 n}$. From Lemma 3.3 (cf., in particular, estimate (3.23)) we immediately conclude that

$$
\begin{equation*}
\left(\chi_{1 n}\left(t_{n}\right), \chi_{2 n}\left(t_{n}\right)\right) \rightarrow\left(\chi_{1 \infty}, \chi_{2 \infty}\right) \text { in } V^{2} \tag{3.37}
\end{equation*}
$$

due to the compect embedding of $W$ into $V$. Next, setting

$$
\begin{gathered}
\left(\left(\xi_{1 n}^{M}, \xi_{2 n}^{M}\right), \boldsymbol{w}_{n}^{M}, \boldsymbol{w}_{n t}^{M}\right)(t):=U_{T_{t n}-M}\left(\boldsymbol{G}_{n}, \vartheta_{n}\right) \\
\left.\left(\left(\xi_{1}^{M}, \xi_{2}^{M}\right), \boldsymbol{w}^{M}, \boldsymbol{w}_{t}^{M}\right)(t):=U_{\left(\boldsymbol{G}_{M}, \vartheta_{M}\right)}(t, 0)\left(\left(\chi_{1 n}, \chi_{2 n}\right), \boldsymbol{u}_{n}, \boldsymbol{u}_{n t}\right)\left(t_{n}-M\right), \boldsymbol{u}_{M}, \boldsymbol{v}_{M}\right)
\end{gathered}
$$

and recalling (3.34) and (3.33), Lemma 3.2 and especially (3.20), (3.17), (3.18) ensure that

$$
\begin{align*}
\xi_{j n}^{M} \rightarrow \xi_{j}^{M} & \text { in } C^{0}([0, M] ; H) \cap L^{2}(0, M ; V), \quad j=1,2,  \tag{3.38}\\
& \boldsymbol{w}_{n}^{M} \stackrel{*}{\rightharpoonup} \boldsymbol{w}^{M} \text { in } L^{\infty}(0, M ; \boldsymbol{V}),  \tag{3.39}\\
& \boldsymbol{w}_{n t}^{M} \stackrel{*}{\rightharpoonup} \boldsymbol{w}_{t}^{M} \text { in } L^{\infty}(0, M ; \boldsymbol{H}) . \tag{3.40}
\end{align*}
$$

To conclude the proof of the asymptotic compactness of the semiprocess, we use the energy functional introduced in (2.37). We remind that $c=1$ throughout this section. Apply the energy identity (2.38) to $\left(\left(\xi_{1 n}^{M}, \xi_{2 n}^{M}\right), \boldsymbol{w}_{n}^{M}, \boldsymbol{w}_{n t}^{M}\right)$ in the time interval $[0, M]$. Thus, by (3.35) we have

$$
\begin{gather*}
\mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}\right)-e^{-M} \mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}-M\right) \\
=\mathcal{E}\left(\boldsymbol{w}_{n t}^{M}, \boldsymbol{w}_{n}^{M}\right)(M)-e^{-M} \mathcal{E}\left(\boldsymbol{w}_{n t}^{M}, \boldsymbol{w}_{n}^{M}\right)(0) \\
=\int_{0}^{M} e^{t-M}\left(\left(T_{t_{n}-M} \boldsymbol{G}_{n}\right)(t), \boldsymbol{w}_{n t}^{M}(t)+\frac{1}{2} \boldsymbol{w}_{n}^{M}(t)\right)_{\Omega} d t \\
+\int_{0}^{M} e^{t-M}\left(\nabla\left(\alpha\left(\left(T_{t_{n}-M} \vartheta_{n}\right)(t)\right) \xi_{2 n}^{M}(t)\right), \boldsymbol{w}_{n t}^{M}(t)+\frac{1}{2} \boldsymbol{w}_{n}^{M}(t)\right)_{\Omega} d t . \tag{3.41}
\end{gather*}
$$

Owing to (3.34), it turns out that (see [20] or, e.g., [18, Théorème 16.7]) $\alpha\left(\left(T_{t_{n}-M} \vartheta_{n}\right) \rightarrow\right.$ $\alpha\left(\vartheta_{M}\right)$ in $L^{2}(0, M ; V)$. Hence, with the help of (3.38), (2.4), (2.9) and possibly using the Lebesgue dominated convergence theorem, one can directly check that

$$
\nabla\left(\alpha\left(\left(T_{t_{n}-M} \vartheta_{n}\right)(t)\right) \xi_{2 n}^{M}\right) \rightarrow \nabla\left(\alpha\left(\vartheta_{M}(t)\right) \xi_{2}^{M}\right) \text { in } L^{2}(0, M ; \boldsymbol{H})
$$

Then, thanks to (3.38)-(3.40) we can pass to the limit in the right hand side of (3.41) by virtue of the strong-weak (star) convergences and find out that

$$
\begin{aligned}
& \lim _{n \nearrow+\infty}\left(\mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}\right)-e^{-M} \mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}-M\right)\right) \\
& \quad=\int_{0}^{M} e^{t-M}\left(\left(\boldsymbol{G}_{M}(t), \boldsymbol{w}_{t}^{M}(t)+\frac{1}{2} \boldsymbol{w}^{M}(t)\right)_{\Omega} d t\right. \\
& \left.+\int_{0}^{M}\left(\nabla\left(\alpha\left(\vartheta_{M}(t)\right) \xi_{2}^{M}(t)\right), \boldsymbol{w}_{t}^{M}(t)+\frac{1}{2} \boldsymbol{w}^{M}(t)\right)_{\Omega}\right) d t
\end{aligned}
$$

which is nothing but

$$
\mathcal{E}\left(\boldsymbol{w}_{t}^{M}, \boldsymbol{w}^{M}\right)(M)-e^{-M} \mathcal{E}\left(\boldsymbol{w}_{t}^{M}, \boldsymbol{w}^{M}\right)(0)=\mathcal{E}\left(\boldsymbol{v}_{\infty}, \boldsymbol{u}_{\infty}\right)-e^{-M} \mathcal{E}\left(\boldsymbol{v}_{M}, \boldsymbol{u}_{M}\right)
$$

thanks to (3.35) and to the identity (2.38) applied to $\left(\left(\xi_{1}^{M}, \xi_{2}^{M}\right), \boldsymbol{w}^{M}, \boldsymbol{w}_{t}^{M}\right)$. Thus, due to the uniform bound in (3.30) and in view of (3.33) one can easily deduce that

$$
\begin{gather*}
\limsup _{n /+\infty} \mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}\right) \\
\leq \lim _{n \nearrow+\infty}\left(\mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}\right)-e^{-M} \mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}-M\right)\right)+C e^{-M} \\
\leq \mathcal{E}\left(\boldsymbol{v}_{\infty}, \boldsymbol{u}_{\infty}\right)+2 C e^{-M} \tag{3.42}
\end{gather*}
$$

Now, since $\mathcal{E}$ is weakly lower semicontinuous in the energy space $\boldsymbol{H} \times \boldsymbol{V}$, by letting $M \nearrow+\infty$ in (3.42) we conclude that

$$
\begin{equation*}
\limsup _{n \nearrow+\infty} \mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}\right) \leq \mathcal{E}\left(\boldsymbol{v}_{\infty}, \boldsymbol{u}_{\infty}\right) \leq \liminf _{n \nearrow+\infty} \mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}\right) \tag{3.43}
\end{equation*}
$$

whence $\mathcal{E}\left(\boldsymbol{u}_{n t}, \boldsymbol{u}_{n}\right)\left(t_{n}\right) \rightarrow \mathcal{E}\left(\boldsymbol{v}_{\infty}, \boldsymbol{u}_{\infty}\right)$. At this point, it is not difficult to check that (3.43) entails the strong convergence $\left(\boldsymbol{u}_{n}, \boldsymbol{u}_{n t}\right)\left(t_{n}\right) \rightarrow\left(\boldsymbol{u}_{\infty}, \boldsymbol{v}_{\infty}\right)$ in $\boldsymbol{V} \times \boldsymbol{H}$. Hence, recalling (3.36) and (3.37), the desired uniform asymptotic compactness for the family $\left\{U_{(\boldsymbol{G}, \vartheta)}(t, \tau), \quad(\boldsymbol{G}, \vartheta) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{\mathbf{1}}, \widehat{\sigma}_{2}\right)\right\}$ follows.

Recalling Definition 2.7, we can now state and prove the main result concerning the long-time behaviour of the solutions to our system.
teo:attr Theorem 3.5 (Uniform attractor). Under the assumptions of Theorem 3.4, the family of semiprocesses $\left\{U_{(\boldsymbol{G}, \vartheta)}(t, \tau), \quad(\boldsymbol{G}, \vartheta) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)\right\}$ possesses a uniform attractor $\mathcal{A}$ in the phase space $\mathcal{X}$. Moreover, the uniform attractor $\mathcal{A}$ is connected.

Proof. From Theorem 2.8 we infer that the family of semiprocesses

$$
\left\{U_{(\boldsymbol{G}, \vartheta)}(t, \tau), \quad(\boldsymbol{G}, \vartheta) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)\right\}
$$

is continuous from $\mathcal{X} \times \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$ to $\mathcal{X}$ for any $t \geq \tau \geq 0$. Theorem 3.1 implies that $\left\{U_{(\boldsymbol{G}, \vartheta)}(t, \tau), \quad(\boldsymbol{G}, \vartheta) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)\right\}$ has a bounded uniformly absorbing set and finally, with help of Theorem 3.4, it results that such a family is also uniformly asymptotically compact. The existence of the compact uniform attractor is thus a consequence of the abstract result in Theorem 2.3. We now give a direct proof of the connectedness of the uniform attractor $\mathcal{A}$. To this end, note that the set $\mathcal{A} \times \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$ yields the global attractor for the semigroup $\boldsymbol{S}_{t}$ E' ANCORA VERA STA COSA ANCHE SE C'E' LA LIMITAZIONE SU $\left(\widehat{\sigma}_{2}\right)_{T}$ ? introduced in (2.30). Now, this attractor is connected. In fact, it turns out that the set $\mathcal{B} \times \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$, where $\mathcal{B}$ is a uniformly absorbing ball in $\mathcal{X}$ (whose existence has been assured by Theorem 3.1), is a bounded absorbing set for the semigroup $\boldsymbol{S}_{t}$. Moreover, since $\mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$ is connected (we recall the general definition (2.26)), then $\mathcal{B} \times \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$ ensues connected. Thus, standard results on semigroups (see, e.g., [17, Proposition 5.2.7]) show that $\mathcal{A} \times \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{\mathbf{1}}, \widehat{\sigma}_{2}\right)$ is connected too. This means that $\mathcal{A}=\Pi_{1}\left(\mathcal{A} \times \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)\right)$ is connected.
completa Remark 3.6. CONTROLLA PER FAVORE SE HO SCRITTO BENE QUESTA REMARK. Our definition of solution $U_{(\boldsymbol{G}, \vartheta)}(t, \tau) z_{0}$ works for $\tau \geq 0$ and for $(\boldsymbol{G}, \vartheta) \in L_{\mathrm{loc}}^{2}(0, \infty ; \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(0, \infty ; W^{1,3}(\Omega)\right)$. However, let us point out that one can extend the notion of solution and semiprocess to values $\tau \in \mathbb{R}$ and to forcing terms

$$
(\boldsymbol{G}, \vartheta) \in L_{\mathrm{loc}}^{2}(\mathbb{R} ; \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(\mathbb{R} ; W^{1,3}(\Omega)\right)
$$

In this case, the uniform attractor $\mathcal{A}$ can be represented as

$$
\mathcal{A}=\left\{\begin{array}{r}
z(0): \quad z(t) \text { is any bounded complete trajectory of } U_{(\boldsymbol{G}, \vartheta)}(t, \tau),  \tag{3.44}\\
\text { that is, } U_{(\boldsymbol{G}, \vartheta)}(t, \tau) z(\tau)=z(t) \quad \forall t \geq \tau, \tau \in \mathbb{R} \\
\text { for some }(\boldsymbol{G}, \vartheta) \in \mathcal{H}\left(\widehat{\boldsymbol{\sigma}}_{\mathbf{1}}, \widehat{\sigma}_{2}\right)
\end{array}\right\}
$$

where $\left(\widehat{\boldsymbol{\sigma}}_{1}, \widehat{\sigma}_{2}\right)$ will be now translation compact in $L_{\text {loc }}^{2}(\mathbb{R} ; \boldsymbol{H}) \times L_{\mathrm{loc}}^{2}\left(\mathbb{R} ; W^{1,3}(\Omega)\right)$ The precise structure of the attractor given in (3.44) is a direct consequence of the known results on uniform attractors (see, e.g., [7, Theorem IV.5.1]).

## References

[1] S. Antman, J. L. Ericksen, and D. Kinderlehrer, editors. Metastability and incompletely posed problems, volume 3 of The IMA Volumes in Mathematics and its Applications. Springer-Verlag, New York, 1987.
ball2 [2] J. M. Ball. Global attractors for damped semilinear wave equations. Discrete Contin. Dynam. Systems., 10(1):31-52, 2004.
[3] E. Bonetti. Global solvability of a dissipative Frémond model for shape memory alloys. I. Mathematical formulation and uniqueness. Quart. Appl. Math., 61(4):759781, 2003.
[4] E. Bonetti. Global solvability of a dissipative Frémond model for shape memory alloys. II. Existence. Quart. Appl. Math., 62(1):53-76, 2004.

Brezis73
hep-vish94
hep-vish02
chu-zhao
colli92
-roc-shi05
roc shios
sma2

## cogigra

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collshi

## Fremond2

lli-pata99
haraux

5] H. Brézis. Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, volume 5 of North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 1973.
[6] V. V. Chepyzhov and M. I. Vishik. Attractors of nonautonomous dynamical systems and their dimension. J. Math. Pures Appl. (9), 73(3):279-333, 1994.
[7] V. V. Chepyzhov and M. I. Vishik. Attractors for equations of mathematical physics, volume 49 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2002.
[8] Y.Y. Chu and L.C.Zhao, editors. Shape memory materials and their applications, volume 394 until 395 of Materials Science Forum. Trans Tech Publications Inc., 2002.
[9] P. Colli. An existence result for a thermomechanical model of shape memory alloys. Adv. Math. Sci. Appl., 1(1):83-97, 1992.
[10] P. Colli, M. Frémond, E. Rocca, and K. Shirakawa. Attractors for a three-dimensional thermo-mechanical model for shape memory alloys. Chinese Ann. Math. Ser. B 27(6):683-700, 2006.
[11] P. Colli, M. Frémond, and A. Visintin. Thermo-mechanical evolution of shape memory alloys. Quart. Appl. Math., 48(1):31-47, 1990.
[12] P. Colli, G. Gilardi, and M. Grasselli. Well-posedness of the weak formulation for the phase-field model with memory. Adv. Differential Equations, 2(3):487-508, 1997.
[13] P. Colli, P. Laurençot, and U. Stefanelli. Long-time behavior for the full onedimensional Frémond model for shape memory alloys. Contin. Mech. Thermodyn., 12(6):423-433, 2000.
[14] P. Colli and K. Shirakawa. Attractors for the one-dimensional Frémond model of shape memory alloys. Asymptot. Anal., 40(2):109-135, 2004.
[15] M. Frémond. Non-smooth thermomechanics. Springer-Verlag, Berlin, 2002.
[16] C. Giorgi, M. Grasselli, and V. Pata. Uniform attractors for a phase-field model with memory and quadratic nonlinearity. Indiana Univ. Math. J., 48(4):1395-1445, 1999.
[17] A. Haraux. Systèmes dynamiques dissipatifs et applications, volume 17 of Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris, 1991.
kavian [18] O. Kavian. Introduction à la théorie des points critiques et applications aux problèmes elliptiques, volume 13 of Mathématiques \& Applications (Berlin) [Mathematics \& Applications]. Springer-Verlag, Paris, 1993.
s-singular
mar-miz
osa-wang04
use-vishik
art1
temam
shiste
simon
[19] J.L. Lions. Perturbations singulières dans les problèmes aux limites et en contrôle optimal, volume 323 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1973.
[20] M. Marcus and V.J. Mizel. Every superposition operator mapping one Sobolev space into another is continuous. J. Funct. Anal., 33(2):217-229, 1979.
[21] I. Moise, R. Rosa, and X. Wang. Attractors for noncompact nonautonomous systems via energy equations. Partial differential equations and applications. Discrete Contin. Dyn. Syst., 10(1-2):473-496, 2004.
[22] V. Pata, G. Prouse, and M. I. Vishik. Traveling waves of dissipative nonautonomous hyperbolic equations in a strip. Adv. Differential Equations, 3(2):249-270, 1998.
[23] A. Segatti. Analysis of a solid-solid phase change model coupling hyperbolic momentum balance and diffusive phase dynamics. Adv. Math. Sci. Appl., 14(1):327-349, 2004.
[24] A. Segatti. Error estimates for a variable time-step discretization of a phase transition model with hyperbolic momentum. Numer. Funct. Anal. Optim., 25(5-6):547-569, 2004.
G. R. Sell. Nonautonomous differential equations and topological dynamics. I. The basic theory. Trans. Amer. Math. Soc., 127:241-262, 1967.
[26] G. R. Sell. Nonautonomous differential equations and topological dynamics. II. Limiting equations. Trans. Amer. Math. Soc., 127:263-283, 1967.
[27] K. Shirakawa and U. Stefanelli. Structure result for steady-state solutions of a onedimensional Frémond model of SMA. Phys. D, 190(3-4):190-212, 2004.
[28] J. Simon. Compact sets in the space $L^{p}(0, T ; B)$. Ann. Mat. Pura Appl. (4), 146:6596, 1987.
[29] R. Temam. Infinite-dimensional dynamical systems in mechanics and physics, volume 68 of Applied Mathematical Sciences. Springer-Verlag, New York, second edition, 1997.

