# Variational Models for Nematic Shells

Antonio Segatti

Dipartimento di Matematica "F. Casorati", Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy.

*E-mail address*, A. Segatti: antonio.segatti@unipv.it

# Contents

# Preface

These are the Lecture Notes of the 8 hours course I gave at the Universidad Autònoma de Madrid in October/November 2015.
This course has been organized within the

*Programa de Posgrado de Excelencia Internacional en Matemticas de la UAM*

which is gratefully acknowledged.

The subject of the course were *Nematic Shells*. Nowadays, this is a lively topic at the frontier between modelling, variational methods, topology and geometry. After a brief introduction on the physics of these structures, I will mainly concentrate on the following basic aspects:

- Intrinsic/Extrinsic energies
- Relation between the functional framework and the topology of the shell and of its boundary conditions, if present
- Existence of minimizers and of the gradient flow
- Refined analysis of the structure of minima and, more generally, of the critical points for toroidal shells.

The material of the notes is the outcome of a research line started in 2012 together with G. Canevari, M. Veneroni and M. Snarski and has been taken from the papers [**50**], [**49**] and [**19**]. In particular, I warmly thank G. Canevari and M. Veneroni for their positive criticism on preliminary versions of the manuscript.

Last but not least, I would like to thank Juan Luis Vázquez and Matteo Bonforte for the kind invitation to Madrid.

CHAPTER I

# Mathematical Modelling

Soft Matter offers many intriguing and fascinating examples of a non trivial interplay between topology, geometry, partial differential equations and physics.

A prominent instance is offered by Liquid Crystals (LC) which manifest several visual representations of the underlying geometric constraints. For instance, the word Nematic itself, identifying Nematic Liquid Crystals (NLC), originates from the Greek word $\nu\eta\mu\alpha$ and refers to a particular type of topological defects that these type of Liquid Crystals exhibit.

It is commonly said that "Liquid Crystals flow like a liquid but its constituent molecules retain some positional order typical of solids". The name itself presents a contradiction but it convey the fact that Liquid Crystals are an intermediate state of matter between solids and liquids, with properties shared with both phases (see e.g. [**55**]).

Liquid crystals have surprising and technologically relevant properties. A prominent application is, for instance, on liquid crystals displays.

Among the different types of LC existing in nature, we are interested in *Nematic Liquid Crystals*. The constituent rod-like molecules have no translational order but retain a certain degree of long range orientational order. In particular, nematic molecules are characterized by the tendency to align parallel to each other along a given direction [**24**].

The mathematical analysis of Liquid Crystals is a lively topic as witnessed by the number of Conferences and Workshops that took place in worldwide renowned institutions together with the huge amount of publications on the topic. Moreover, it is intriguing and fascinating as it is a truly interdisciplinary research combining tools from different fields such as Partial Differential Equations, Calculus of Variations, Topology, Differential Geometry.

## I.1. Mathematics for Liquid Crystals

The basic mathematical description of NLC (for other, more refined theories, we refer to the works of De Gennes [**24**] and the works of J. M. Ball & his school [**4**], [**39**], [**60**], [**30**]) is given in terms of a vector field of unit norm, named *director*, representing the local preferred orientation of the molecules.

The resulting energy governing the statics of a NLC sample occupying a bounded and regular domain $\Omega \subset \mathbb{R}^3$ is the so called Oseen, Zocher and Frank energy (see e.g. [**55**])

$$W^{OZF}(\mathbf{n}, \Omega) := \frac{1}{2}\int_\Omega \left[k_1(\operatorname{div}\mathbf{n})^2 + k_2(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + k_3|\mathbf{n}\times\operatorname{curl}\mathbf{n}|^2\right]\,\mathrm{d}x, \tag{I.1.1}$$

where $k_1$, $k_2$, $k_3$ are the so called Frank's constant each of which representing a particular type of distortion of the of the crystal, namely pure splay, pure twist and pure bend, respectively. Consequently, these positive constant are known respectively as the splay, twist and bend moduli. In the picture I.1 we depicted these three types of deformations.

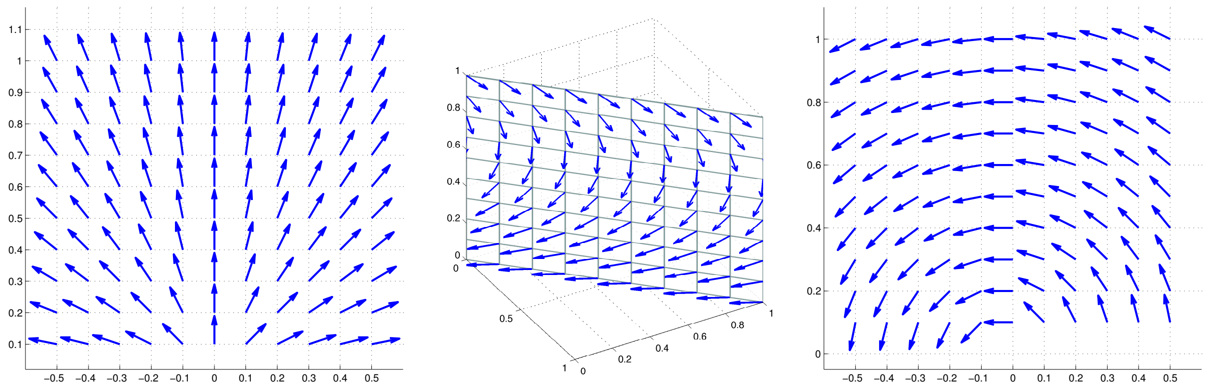


FIGURE 1. Left picture: splay deformation. Center picture: twist deformation. Right picture: bend deformation

When the Frank's constants have the same order of magnitude, a typical approximation consists in setting $k = k_1 = k_2 = k_3$. Thus the energy (I.1.1) becomes

$$W^{OZF}(\mathbf{n}, \Omega) = \frac{k}{2} \int_{\Omega} |\nabla \mathbf{n}|^2 \mathrm{d}x. \tag{I.1.2}$$

This is the so called *one constant approximation*. It is interesting to note that in this regime an equilibrium configuration for a NLC is a map $\mathbf{n} : \Omega \to \mathbb{S}^2$ minimizing (I.1.2). Thus it is an *harmonic map* into sphere. The mathematical analysis of the energy (I.1.1) has produced a huge number of different contribution in the last 30 years, including existence of minimizers and (partial) regularity results (see, among the others, [**15**], [**27**], [**6**], [**13**], [**14**]). Moreover, the connection with the theory of *harmonic maps* gives a more general perspective to the analysis of NLC thus permitting the use of geometric and topological techniques (see, e.g., [**29**], [**37**]) in the study of NLC.

To have a better insight on how the energy (I.1.2) selects the minimizers, let us consider a microscopic two dimensional spin model, which represents an approximation of the continuum energy in the one constant approximation (see [**11**]). This model is usually called $XY$ model and, despite its simplicity, captures many important issues of nematics and it is used, in its variants, in Monte Carlo simulations (see e.g. [**35**]). The energy we are considering is defined on a two dimensional lattice $\varepsilon\mathbb{Z}^2$, with characteristic lenght $\varepsilon > 0$ deposited on an open domain $\Omega \subset \mathbb{R}^2$.

Any point $i \in \varepsilon\mathbb{Z}^2 \cap \Omega$ is occupied by a vector $\mathbf{n}(i)$ with unit lenght, named spin vector. The energy takes the form (we refer to, e.g., the lecture notes [**12**])

$$E_\varepsilon(\mathbf{n}) = -\sum_{<i,j>} \varepsilon^2 (\mathbf{n}(i), \mathbf{n}(j)). \tag{I.1.3}$$

where $< \cdot, \cdot >$ indicates the nearest neighboors, namely those sites at distance $\varepsilon$ from $i$. Since the norm of the spin vectors is fixed to be equal to one, the energy penalizes only alignment. Once we have fixed a proper functional setting (see the next Chapter II), one may compute the $\Gamma$-limit as $\varepsilon \searrow 0$ (see V.2 for the definition) and get (see [**2**] and [**1**])

$$E_\varepsilon \xrightarrow{\Gamma} -2|\Omega| \tag{I.1.4}$$

A more interesting description appears when we analyze the asymptotics of the energy $F_\varepsilon$ defined by

$$F_\varepsilon(\mathbf{n}) := \frac{E_\varepsilon(\mathbf{n}) - \min E_\varepsilon}{\varepsilon^2}. \tag{I.1.5}$$

Note that $\min E_\varepsilon = -\sum_{<i,j>} \varepsilon^2$, thus he above considerations give that

$$F_\varepsilon(\mathbf{n}) = \sum_{<i,j>} (1 - (\mathbf{n}(i), \mathbf{n}(j)) = \frac{1}{2} \sum_{<i,j>} \varepsilon^2 \Big| \frac{\mathbf{n}(i) - \mathbf{n}(j)}{\varepsilon} \Big|^2 \approx \frac{1}{2} \int_\Omega |\nabla U|^2 \mathrm{d}x, \tag{I.1.6}$$

where $U$ is the linear affine interpolation of the discrete values $\mathbf{n}(i)$. Thus, it is possible to prove that (see [**1**] and [**11**])

$$\Gamma - \lim_{\varepsilon \searrow 0} F_\varepsilon = W^{OZF}, \tag{I.1.7}$$

where $W^{OZF}$ is the energy (I.1.2). The above considerations do not take into account the boundary conditions or rather suppose that the boundary conditions are chosen properly. The choice of the boundary conditions has indeed a crucial role in fixing the correct functional framework. More precisely, particular choices of the boundary conditions (actually of the "topology" of the boundary conditions) force an infinite limit in (I.1.7). This is the occurrence of (point) defects (see [**7**] and [**1**]) which make the analysis challenging and complicated. However, the above arguments show that, once we correctly fix the boundary conditions, the energy (I.1.2) favors alignment of the director field. Hence a similar behavior is expected for the continuum energy (I.1.2) Let us anticipate that in the case of Nematic Shells, we will observe a competition between the alignment of the director field and the geometry of the substrate.

## I.2. Nematic Shells: modelling and energetics

A *Nematic Shell* is a rigid colloidal particle with a typical dimension in the micrometer scale coated with a thin film of nematic liquid crystal whose molecular orientation is subjected to a tangential anchoring. The study of these structures has recently received a good deal of interest. As suggested by Nelson [**44**], the interest in *Nematic Shells* is related to the possibility of using them as building blocks of mesoatoms with a controllable valence.

From a mathematical point of view, a *Nematic shell* is usually represented as a two dimensional surface $\Sigma$ embedded in $\mathbb{R}^3$ with the local orientation of the molecules described via a unit norm tangent vector field, named director in analogy with the "flat" case.
The study of these structures offers a non trivial interplay between the geometry and the topology of the fixed substrate and the tangential anchoring constraint. Indeed, as observed in [**57**] and [**9**], the liquid crystal equilibrium (and all its stable configurations, in general) is the result of the competition between two driving principles: on the one hand the minimization of the "curvature of the texture" penalized by the elastic energy, and on the other the frustration due to constraints of geometrical and topological nature, imposed by anchoring the nematic to the surface of the underlying particle.

Different theoretical approaches for the treatment of *Nematic shells* are available. Differences arise in the choice of the form of the elastic part of the free energy which could be of *intrinsic* or *extrinsic* nature. More precisely, theories which employ only covariant derivatives will be named *intrinsic* (see [**38**], [**53**], [**57**], [**52**]) while theories that comprise also how the shell sits in the three dimensional space will be named *extrinsic* (see [**42**] and [**43**]). When restricting to the simpler *one* *constant approximation*, the *extrinsic energy* has the form

$$W_{extr}(\mathbf{n}) := \frac{\kappa}{2}\int_\Sigma |D\mathbf{n}|^2 + |\mathfrak{B}\mathbf{n}|^2 \,\mathrm{dVol}, \tag{I.2.1}$$

while the *intrinsic energy* has the form

$$W_{intr}(\mathbf{n}) := \frac{\kappa}{2}\int_\Sigma |D\mathbf{n}|^2 \,\mathrm{dVol}. \tag{I.2.2}$$

In the definitions above $\mathbf{n}$ is a tangent vector field with unit norm, $\kappa$ is a positive constant (often taken equal to one for simplicity), the symbol $D$ denotes the covariant derivative on $\Sigma$, and $\mathfrak{B}$ is the shape operator. We refer to the quantity $\int_\Sigma |D\mathbf{n}|^2$ as the *Dirichlet (or elastic) energy* of $\mathbf{n}$.
The extrinsic energy (I.2.1) has been derived by Napoli & Vergori (see [**42**] and [**43**]) by using a formal dimension reduction. More precisely, starting from a tubular neighborhood $\Sigma_h$ of thickness $h$ (satisfying a suitable constraint related to the curvature of $\Sigma$), Napoli and Vergori obtain that $W_{extr}(\mathbf{n})$ in (I.2.1) is given by

$$W_{extr}(\mathbf{n}) = \lim_{h\searrow 0}\frac{1}{h}W^{OZF}(\mathbf{n},\Sigma_h). \tag{I.2.3}$$

The limit above holds for any fixed and sufficiently smooth field $\mathbf{n}$ with the property of being independent of the thickness direction and tangent to leaf of the foliation $\Sigma_h$. Actually, by performing the above limit starting from the full energy (I.1.1), they obtained the energy

$$W_{extr}(\mathbf{n}) := \frac{1}{2}\int_\Sigma K_1(\mathrm{div_s}\,\mathbf{n})^2 + K_2(\mathbf{n}\cdot\mathrm{curl_s}\,\mathbf{n})^2 + K_3|\mathbf{n}\times\mathrm{curl_s}\,\mathbf{n}|^2 \,\mathrm{dVol}, \tag{I.2.4}$$

where the various differential operators coming into play are defined in the Appendix. Differently from the *intrinsic* theories, the twist term is still present in the surface energy.

It is worthwhile noting that the above formal argument can be made rigorous using the theory of $\Gamma$-convergence in the spirit of [**34**]. We will discuss this limit in Section III.2.

The difference between the two energies, *intrinsic* and *extrinsic* lies in the choice of the distortion element or, more precisely, on the choice of the type of *parallel transport* (see [**36**] for the definition of parallel transport). More precisely, as it has been done in the flat case one may imagine to study a discrete energy defined on lattice deposited on the given shell $\Sigma$. Suppose for simplicity that $\Sigma$ can be represented with a single chart $\Phi:\Omega\to\Sigma$, with $\Omega\subset\mathbb{R}^2$. We denote with $g$ the standard metric on $\Sigma$. We first consider a uniform square lattice with characteristic size $\varepsilon>0$ and then map it on the shell $\Sigma$ using the map $\Phi$. With this respect, once we have two nearest neighbors $i$ and $j$ in $\mathbb{Z}^2_\varepsilon$, we have that $p_i^\varepsilon=\Phi(i)$ and $p_j^\varepsilon=\Phi(j)$ are nearest neighbors on the curved lattice on $\Sigma$. Now, as in the flat case one considers the reference square $W:=[0,1)^2$ in $\mathbb{R}^2$ and then considers the set $\Omega_\varepsilon$ defined as the union over all the $i\in\mathbb{Z}^2$ of the squares $Q_i^\varepsilon:=\{i+\varepsilon W\}$ such that $\{i+\varepsilon W\}\subset\subset\Omega$. The square $Q_i^\varepsilon$ is mapped, via the chart $\Phi$, to a curvilinear square $\tilde{Q}_i^\varepsilon$ in $\Sigma$. Note that the area of square $Q_i^\varepsilon$ is in general different from the area of $\tilde{Q}_i^\varepsilon$.

Now, the energy we aim to consider is defined as follows (compare with (I.1.6))

$$F_\varepsilon(\mathbf{n}) := \sum_{<i,j>}\sqrt{\det g(p_i^\varepsilon)}\big(1-(\mathbf{n}(p_i^\varepsilon),\mathbf{n}(p_j^\varepsilon))_{\mathbb{R}^3}. \tag{I.2.5}$$

Now, in the definition above, it is not yet specified how the scalar product is computed, since $\mathbf{n}(p_i^\varepsilon)$ and $\mathbf{n}(p_j^\varepsilon)$ may not lie in the same tangent space. More precisely, a first possibility is that one may "forget" the curved substrate $\Sigma$ and imagine that the vector $\mathbf{n}(p_i^\varepsilon)$ sees $\mathbf{n}(p_j^\varepsilon)$ in $\mathbb{R}^3$. Consequently, we fix the scalar product in (I.2.5) to be the scalar product between $\mathbf{n}(p_i^\varepsilon)$ and $\mathbf{n}(p_j^\varepsilon)$, thought as vectors in $\mathbb{R}^3$ and constrained to be tangent to $\Sigma$. In a way, we are thinking of transporting $\mathbf{n}(p_j^\varepsilon)$ at the site $p_i^\varepsilon$ by a rigid motion on the segment joining $p_i^\varepsilon$ and $p_j^\varepsilon$. The resulting energy is named *extrinsic* discrete energy and denoted with $F_{\varepsilon,extr}$. Alternatively, one may decide to transport the vector $\mathbf{n}(p_j^\varepsilon)$ at the point $p_i^\varepsilon$ via the parallel transport on $\Sigma$. This is a purely intrinsic operation as it can be expressed using quantity defined only on $\Sigma$ without taking care of the way in which $\Sigma$ sits in $\mathbb{R}^3$. We denote with $\mathcal{T}_j^i\mathbf{n}$, the parallel transported $\mathbf{n}$ from the point $P_j^\varepsilon$ to the point $P_i^\varepsilon$. By construction, there holds that $\mathcal{T}_j^i\mathbf{n}$ lies in the same tangent plane of $\mathbf{n}(P_i^\varepsilon)$. With this notation, the intrinsic version of the discrete energy (I.2.5) becomes

$$F_{\varepsilon,intr}(\mathbf{n}) := \sum_{<i,j>} \sqrt{\det g(p_i^\varepsilon)}\big(1-(\mathbf{n}(p_i^\varepsilon),\mathcal{T}_j^i\mathbf{n}(p_j^\varepsilon)\big)_{\mathbb{R}^3}. \tag{I.2.6}$$

Now, when we try to obtain a macroscopic energy from the lattice energy (as in the flat case), we have the following scenario (see Section III.3) [1]:

$$F_{\varepsilon,intr}(\mathbf{n}) \xrightarrow{\Gamma} W_{intr} \tag{I.2.7}$$

$$F_{\varepsilon,extr}(\mathbf{n}) \xrightarrow{\Gamma} W_{extr} \tag{I.2.8}$$

It is interesting to compare the two possible choices of the discrete energies (and hence of the two possible continuum one) on a cylindrical shell. In fact, even if the geometry of the cylinder is particularly simple since its Gaussian curvature is zero, it shows some interesting features. Thus, we set $\Sigma$ to be a cylinder with height $2\pi$ and radius one. The tangent plane $T_p\Sigma$ at the give point $p=(p_1,p_2,p_3)$ is spanned by the two vectors $\mathbf{e}_1 := (0,0,1)$ and $\mathbf{e}_2 = (-p_2,p_1,0)$. Recall that the cylinder is usually parametrized with

$$\sigma(t,\vartheta) = (\cos(\vartheta),\sin(\vartheta),t), \qquad t\in[0,2\pi],\ \vartheta\in[0,2\pi],$$

and thus the vector $\mathbf{e}_2$ is

$$\mathbf{e}_2(t,\vartheta) = (-\sin(\vartheta),\cos(\vartheta),0).$$

Now, let us deposit a square lattice on the cylinder and consider, at any point of this lattice, the unit norm tangent vector fields $\mathbf{n}=\mathbf{e}_1$ and $\mathbf{n}=\mathbf{e}_2$ (see Figure 2). As before, this is obtained by considering a square lattice with lattice spacing $\varepsilon$ on $[0,2\pi]\times[0,2\pi]$. Note that we have that $\varepsilon=\frac{2\pi}{k}$, with $k$ being the number of subintervals in which $[0,2\pi]$ is divided. First of all, we obtain that the configuration $\mathbf{e}_1$ minimizes the *extrinsic* discrete energy as

$$F_{\varepsilon,extr}(\mathbf{e}_1)=0.$$

Moreover, since the angle $\alpha$ between two nearest neighbors is $\alpha=\pi-\beta=\varepsilon$, where $\beta=\pi(k-2)/k$ is the internal angle in a regular polygon with $k$ sides, a simple computation reveals that

$$F_{\varepsilon,extr}(\mathbf{e}_2)=\frac{4\pi^2}{\varepsilon^2}(1-\cos(\varepsilon))>0.$$

Note that when $\varepsilon\searrow 0$, we obtain that $F_{\varepsilon,extr}(\mathbf{e}_2)\to 2\pi^2$, that is the value of the continuum extrinsic energy (I.2.1) at $\mathbf{n}=\mathbf{e}_2$ (see II.1.16 in Lemma II.1.3).

The situation changes when we consider the intrinsic discrete energy. A straightforward calculation shows that

$$F_{\varepsilon,intr}(\mathbf{e}_1)=F_{\varepsilon,intr}(\mathbf{e}_2)=0.$$

[1]The convergence is intended to be the $\Gamma$ convergence (see Section V.2 for the details), even if we do not specify the functional framework

Actually, since the Gaussian curvature of the cylinder is zero much more is true: the *intrinsic* discrete energy is zero for all the configurations in which $\mathbf{n}$ forms a constant angle with, say, $\mathbf{e}_1$. On the other hand, when we use the *extrinsic* discrete energy the configuration $\mathbf{n} = \mathbf{e}_2$ (and generally all the other configurations different from $\mathbf{n} \equiv \mathbf{e}_1$) experiences a "three dimensional bending" when the scalar product is formed. As it apparent from the picture below this is due to the fact that we consider the cylinder as immersed in $\mathbb{R}^3$ and the vector field $\mathbf{n}$ as a vector field in $\mathbb{R}^3$, though tangent to $\Sigma$. As a consequence, the *extrinsic* discrete energy selects a unique minimizer (i.e. $\mathbf{n} = \mathbf{e}_1$) while the *intrinsic* discrete energy has an infinite number of minima (for instance $\mathbf{n} = \mathbf{e}_1, \mathbf{n} = \mathbf{e}_2$ and their linear combinations).

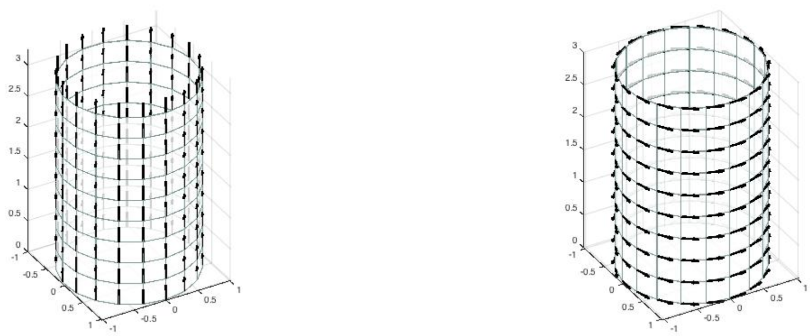


FIGURE 2. Examples of unitary vector fields on a cylinder in $\mathbb{R}^3$: left picture $\mathbf{n} = \mathbf{e}_1$, right picture $\mathbf{n} = \mathbf{e}_2$

The discussion above can be repeated at the continuum level (see [**42**] for a complete discussion). The main observation is the fact that the extrinsic energy (I.2.1) has the form of a "phase transition energy" since it is the sum of a Dirichlet part and of a (vectorial) double well potential part. In fact (see Section V.1) the purely extrinsic part $|\mathfrak{B}\mathbf{n}|^2$ is minimized when $\mathbf{n}$ is oriented along the direction of minimal principal curvature (i.e. minimal normal curvature). Thus, the extrinsic energy favors a parallel configuration (i.e. a vector field such that $D\mathbf{n} = 0$) in the direction of minimal principal curvature. Already considering only the Dirichlet part (i.e. the intrinsic energy) the minimization experiences an interesting frustration of geometric nature due to the fact that the existence of globally defined unit norm parallel vector fields requires the Gaussian curvature to vanish (see Corollary II.4). On the other hand, the effect of the competition between the two terms of the *extrinsic* energy is particularly simple on the cylinder. In this case, the costless configuration for the extrinsic energy is the one in which $\mathbf{n}$ is everywhere oriented like $\mathbf{e}_1$, namely the principal direction of curvature. On the other hand, when considering the intrinsic energy (I.2.2) all the uniform configuration (i.e. the director $\mathbf{n}$ is everywhere oriented with fixed direction) have the same minimal energy. In Chapter IV we will analyze the extrinsic energy on the axisymmetric torus and compare its minimization with the one of the intrinsic energy.

A crucial step in the analysis of a variational problem is the understanding of the correct functional framework where to set, for example, the minimization of the given energy.

A closer inspection of the energy (I.2.4) reveals that there exist constants such that

$$\frac{K_*}{2}\int_\Sigma \big(|D\boldsymbol{u}(x)|^2 + |\mathfrak{B}\boldsymbol{u}(x)|^2\big)\mathrm{dVol} \le W_{extr}(\boldsymbol{u}) \le \frac{K^*}{2}\int_\Sigma \big(|D\boldsymbol{u}(x)|^2 + |\mathfrak{B}\boldsymbol{u}(x)|^2\big)\mathrm{dVol}, \tag{I.2.9}$$

Consequently, the natural choice for the functional framework would be to set its analysis in the space of tangent vector fields such that $|\mathbf{n}|$ and $|D\mathbf{n}|$ belong to $L^2(\Sigma)$, which means

$$L^2_{\tan}(\Sigma) := \left\{ \boldsymbol{u} \in L^2(\Sigma;\mathbb{R}^3) : \boldsymbol{u}(x) \in T_x\Sigma \text{ a.e.} \right\},$$
$$H^1_{\tan}(\Sigma) := \left\{ \boldsymbol{u} \in L^2_{tan}(\Sigma) : |D\boldsymbol{u}| \in L^2(\Sigma) \right\}.$$

However, the topology of the surface may introduce possibile obstructions to this program. In particular, it may well happen that

$$H^1_{\tan}(\Sigma, \mathbb{S}^2) := \left\{ \boldsymbol{u} \in H^1_{\tan}(\Sigma) : |\boldsymbol{u}| = 1 \text{ a.e.} \right\}, \tag{I.2.10}$$

is empty. This could be heuristically explained as follows. Let $\boldsymbol{v}$ be a smooth tangent vector field on $\Sigma$, with finitely many zeroes. The index $m \in \mathbb{Z}$ of a zero $\bar{x} \in \Sigma$ is, intuitively, the number of counterclockwise rotations that the vector completes around a small circle around $\bar{x}$. So, if $m \neq 0$, the corresponding unit-length vector field $\boldsymbol{v}/|\boldsymbol{v}|$ has a discontinuity at $\bar{x}$ (see Figure 3). These discontinuities are named (point) defects. By the Poincaré-Hopf index Theorem [**26**, Chapter 3], the global sum of the indices of the zeroes of $\boldsymbol{v}$ equals the Euler characteristic $\chi(\Sigma)$ (hence it is only related to the topology of $\Sigma$) and therefore it is possible to find a smooth field $\mathbf{n}$ with $|\mathbf{n}| \equiv 1$ on $\Sigma$ if and only if $\chi(\Sigma) = 0$, i.e. if $\Sigma$ is a genus-1 surface ("hairy ball Theorem"). Moreover, a direct computation (say for $m = 1$) shows that the Dirichlet energy of $\boldsymbol{v}/|\boldsymbol{v}|$ in any small enough annulus centered at $\bar{x}$, with internal radius $\rho$, scales like $|\log(\rho)|$ as $\rho \to 0$. Therefore, one would expect the topological constraint of the hairy ball Theorem to hold also for $H^1$-regular vector fields.

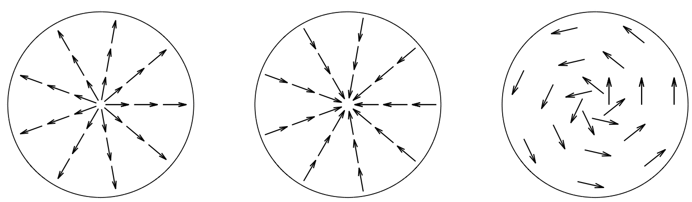


FIGURE 3. Examples of unitary vector fields on a disc in $\mathbb{R}^2$, showing topological defects with index 1.

This is indeed the case and in Chapter II we will prove the following (see also [**49**])

THEOREM I.1. *Let $\Sigma$ be a compact smooth surface without boundary, embedded in $\mathbb{R}^3$. Let $\chi(\Sigma)$ be the Euler characteristic of $\Sigma$. Then*

$$H^1_{\tan}(\Sigma, \mathbb{S}^2) \neq \emptyset \ \Leftrightarrow \chi(\Sigma) = 0.$$

This Theorem is an $H^1$-version of the celebrated Hairy Ball Theorem and can be generalized to compact manifolds without boundary. Moreover, Theorem I.1 can be generalized to the case of a shell with boundary and to vector and line fields with VMO regularity. We refer to [**19**] for the analysis of this issue. Theorem I.1 provides a "non flat" version of a well know result of Bethuel [**5**] that gives conditions for the non emptiness of the space

$$H^1_g(\Omega;\mathbb{S}^1) := \left\{ v \in H^1(\Omega;\mathbb{R}^2) : |v(x)| = 1 \text{ a.e. in } \Omega \text{ and } v \equiv g \text{ on } \partial\Omega \right\},$$

where $\Omega$ is a simply connected bounded domain in $\mathbb{R}^2$ and $g$ is a prescribed smooth boundary datum with $|g| = 1$. The non-emptiness of $H^1_g(\Omega;\mathbb{S}^1)$ is related to a topological condition on the Dirichlet datum $g$ (see [**5**] and [**7**]) while Theorem I.1 the topological constraint is on the genus of the surface.

Note that the exponent 2 in (I.2.1) is a limit-case, as it is possible to construct unitary fields such that $|D\boldsymbol{v}| \in L^p(\Sigma)$ for any $p \in [1, 2)$, on any smooth compact surface $\Sigma$.

In view of Theorem I.1, we restrict our study to genus-1 surfaces, where the underlying geometry of the substrate does not force the creation of defects. A rigorous analysis of the distribution and evolution of defects on nematic surfaces is an interesting problem which is beyond the scope of our analysis. Due to its large potential impact on the design of new generation metamaterial structures (see [**44, 59**]), this question has garnered a huge deal of interest within the physics community (see [**32, 47, 51, 45, 57**]). To the best of our knowledge it still lacks a rigorous mathematical treatment. For instance a crucial problem is to understand which should be the form of an energy functional capable of describing configurations with defects. We will (briefly) discuss about this problem in Chapter III and we refer to the paper in preparation [**18**] for a more detailed analysis.

## I.3. Presentation of the Results

**A word on the notation**
From now on we will denote with $W$ the surface extrinsic energy. In particular, we will use the same notation to indicate the full energy (I.2.4) and the one constant approximation (I.2.1). Moreover, since for most of the results the exact value of the constant $\kappa$ in (I.2.1) is irrelevant, we will assume that, if not otherwise stated, $\kappa = 1$.

These Lectures are thus organized as follows

In Chapter II we will explore the relations between the functional framework and the topology of the shell, which will be assumed to be without boundary. In particular, we will prove Theorem I.1 that will fix the shell $\Sigma$ to be a two dimensional compact surface of genus 1 without boundary of genus. The proof of this result will make use of a construction that will be useful throughout the notes, that is the possibility of representing a unit norm vector field $\mathbf{n}$ with some regularity in terms of a scalar parameter $\alpha$ which measures the rotation of $\mathbf{n}$ with respect to a given orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2\}$, i.e.

$$\mathbf{n} = (\cos\alpha)\mathbf{e}_1 + (\sin\alpha)\mathbf{e}_2.$$

This angle is usually called deviation angle. The local existence of such a representation is straightforward (see [**25**]). On the other hand, a global one on $\Sigma$ is in general not possible (even when the topology of $\Sigma$ allows for $H^1$-fields). A way to encompass this problem is to pass to a universal covering of $\Sigma$. This means that for every $H^1$-regular unit-norm vector field $\mathbf{n}$ there exists a representation $\alpha \in H^1_{\text{loc}}(\mathbb{R}^2)$ defined on the universal covering of $\Sigma$ (see [**49**]). This representation takes into account the possibility of having a vector field with non zero *winding number*, that is the number that "counts" how many times $\mathbf{n}$ wraps around $\Sigma$. The deviation angle representation of $\mathbf{n}$ is at the basis of the proof of I.1 in Chapter II. Moreover, once we have represented $\mathbf{n}$ via the scalar $\alpha$, we rewrite the energies in terms of $\alpha$.
Finally, in the last part of the chapter, we will extend Theorem I.1 to vector fields with VMO regularity on a $n$ dimensional hypersurface in $\mathbb{R}^{n+1}$ without boundary.

In Chapter III using the direct method of the calculus of variations, we prove existence of a minimizer of (I.2.4). Then, we justify the dimensional reduction (I.2.3) using $\Gamma$ convergence in the one constant approximation regime. This is indeed an application of the results of Le Dret and Raoult [**34**] to our setting. We decided to present them with complete proofs just for the sake of completeness. The justification of (I.2.4) using $\Gamma$-convergence is currently under investigation.

Subsequently, we present the justification of (I.2.4) using a discrete to continuum convergence as discussed in (I.2.8).

We then focus on the $L^2$-gradient flow of the one constant approximation energy (I.2.1). The study of the gradient flow for the energy (I.2.1) could be seen as a starting point for the analysis of an Ericksen-Leslie type model for nematic shells. This problem has already been addressed in [**52**] where various wellposedness and long time behavior results have been obtained for an Ericksen-Leslie type model on Riemannian manifolds. However, it should be pointed out that the model in [**52**] is purely intrinsic and does not take into account the way the substrate on which the nematic is deposited sits in the three-dimensional space. Moreover, in the equation describing the evolution of $\mathbf{n}$ (there called $\mathbf{d}$) the constraint $|\mathbf{n}| = 1$ is not considered.

More precisely, we prove (see Theorem III.3) the well-posedness of the $L^2$-gradient flow of (I.2.1), i.e.

$$\partial_t \mathbf{n} - \Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n} = |D\mathbf{n}|^2 \mathbf{n} + |\mathfrak{B}\mathbf{n}|^2 \mathbf{n} \quad \text{in } \Sigma \times (0, +\infty). \tag{I.3.1}$$

Here $\Delta_g$ is the *rough Laplacian*, $D$ is the covariant derivative and $\mathfrak{B}$ is the shape operator on $\Sigma$ (see Section V.1). The right-hand side of (I.3.1) is a result of the unit-norm constraint on the director $\mathbf{n}$. A proof of the existence relying on *i) discretization, ii) a priori estimates, iii) convergence of discrete solutions*, would encounter a difficulty here, as the nonlinear term $|D\mathbf{n}|^2$ in the right-hand side of (I.3.1) is not continuous with respect to the weak-$H^1$ convergence expected from the a priori estimates. We overcome this problem with techniques employed in the study of the heat flow for harmonic maps (see [**21, 22**]): we first relax the unit-norm constraint with a Ginzburg-Landau approximation, i.e., we allow for vectors $\mathbf{n}$ with $|\mathbf{n}| \neq 1$, but we penalize deviations from unitary length at the order $1/\varepsilon^2$, for a small parameter $\varepsilon > 0$. In this way, it is possible to build a sequence of fields $\mathbf{n}^\varepsilon$, with $|\mathbf{n}^\varepsilon| \to 1$ as $\varepsilon \to 0$, which solve an approximation of (I.3.1), with zero right-hand side. The crucial remark, in order to recover (I.3.1) in the limit, is that for a smooth unit-norm field $\mathbf{n}$, (I.3.1) is equivalent to

$$(\partial_t \mathbf{n} - \Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n}) \times \mathbf{n} = 0.$$

When passing to the limit, the non-trivial term is $\Delta_g \mathbf{n} \times \mathbf{n}$, which can be shown to have a divergence-like structure thus resulting continuous with respect to the weak convergence involved.
In the second part of the Chapter, we will discuss some finer properties of the solution of the gradient flow. More precisely, thanks to the deviation angle representation we are able to prove regularity results and existence of evolutions with fixed *winding number*.

In Chapter IV we fix $\Sigma$ to be the axisymmetric torus. Using the deviation angle representation, we explicitly calculate the value of the energy (I.2.4) on constant deviations $\alpha$. The interest lies in understanding the dependence of the energy on the mechanical parameters $K_i$ and on the aspect ratio of the torus, even on a special set of configurations. The constant configurations $\alpha_m := 0$ and $\alpha_p := \pi/2$ (see Figure 4) are of particular interest, as, up to an additive constant, the $\alpha$-representation of (I.2.1) is

$$W(\alpha) = \frac{1}{2} \int_Q \left\{ |\nabla_s \alpha|^2 + \eta \cos(2\alpha) \right\} \,\mathrm{dVol}, \tag{I.3.2}$$

where $\eta$ is a function which depends only on the geometry of the torus. This structure, a Dirichlet energy plus a double (modulo $2\pi$) well potential, is well-studied in the context of Cahn-Hilliard phase transitions. Depending on the torus aspect ratio, the sign of $\eta$ may not be constant on $Q$, thus forcing a smooth transition between the states $\alpha_m$, where $\eta < 0$, and $\alpha_p$, where $\eta > 0$. With the aid of numerical simulations, in Chapter IV we discuss some qualitative behavior of the minimizers of (I.3.2), such as stability, depending on the aspect ration of the torus. Moreover, we show the emergence of a new type of solution "interpolating" the constant states $\alpha_p$ and $\alpha_m$. This solution was never observed before (see [**50**] and [**49**]) and is the result of the combination between *intrinsic* and *extrinsic* effects in the energy (I.2.1).

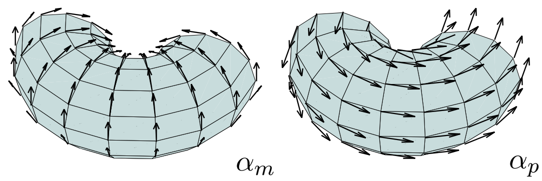


FIGURE 4. The constant states $\alpha_m \equiv 0$ (director oriented along the meridians of the torus), $\alpha_p \equiv \pi/2$ (director oriented along the parallels of the torus).

Finally, in Chapter V we collect (without proofs) some of the mathematical tools we use in the Lecture Notes. In particular, in Section V.1 we recall some facts of differential geometry while in Section V.2 we introduce the notion of Γ-convergence.

CHAPTER II

# Functional Framework

In this Chapter we want to discuss the functional framework for the variational analysis of the energy

$$W(\mathbf{n}) := \frac{1}{2}\int_\Sigma K_1(\mathrm{div_s}\,\mathbf{n})^2 + K_2(\mathbf{n}\cdot\mathrm{curl_s}\,\mathbf{n})^2 + K_3|\mathbf{n}\times\mathrm{curl_s}\,\mathbf{n}|^2\,\mathrm{dVol}. \tag{II.0.3}$$

and of its one constant approximation

$$W(\mathbf{n}) = \frac{1}{2}\int_\Sigma |D\mathbf{n}|^2 + |\mathfrak{B}\mathbf{n}|^2\mathrm{dVol}.$$

Recall that the shell $\Sigma$ is supposed to be a two dimensional compact surface without boundary.

As anticipated in the introduction the problem consists in understanding the conditions, if any, on the topology of $\Sigma$ that guarantee that the set (if we want to remain at a Sobolev level)

$$H^1_{\tan}(\Sigma,\,\mathbb{S}^2) := \left\{\boldsymbol{u}\in H^1_{\tan}(\Sigma) : |\boldsymbol{u}| = 1 \text{ a.e.}\right\},$$

is not empty. The following Theorem clarifies the situation

Theorem II.1. *Let $\Sigma$ be a compact smooth surface without boundary, embedded in $\mathbb{R}^3$. Let $\chi(\Sigma)$ be the Euler characteristic of $\Sigma$. Then*

$$H^1_{\tan}(\Sigma,\,\mathbb{S}^2) \neq \emptyset \;\Leftrightarrow \chi(\Sigma) = 0.$$

Note that the topological condition on $\Sigma$ (roughly speaking related to number of "holes" in $\Sigma$) is exactly the same of the condition for smooth vector fields in the classical Hairy Ball Theorem.

In particular, we have that the two-dimensional sphere cannot be combed with $H^1$-regular vector fields. On the other hand, the above Theorem (as well as its smooth classical counterpart) does not hold for odd-dimensional spheres as the following example shows. Take $x = (x_1,\ldots,x_{2N})\in\mathbb{S}^{2N-1}$. The vector field $\boldsymbol{u}$ given by

$$\boldsymbol{u}(x) = (x_2,-x_1,\ldots,x_{2i},-x_{2i-1},\ldots,x_{2N},-x_{2N-1})$$

is smooth, tangent, and with unit norm.

In the last part of the Chapter we will deal with the problem of combing an hypersurface with VMO tangent vector fields and we will prove the following

Theorem II.2. *Let $N$ be a compact, connected $n$-dimensional submanifold of $\mathbb{R}^{n+1}$, without boundary. There exists $v\in\mathrm{VMO}(N,\,\mathbb{R}^{n+1})$ satisfying*

$$v(x)\in T_xN \quad \textit{and} \quad c_1 \le |v(x)| \le c_2, \quad \textit{a.e. in } N, \quad \textit{with } c_1, c_2 > 0 \tag{II.0.4}$$

*if and only if $\chi(N) = 0$.*

Since a vector field $\boldsymbol{v}\in H^1_{\tan}(\Sigma,\,\mathbb{S}^2)$ satisfies (II.0.4), we have that Theorem II.1 follows from II.2. However, it is instructive to present them separately because the proof of Theorem II.1 employs some tools that we will need also in the sequel and uses rather elementary notions of differential geometry.

More precisely, the proof uses the representation of a unitary vector field in terms of its deviation angle, namely the angle that the vector field forms with a direction, typically one of the two elements of the base for the tangent plane of $\Sigma$.
In the first section of this Chapter we describe this representation. Then, in the remaining two sections we will describe the two different proofs of the Theorem.

We conclude by noting that the above Theorems II.1 and II.2 are indeed corollaries of the following result proved in [**19**]

THEOREM II.3 (Morse Index formula in VMO). *Let $N$ be a compact, connected and orientable submanifold of $\mathbb{R}^d$, with boundary. Let $g \in \mathrm{VMO}(\partial N, \mathbb{R}^d)$ be a boundary datum which fulfills*

$$g(x) \in T_x N \qquad \text{and} \qquad c_1 \leq |g(x)| \leq c_2$$

*for some constants $c_1$, $c_2 > 0$ and $\mathcal{H}^{n-1}$-a.e. $x \in \partial N$. If $v \in \mathrm{VMO}(N, \mathbb{R}^d)$ is a map with trace $g$ at the boundary, satisfying*

$$v(x) \in T_x N$$

*for a.e. $x \in N$, then*

$$\mathrm{ind}(v, N) + \mathrm{ind}_-(v, \partial N) = \chi(N).$$

The Theorem above is the VMO version of the classical result of Morse [**41**] for smooth vector fields.

## II.1. Representation of vector fields n via local deviation $\alpha$

Given a smooth unitary vector field $\mathbf{n}$ on a two dimensional surface or on a portion of it, we aim at representing it in terms of the so called deviation angle $\alpha$. Namely, we are interested in the following representation

$$\mathbf{n} = (\cos\alpha)\mathbf{e}_1 + (\sin\alpha)\mathbf{e}_2, \tag{II.1.1}$$

in an open subset $U \subset \Sigma$. In the representation above, $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote a (possibly local) smooth orthonormal frame for the tangent plane of $\Sigma$. Consequently, $\alpha$ is the angle that $\mathbf{n}$ forms with $\mathbf{e}_1$. Once we fix $U$ to be simply connected and we restrict to smooth vector fields (actually, by approximation, one can prove the very same result for $H^1$ vector fileds (see [**8**] and [**49**]), the representation above is a standard result in the differential geometry of surfaces (see [**25**]). On the other hand, when we are interested in a *global representation*[1], problems may occur for non simply connected surfaces and for vector fields with non vanishing *winding number*. The following example clarifies the situation. Consider the axisymmetric torus $\mathbb{T}$ with its standard parametrization $X : [0, 2\pi] \times [0, 2\pi] \to \mathbb{T}$ (see Section V.1) and the unit norm vector field $\mathbf{n}$ such that $\mathbf{n}(\theta, \phi) := \cos(\theta)\mathbf{e}_1(\theta, \phi) + \sin(\theta)\mathbf{e}_2(\theta, \phi)$. Let us denote with $\alpha$ the angle that $\mathbf{n}$ forms with $\mathbf{e}_1$. We immediately obtain that the only possible $\alpha$ is clearly $\alpha(\theta, \phi) = \theta + 2h\pi$, for $h \in \mathbb{Z}$, which cannot be continuously extended to $[0, 2\pi] \times [0, 2\pi]$ since $2h\pi = \lim_{t\to 0^+} \alpha(t, \phi) \neq \lim_{t\to 2\pi^-} \alpha(t, \phi) = 2\pi(1 + h)$. The integer number $h$ is related to the *winding number* which is an homotopy invariant that measures, roughly speaking, how many times the given vector field "wraps around" $\Sigma$. When $\Sigma$ is the torus, following Bethuel and Zheng [**8**], we can prove (see [**49**])

[1]By *global* we mean that the representation (II.1.1) holds globally in $\Sigma$ and that the vector field $\mathbf{n}$ and its deviation angle $\alpha$ have the same regularity.

PROPOSITION II.1.1. *Let $h = (h_\theta, h_\phi) \in \mathbb{Z}^2$ and define*

$$\mathscr{A}_h := \left\{ \alpha \in H^1(Q) : \alpha_{|\{x_j=2\pi\}} = \alpha_{|\{x_j=0\}} + 2\pi h_{x_j},\ \textit{for } x_j = \theta, \phi \right\}, \qquad \mathscr{A} := \bigcup_{h\in\mathbb{Z}^2} \mathscr{A}_h, \tag{II.1.2}$$

*Then, the map $\Phi : \alpha \mapsto \mathbf{e}_1 \cos\alpha + \mathbf{e}_2 \sin\alpha$ defines a bijection*

$$\Phi : \mathscr{A}/2\pi\mathbb{Z} \to H^1_{\tan}(\mathbb{T}, \mathbb{S}^2), \tag{II.1.3}$$

*and we have $\Phi^{-1}[\mathbf{n}] \subset \mathscr{A}_{h(\mathbf{n})}$.*

The couple of integer numbers in the Proposition above is the *winding number* of the vector field. In particular, when the *winding number* is $(0,0)$ then the deviation angle is periodic. In all the other cases we have to take into account the boundary conditions in the definition of $\mathscr{A}$. This will be important when we will discuss the existence for the gradient flow in the next Chapter III.

The structure of $\mathscr{A}_h$ comes from the following observation. Let $\mathbf{n} \in H^1_{\tan}(\mathbb{T}, \mathbb{S}^2)$ be fixed, and let us assume that $\mathbf{n}$ is also continuous. For a general vector field, we cannot expect the corresponding $\alpha$ to be periodic on $Q = [0, 2\pi] \times [0, 2\pi]$ on the other hand we observe that the vector field $\mathbf{n}$ is continuous if and only if there exist $m, n \in \mathbb{Z}$ such that

$$\alpha(2\pi, \phi) = \alpha(0, \phi) + 2m\pi, \qquad \alpha(\theta, 2\pi) = \alpha(\theta, 0) + 2n\pi, \qquad \forall\, (\theta, \phi) \in Q.$$

By continuity of $\mathbf{n}$, $m$ and $n$ do not depend on the choice of $\theta$ and $\phi$. Moreover, since $\alpha$ is unique up to an additive constant, $m$ and $n$ are also independent of the choice of $\alpha$ which represents $\mathbf{n}$. Therefore, the *winding number* $\mathbf{n}$ *on* $\mathbb{T}$ is defined to be the the couple of indices $h(\mathbf{n}) = (h_\theta, h_\phi) \in \mathbb{Z} \times \mathbb{Z}$, given by

$$h_\theta := \frac{\alpha(2\pi, 0) - \alpha(0, 0)}{2\pi}, \qquad h_\phi := \frac{\alpha(0, 2\pi) - \alpha(0, 0)}{2\pi}. \tag{II.1.4}$$

Moreover, if $\mathbf{n}, \mathbf{v} \in H^1_{tan}(\mathbb{T}; \mathbb{S}^2)$ are homotopic, then $h(\mathbf{n}) = h(\mathbf{v})$.

**II.1.1. Formulas for the deviation $\alpha$.** In this subsection, we perform the formal computations which lead to the representation of $\nabla_s \mathbf{n}$, in terms of $\alpha$. Thus, we suppose that, at least in a local open and simply connected neighborhood $U \subset \Sigma$, we have

$$\mathbf{n} = (\cos\alpha)\mathbf{e}_1 + (\sin\alpha)\mathbf{e}_2 \quad \text{in } U. \tag{II.1.5}$$

First of all, we introduce the *spin connection* $\mathbb{A}$, which, for a two-dimensional manifold $\Sigma$ embedded in $\mathbb{R}^3$, can be expressed using the 1-form $\omega$ defined as

$$\omega(\boldsymbol{v}) = (\mathbf{e}_1, D_{\boldsymbol{v}}\mathbf{e}_2)_{\mathbb{R}^3} \quad \forall \boldsymbol{v} \in T_p\Sigma, \tag{II.1.6}$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a local orthonormal frame for $T\Sigma$. A simple computation reveals that $\omega(\boldsymbol{v}) = -(\mathbf{e}_2, D_{\boldsymbol{v}}\mathbf{e}_1)_{\mathbb{R}^3}$ for any $\boldsymbol{v}$ tangent and that $(\mathbf{e}_1, D_i\mathbf{e}_1)_{\mathbb{R}^3} = (\mathbf{e}_2, D_i\mathbf{e}_2)_{\mathbb{R}^3} = 0$ for $i = 1, 2$. We define the spin connection $\mathbb{A}$ to be the tangent vector field $\mathbb{A} := \omega^\sharp$, that is $\mathbb{A}^i = g^{ij}\omega_j$. In what follows we will unambiguously refer to $\mathbb{A}$ and to $\omega$ as the spin connection. Let $\kappa_1, \kappa_2$ be the geodesic curvatures of the flux lines of $\mathbf{e}_1$, $\mathbf{e}_2$, respectively. By the definition of geodesic curvature (see V.1), it is immediate to see that

$$\mathbb{A} = -\kappa_1\mathbf{e}_1 - \kappa_2\mathbf{e}_2. \tag{II.1.7}$$

The curl of $\mathbb{A}$ is related to the Gaussian curvature of $\Sigma$. In fact, following [**33**] we have that

$$(\mathrm{curl}\mathbb{A}, \boldsymbol{\nu})_{\mathbb{R}^3} = \kappa_1\kappa_2, \tag{II.1.8}$$

where $\mathrm{curl}\boldsymbol{v} = -\epsilon D\boldsymbol{v}$ ($\epsilon$ denotes the Ricci alternator defined in Section V.1).

Now we show how the spin connection $\mathbb{A}$ and its related 1-form $\omega$ change when we change the orthonormal frame. In particular, it will be important to be able to choose a local orthonormal frame with divergence-free spin connection (see [**37**, Lemma 3.2.9] for a similar result). Thus, let $\{\mathbf{f}_1, \mathbf{f}_2\}$ be another smooth local orthonormal frame centered $U$. We denote with $\beta$ the angle that $\mathbf{f}_1$ forms with $\mathbf{e}_1$. Thus, have

$$\begin{aligned}\mathbf{f}_1 &= \cos\beta\mathbf{e}_1 + \sin\beta\mathbf{e}_2,\\ \mathbf{f}_2 &= -\sin\beta\mathbf{e}_1 + \cos\beta\mathbf{e}_2.\end{aligned}$$

LEMMA II.1.1. *Let $\omega'$ denote the spin connection in the frame $\{\mathbf{f}_1, \mathbf{f}_2\}$, namely the 1-form $\omega'(\boldsymbol{v}) = (\mathbf{f}_1, D_{\boldsymbol{v}}\mathbf{f}_2)_{\mathbb{R}^3}$ for $\boldsymbol{v}$ tangent. Then there holds*

$$\omega'(\boldsymbol{v}) = \omega(\boldsymbol{v}) - \mathrm{d}\beta(\boldsymbol{v}). \tag{II.1.9}$$

*Moreover, if $\mathbb{A}' = (\omega')^\sharp$, we have*

$$\mathrm{div}_s\mathbb{A}' = \mathrm{div}_s\mathbb{A} - \Delta_s\beta. \tag{II.1.10}$$

We are going to prove the following

LEMMA II.1.2. *Let $U \subset \Sigma$ be open and simply connected and let $\mathbf{n} \in H^1_{tan}(U;\mathbb{S}^2)$. Then, for a.a. $x \in U$,*

$$|D\mathbf{n}|^2 = |\nabla_s\alpha - \mathbb{A}|^2, \tag{II.1.11}$$

$$|\nabla_s\mathbf{n}|^2 = |\nabla_s\alpha - \mathbb{A}|^2 + |\mathfrak{B}\mathbf{e}_1|^2\cos^2\alpha + |\mathfrak{B}\mathbf{e}_2|^2\sin^2\alpha + 2(\mathfrak{B}\mathbf{e}_1, \mathfrak{B}\mathbf{e}_2)_{\mathbb{R}^3}\sin\alpha\cos\alpha. \tag{II.1.12}$$

The expression (II.1.12) further simplifies if we choose, for any point $x \in \Sigma$, $\{\mathbf{e}_1, \mathbf{e}_2\}$ to be the *principal directions* of $\Sigma$ at $x$. In particular, $\{\mathbf{e}_1, \mathbf{e}_2\}$ are orthonormal eigenvectors of $\mathfrak{B}$. The relative eigenvalues $c_1$ and $c_2$ are named *principal curvatures* of $\Sigma$ at $x$ ([**25**]). As a result, we have

$$\begin{aligned}|\nabla_s\mathbf{n}|^2 &= |\nabla_s\alpha - \mathbb{A}|^2 + |\mathfrak{B}\mathbf{e}_1|^2\cos^2\alpha + |\mathfrak{B}\mathbf{e}_2|^2\sin^2\alpha\\ &= |\nabla_s\alpha - \mathbb{A}|^2 + \frac{(c_1^2 - c_2^2)}{2}\cos(2\alpha) + \frac{(c_1^2 + c_2^2)}{2}.\end{aligned} \tag{II.1.13}$$

Note that $\frac{(c_1^2+c_2^2)}{2} = (\mathrm{tr}_g\mathfrak{B})^2 = 2H$, where $H$ is the mean curvature of $\Sigma$.

Thanks to (II.1.11) we have that if $\mathbf{n}$ is a parallel vector field with unit norm in some simply connected region $U$ then $|\nabla_s\alpha - \mathbb{A}|^2 = 0$, namely $\mathbb{A} = \nabla_s\alpha$. The Poincaré Lemma gives that the condition above is satisfied if and only if $\mathrm{curl}\,\mathbb{A} = 0$, that is the Gaussian Cruvature should vanish. We thus have proved the following.

COROLLARY II.4. *Let $U \subset \Sigma$ be open and simply connected and let $\mathbf{n} \in H^1_{tan}(U;\mathbb{S}^2)$. Then,*

$$|D\mathbf{n}|^2 = 0 \textit{ for a.a. } x \in U \quad \textit{implies that} \quad \textit{curl}\mathbb{A} = 0, \tag{II.1.14}$$

*namely the Gaussian curvature vanishes on $U$.*

Relying on the computations above, we rewrite the energy $W$ in (II.0.3) in terms of the deviation angle $\alpha$. We have

LEMMA II.1.3. *Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal frame provided by the principal directions on $\Sigma$. Let $c_1, c_2$ be the corresponding principal curvatures and let $\kappa_1, \kappa_2$ be the corresponding geodesic curvatures. The energy* (III.0.6) *of a director field $\mathbf{n}$, in terms of the deviation angle $\alpha$ characterized* *by* $\mathbf{n} = \cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2$ *and of the spin connection* (II.1.7) *is*

$$\begin{aligned} W(\mathbf{n}) = \frac{1}{2}\int_\Sigma \big\{ & K_1((\nabla_s\alpha - \mathbb{A})\cdot\mathbf{t})^2 + K_2(c_1-c_2)^2\sin^2\alpha\,\cos^2\alpha \\ & + K_3((\nabla_s\alpha - \mathbb{A})\cdot\mathbf{n})^2 + K_3(c_1\cos^2\alpha + c_2\sin^2\alpha)^2 \big\}\,\mathrm{dVol}. \end{aligned} \tag{II.1.15}$$

*The corresponding one-constant approximation ($\kappa = K_1 = K_2 = K_3$) is*

$$W(\mathbf{n}) = \frac{\kappa}{4}\int_\Sigma \{c_1^2 + c_2^2\}\,\mathrm{dVol} + \frac{\kappa}{2}\int_\Sigma \left\{ |\nabla_s\alpha - \mathbb{A}|^2 + \frac{1}{2}(c_1^2 - c_2^2)\cos(2\alpha) \right\}\mathrm{dVol}. \tag{II.1.16}$$

## II.2. Combing a surface in $H^1$

In this Section we finally prove Theorem II.1. The proof is based on a contradiction argument and makes use of the representation of a unit norm vector field via the angle $\alpha$ discussed above. As it will be clear from the proof the argument works for a two dimensional surface in $\mathbb{R}^3$.

Let $\Sigma$ be given, as in the hypothesis of Theorem II.1. We consider $E := H^1_{\tan}(\Sigma, \mathbb{S}^2)$ as a subset of the Hilbert space $X := H^1_{tan}(\Sigma)$. Assume that $E \neq \emptyset$, we need to prove that $\chi(\Sigma) = 0$. We study the minimization problem related to the energy

$$\mathfrak{E} : X \to \mathbb{R}, \qquad \mathfrak{E}(\boldsymbol{u}) := \frac{1}{2}\int_\Sigma |D\boldsymbol{u}|^2 \mathrm{dVol}. \tag{II.2.1}$$

Since the function $f : \Sigma \times \mathbb{R}^3 \to \mathbb{R}$, $f(x,\xi) = g_x(\xi,\xi)\sqrt{g_x}$ is continuous and convex in $\xi$ for all $x \in \Sigma$, the energy $\mathfrak{E}$ is weakly lower semicontinuous on $X$. As the constraint "$|\boldsymbol{u}| = 1$ a.e. on $\Sigma$" is continuous with respect to the $L^2$ convergence, we deduce that sublevel sets of $\mathfrak{E}$ in $E$ are sequentially weakly compact in $X$. Hence, using the direct method of the calculus of variations we can find a field $\boldsymbol{u}^* \in E$ which minimizes $\mathfrak{E}$ on $E$. We get a contradiction as soon as we prove that $\boldsymbol{u}^*$ is actually more regular (say continuous) hence violating the classical Poincaré-Hopf Theorem (see [**40**]). Now, thanks to [**8**] for any given point $x \in \Sigma$ we can find an open neighbourhood $U \subset \Sigma$ and a real function $\alpha : U \to \mathbb{R}$ such that any vector field $\boldsymbol{u} \in E$ can be locally represented as $\boldsymbol{u} = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2$ a.e. in $U$. Here $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a smooth local orthonormal frame for $T_x\Sigma$ for all $x \in U$, and $\alpha \in H^1(U)$ is the angle that $\boldsymbol{u}$ forms with $\mathbf{e}_1$. Owing to Lemma II.1.1, it is not restrictive to assume that the spin connection $\mathbb{A}$ corresponding to $\{\mathbf{e}_1, \mathbf{e}_2\}$ is divergence-free: indeed if $\mathrm{div_s}\,\mathbb{A} \neq 0$, we can define a new orthonormal frame by rotating $\{\mathbf{e}_1, \mathbf{e}_2\}$ of an angle $\beta$ such that $\Delta_s\beta = \mathrm{div_s}\,\mathbb{A}$ in $U$. The spin connection $\mathbb{A}'$ in the new frame, owing to (II.1.10), satisfies then $\mathrm{div_s}\,\mathbb{A}' = \mathrm{div_s}\,\mathbb{A} - \Delta_s\beta = 0$.

Now, since $\boldsymbol{u}^*$ minimizes (II.2.1) on $E$, by Lemma II.1.2 any function $\alpha^* \in H^1(U)$, such that $\boldsymbol{u}^* := \cos\alpha^*\,\mathbf{e}_1 + \sin\alpha^*\,\mathbf{e}_2$ on $U$, minimizes

$$\mathfrak{F} : H^1(U) \to \mathbb{R}, \qquad \mathfrak{F}(\alpha) := \frac{1}{2}\int_U |\nabla_s\alpha - \mathbb{A}|^2 \mathrm{dVol}, \tag{II.2.2}$$

on the set $\{\alpha \in H^1(U) : \alpha_{|\partial U} = \alpha^*_{|\partial U}\}$. As a result, $\alpha^*$ is a stationary point of (II.2.2), with respect to variations in $H^1_0(U)$, and hence it solves

$$\Delta_s\alpha^* = 0 \quad \text{ in } U.$$

As the Laplace Beltrami operator on a smooth compact manifold is an elliptic operator with smooth coefficients, we have that $\alpha^*$, hence $\boldsymbol{u}^*$, is smooth in $U$. Being the choice of the point $x$ completely arbitrary, we have proved that $\boldsymbol{u}^*$ is a unit norm vector field which is smooth everywhere in $\Sigma$. Thanks to the classical Poincaré-Hopf Theorem, $\Sigma$ must be a genus-1 surface, i.e. $\chi(\Sigma) = 0$. The opposite implication is straightforward. More precisely, assuming that $\chi(\Sigma) = 0$, classical results give the existence of a smooth vector field on $\Sigma$ with unit norm, which, in particular, belongs to $H^1_{\tan}(\Sigma, \mathbb{S}^2)$.

## II.3. Combing an hypersurface in VMO

In this Section we extend Theorem II.1 to VMO unit lenght vector fields on a $n$ dimensional compact, connected submanifold of $\mathbb{R}^{n+1}$ without boundary. These results are taken from [**19**] to which we refer for all the details.

Theorem II.2 is formulated for vector fields in VMO, namely the space of Vanishing Mean Oscillation (VMO) functions, introduced by Sarason in [**48**]. These constitute a special subclass of Bounded Mean Oscillations functions, defined by John and Nirenberg in [**31**]. We recall the definitions and some properties of these objects later on, but we immediately note that VMO contains the critical spaces with respect to Sobolev embeddings, that is,

$$W^{s,p}(\mathbb{R}^n) \subset \mathrm{VMO}(\mathbb{R}^n) \qquad \text{when } sp = n,\ 1 < s < n. \tag{II.3.1}$$

In a sense, VMO functions are a good surrogate for the continuous functions, because some classical topological constructions can be extended, in a natural way, to the VMO setting. In particular, we recall here the VMO degree theory, which has been developed after Brezis and Nirenberg's seminal papers [**16**] and [**17**]. We anticipate that this will be the main ingredient in the proof. Besides relaxing the regularity on the vector field, we will consider $n$-dimensional compact and connected submanifolds of $\mathbb{R}^{n+1}$ and, instead of fixing the length of the vector field to be 1, we will look for vector fields which are bounded and uniformly positive.

For the reader's convenience, we recall here the basic definitions about VMO functions, following the presentation of [**17**] (to which the reader is referred, for more details). All the functions we consider here take values in $\mathbb{R}^d$, so functional spaces such as, e.g., $L^1(N, \mathbb{R}^d)$ or $\mathrm{VMO}(N, \mathbb{R}^d)$ will be simply written as $L^1(N)$ or $\mathrm{VMO}(N)$.

Recall that $N$ is endowed with a Riemannian measure $\sigma$. For any $\varepsilon > 0$, we denote with $N_\varepsilon$ the set

$$N_\varepsilon := \{x \in N : \mathrm{dist}(x, \partial N) > \varepsilon\}.$$

For $u \in L^1(N)$ (with respect to $\sigma$), define

$$\|u\|_{\mathrm{BMO}} := \sup_{\varepsilon \le r_0,\, x \in N_{2\varepsilon}} \fint_{B_\varepsilon(x)} |u(y) - \bar{u}_\varepsilon(x)| \,\mathrm{d}\sigma(y), \tag{II.3.2}$$

where

$$\bar{u}_\varepsilon(x) := \fint_{B_\varepsilon(x)} u(y)\,\mathrm{d}\sigma(y), \qquad \text{for } x \in N_{2\varepsilon}. \tag{II.3.3}$$

The set of functions with $\|u\|_{\mathrm{BMO}} < +\infty$ will be denoted $\mathrm{BMO}(N)$, and (II.3.2) defines a norm on $\mathrm{BMO}(N)$ modulo constants. Using cubes instead of balls leads to an equivalent norm. Moreover, if $\varphi\colon X_1 \to X_2$ is a $\mathscr{C}^1$ diffeomorphism between two unbounded manifolds, then $u \in \mathrm{BMO}(X_2)$ implies $u \circ \varphi \in \mathrm{BMO}(X_1)$ and

$$\|u \circ \varphi\|_{\mathrm{BMO}(X_1)} \le C\, \|u\|_{\mathrm{BMO}(X_2)}.$$

Bounded functions (in particular, continuous functions) belong to BMO. Following Sarason, we define $\mathrm{VMO}(N)$ as the closure of $\mathscr{C}(N)$ with respect to the BMO norm. Functions in $\mathrm{VMO}(N)$ are then defined in this way (see [**16**, Lemma 3]):

Definition II.3.1. A function $u \in \mathrm{BMO}(N)$ is in $\mathrm{VMO}(N)$ if and only if

$$\lim_{\varepsilon \to 0} \sup_{x \in N_{2\varepsilon}} \fint_{B_\varepsilon(x)} |u(y) - \bar{u}_\varepsilon(x)| \,\mathrm{d}\sigma(y) \to 0.$$

Sobolev spaces provide an interesting class of functions in VMO, since, for critical exponents, the embeddings which fail to be in $L^\infty$ hold true in VMO:

$$W^{s,p}(N) \subset \mathrm{VMO}(N) \qquad \text{whenever } 0 < s < n,\ sp = n.$$

We present the proof of Theorem II.2. The argument is inspired by [**28**, Theorem 2.28]. We assume that $N$ is a compact, connected $n$-manifold *without boundary*, embedded as an hypersurface of $\mathbb{R}^{n+1}$.

Proof of Theorem II.2. It is well-known that, if $\chi(N) = 0$, then a nowhere vanishing, smooth (hence VMO) vector field on $N$ exists. The details of this argument are in [**19**].

Let us prove the other side of the Theorem: we suppose that a tangent vector field $v \in \mathrm{VMO}(N)$ such that $\operatorname{ess\,inf}_N |v| > 0$ exists, and we claim that $\chi(N) = 0$. Every compact hypersurface of $\mathbb{R}^{n+1}$ is orientable, so there is a smooth unit vector field $\gamma\colon N \to \mathbb{R}^{n+1}$ such that $\gamma(x) \perp T_x N$ for all $x \in N$. The choice of such a map induces an orientation on $N$, and $\gamma$ is called the Gauss map of the oriented manifold $N$. We can also assume that $n$ is even, since $\chi(N) = 0$ whenever $N$ is a compact, unbounded manifold of odd dimension (see, e.g., [**28**, Corollary 3.37]).

Consider the function $H\colon N \times [0, \pi] \to \mathbb{R}^{n+1}$ given by

$$H(x, t) := (\cos t)\gamma(x) + (\sin t) w(x),$$

where $w := \frac{v}{|v|}$. The function $H$ is clearly well defined since $\operatorname{ess\,inf}_N |v| > 0$ by assumption. Moreover, it is readily checked that $|H(x,t)|^2 = 1$ for all $(x,t) \in N \times [0, \pi]$. We claim that

$$H \in \mathscr{C}\left([0, \pi],\, \mathrm{VMO}(N, \mathbb{S}^n)\right). \tag{II.3.4}$$

Indeed, $H(\cdot, t)$ is the linear combination of functions in $\mathrm{VMO}(N)$ and hence belongs to $\mathrm{VMO}(N)$, for all $t$. On the other hand, for all $t_1,\, t_2 \in [0, \pi]$

$$\|H(\cdot,\, t_1) - H(\cdot,\, t_2)\|_{\mathrm{BMO}} \leq |\cos t_1 - \cos t_2|\, \|\gamma\|_{\mathrm{BMO}} + |\sin t_1 - \sin t_2|\, \|w\|_{\mathrm{BMO}},$$

whence the claimed continuity (II.3.4) follows.

We recall that, if $N$ is a compact, connected and orientable $n$-submanifold with boundary, $M$ is a connected, orientable manifold without boundary, of the same dimension as $N$, and $\varphi\colon N \to M$ is a smooth map, then the degree of $\varphi$ is defined as

$$\deg(\varphi,\, N,\, M) = \frac{1}{\tau(M)} \int_N \det D\varphi(x) \,\mathrm{d}\sigma(x). \tag{II.3.5}$$

Since the degree is a continuous function $\mathrm{VMO}(N, \mathbb{S}^n) \to \mathbb{Z}$ (see [**16**, Theorem 1]), we infer that

$$\deg(H(\cdot,\, 0),\, N,\, \mathbb{S}^n) = \deg(H(\cdot,\, \pi),\, N,\, \mathbb{S}^n).$$

On the other hand, $H(\cdot,\, 0) = \gamma$ and $H(\cdot,\, \pi) = -\gamma$. By standard properties of the degree (in particular, [**28**, Properties (d, f) p. 134]), and since we have assumed that $n$ is even, we have

$$\deg(-\gamma,\, N,\, \mathbb{S}^n) = (-1)^{n+1} \deg(\gamma,\, N,\, \mathbb{S}^n) = -\deg(\gamma,\, N,\, \mathbb{S}^n),$$

hence

$$\deg(\gamma,\, N,\, \mathbb{S}^n) = -\deg(\gamma,\, N,\, \mathbb{S}^n).$$

By the degree formula (II.3.5) and Gauss-Bonnet Theorem (see, e.g., [**26**, page 196]), for an even-dimensional hypersurface $N$

$$\deg(\gamma,\, N,\, \mathbb{S}^n) = \deg(\gamma,\, N,\, \mathbb{S}^n) \fint_{\mathbb{S}^n} \mathrm{d}\sigma_n = \frac{1}{\omega_n} \int_N \gamma^*(\mathrm{d}\sigma_n) = \frac{1}{\omega_n} \int_N \kappa \, \mathrm{d}\sigma = \frac{1}{2}\chi(N),$$

where $\mathrm{d}\sigma_n$ is the volume form of $\mathbb{S}^n$, $\omega_n := \int_{\mathbb{S}^n} \mathrm{d}\sigma_n$ is the volume of $\mathbb{S}^n$, and $\kappa$ is the Gaussian curvature of $N$. Since $\deg(\gamma,\, N,\, \mathbb{S}^n) = 0$ by the above construction, this shows that $\chi(N) = 0$ and thus completes the proof. $\square$

Remark II.5. When $\chi(N) \neq 0$, Theorem II.2 shows that there is no unit vector field in the critical Sobolev space $W^{s,p}(N)$, for $0 < s < n$ and $sp = n$. In contrast, when $sp < n$ it is not difficult to construct unit vector fields in $W^{s,p}(N)$. For instance, on $N = \mathbb{S}^{2k}$ one may consider a field with two "hedgehog" singularities, of the form $x \mapsto x/\,|x|$, located at the opposite poles of the sphere.

CHAPTER III

# Justification and Analysis of the surface energy

In this Chapter we take advantage of the results of the Chapter II and we analyze the energy

$$W(\mathbf{n}) := \frac{1}{2}\int_\Sigma K_1(\mathrm{div_s}\,\mathbf{n})^2 + K_2(\mathbf{n}\cdot\mathrm{curl_s}\,\mathbf{n})^2 + K_3|\mathbf{n}\times\mathrm{curl_s}\,\mathbf{n}|^2\,\mathrm{dVol}. \tag{III.0.6}$$

More precisely, in Section III.1 we discuss the existence of minimizers in $H^1_{\tan}(\Sigma, \mathbb{S}^2)$.

Then, in Section III.2 we justify the dimensional reduction limit for the one constant approximation energy

$$W(\mathbf{n}) = \lim_{h\searrow 0} W^{OZF}(\mathbf{n}, \Sigma_h), \quad \Sigma_h \text{ being a tubular neighbor of } \Sigma,$$

in terms of $\Gamma$-convergence, where

$$W(\mathbf{n}) = \frac{1}{2}\int_\Sigma |D\mathbf{n}|^2 + |\mathfrak{B}\mathbf{n}|^2\,\mathrm{dVol}, \tag{III.0.7}$$

In Section III.3 we discuss the microscopic derivation, again using $\Gamma$-convergence of the surface energy (III.0.7), namely we prove the limit (I.2.8).

Finally, in Section III.4 we present a detailed analysis of the gradient flow of the energy (III.0.7) on shells with $\chi(\Sigma) = 0$. In particular, we will first prove the existence of an energy solution (roughly speaking a solution that emanates from $H^1_{\tan}(\Sigma, \mathbb{S}^2)$ initial conditions). Note that this existence result can be easily extended to an $n$ dimensional hypersurface with $\chi(\Sigma) = 0$. Secondly, we will specialize to a two dimensional toroidal shell and we will prove uniqueness and regularity results for the evolution.

## III.1. Minimization of (III.0.6)

There are two major problems to address before discussing the existence of minimizers of (III.0.6):

- Choice of the topology of the surface $\Sigma$. This is related to the choice of the functional space thanks to the $H^1$-version of the hairy ball Theorem (see I.1) (see the next Theorem I.1).
- Choice of the boundary conditions. Given a boundary datum $\mathbf{n}_b$ in some functional class, we have to show that the set of competitors $\mathcal{A}(\mathbf{n}_b)$ is not empty, where

  $$\mathcal{A}(\mathbf{n}_b) := \left\{\boldsymbol{u} \in H^1_{\tan}(\Sigma, \mathbb{S}^2) : \boldsymbol{u} = \mathbf{n}_b \text{ on } \partial\Sigma\right\}.$$

  This fact is related to some precise topological properties of $\mathbf{n}_b$, which are discussed and analyzed in [**19**].

Here, we restrict to the case of a smooth surface without boundary.

According to Theorem I.1, in this Chapter, unless otherwise stated, we will consider a $\Sigma$ to be a compact and smooth two-dimensional surface without boundary such that

$$\boxed{\texttt{$\Sigma$ has Euler characteristic equal to zero, that is } \chi(\Sigma) = 0.} \tag{III.1.1}$$

Choosing $\Sigma$ satisfying (III.1.1), namely in such a way that $H^1_{\tan}(\Sigma,\mathbb{S}^2)\neq\emptyset$, we have the following (see [**27**] for the flat case)

PROPOSITION III.1.1. *Let $\Sigma$ be a smooth, compact surface in $\mathbb{R}^3$, without boundary, satisfying* (III.1.1) *and let $W : H^1_{\tan}(\Sigma,\mathbb{S}^2)\to\mathbb{R}$ be the energy functional defined in* (III.0.6). *Set $K_* := \min\{K_1,K_2,K_3\}$ and $K^* := 3(K_1+K_2+K_3)$. We have that*

$$\frac{K_*}{2}\int_\Sigma\left(|D\boldsymbol{u}(x)|^2+|\mathfrak{B}\boldsymbol{u}(x)|^2\right)\mathrm{dVol}\le W(\boldsymbol{u})\le\frac{K^*}{2}\int_\Sigma\left(|D\boldsymbol{u}(x)|^2+|\mathfrak{B}\boldsymbol{u}(x)|^2\right)\mathrm{dVol}.$$

*Moreover, the energy $W$ is lower semicontinuous with respect to the weak convergence of $H^1(\Sigma;\mathbb{R}^3)$.*

PROOF. The upper and the lower bound follow by the *one-constant approximation* (see (V.1.3)) and the equality (V.1.5). The lower semicontinuity can be proved by noting that all the terms in (III.0.6) are indeed weakly lower semicontinuous in $H^1(\Sigma;\mathbb{R}^3)$ and are multiplied by the positive constants $K_1, K_2$ and $K_3$. □

Thus, the existence of a minimizer of the energy $W$ follows from the direct method of calculus of variations

PROPOSITION III.1.2. *There exists $\mathbf{n}\in H^1_{\tan}(\Sigma,\mathbb{S}^2)$ such that $W(\mathbf{n})=\inf_{\boldsymbol{u}\in H^1_{\tan}(\Sigma,\mathbb{S}^2)}W(\boldsymbol{u})$.*

PROOF. Let $\boldsymbol{u}_n$ be a minimizing sequence uniformly bounded in $H^1_{\tan}(\Sigma,\mathbb{S}^2)$. This means that $|\boldsymbol{u}_n|=1$ and that $\{\boldsymbol{u}_n\}$ is uniformly bounded in $H^1_{\tan}(\Sigma)$. Thus, up to a not relabeled subsequence of $n$, we have that there exists a vector field $\mathbf{n}\in H^1_{\tan}(\Sigma)$ with $|\mathbf{n}|=1$ such that

$$\boldsymbol{u}_n\xrightarrow{n\nearrow+\infty}\mathbf{n}\text{ weakly in }H^1_{\tan}(\Sigma,\mathbb{S}^2)\text{ and strongly in }L^2(\Sigma).$$

Thus, the lower semicontinuity of $W$ gives that $\inf_{\boldsymbol{u}\in\mathcal{A}}W(\boldsymbol{u})=\liminf_{n\nearrow+\infty}W(\boldsymbol{u}_n)\ge W(\mathbf{n})$ which means that $\mathbf{n}$ is a minimizer for $W$.

□

Now, in the case of the one-constant approximation, we compute, the Euler Lagrange equation associated to the minimization of (III.0.7) (see also). Incidentally, note that up to technical modifications, the same computations are valid for an $n$-dimensional hypersurface. Thus, let $\mathbf{n}\in H^1_{\tan}(\Sigma,\mathbb{S}^2)$ be a minimizer for (III.0.7). Take a smooth $\boldsymbol{v}\in H^1_{\tan}(\Sigma,\mathbb{S}^2)$ and consider the family of deformations $\varphi(t):=\frac{\mathbf{n}+t\boldsymbol{v}}{|\mathbf{n}+t\boldsymbol{v}|}$, for $t\in(0,1)$. Note that $|\varphi|=1$ by construction and that $\varphi\in H^1_{\tan}(\Sigma,\mathbb{S}^2)$. Moreover, $\varphi(0)=\mathbf{n}$ and $\dot\varphi(0)=\boldsymbol{v}-(\boldsymbol{v},\mathbf{n})\mathbf{n}$ and thus $W(\varphi(t))$ has a minimum at $t=0$. Hence, we have

$$\begin{aligned}0=\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}W(\varphi(t))&=\kappa\int_\Sigma(D\varphi(0),D\dot\varphi(0))_{\mathbb{R}^3}\mathrm{dVol}+\int_\Sigma(\mathfrak{B}\varphi(0),\mathfrak{B}\dot\varphi(0))_{\mathbb{R}^3}\mathrm{dVol}\\&=\kappa\int_\Sigma(D\mathbf{n},D\boldsymbol{v})_{\mathbb{R}^3}\mathrm{dVol}+\int_\Sigma(\mathfrak{B}\mathbf{n},\mathfrak{B}\boldsymbol{v})_{\mathbb{R}^3}\mathrm{dVol}\\&\quad-\int_\Sigma|D\mathbf{n}|^2(\mathbf{n},\boldsymbol{v})_{\mathbb{R}^3}\mathrm{dVol}-\int_\Sigma|\mathfrak{B}\mathbf{n}|^2(\mathbf{n},\boldsymbol{v})_{\mathbb{R}^3}\mathrm{dVol},\end{aligned}$$

where we have used that, being $|\mathbf{n}|=1$, there holds that $(D\mathbf{n},\mathbf{n})_{\mathbb{R}^3}=0$, and the fact that $\mathfrak{B}[\mathbf{n}(\mathbf{n},\boldsymbol{v})_{\mathbb{R}^3}]=-\nabla_{\mathbf{n}(\mathbf{n},\boldsymbol{v})_{\mathbb{R}^3}}\boldsymbol{\nu}=-(\mathbf{n},\boldsymbol{v})_{\mathbb{R}^3}\nabla_{\mathbf{n}}\boldsymbol{\nu}=(\mathbf{n},\boldsymbol{v})_{\mathbb{R}^3}\mathfrak{B}\mathbf{n}$. Now, since the shape operator $\mathfrak{B}$ is self-adjoint, we may introduce the operator $\mathfrak{B}^2$ given by

$$(\mathfrak{B}^2\boldsymbol{u},\boldsymbol{v})_{\mathbb{R}^3}:=(\mathfrak{B}\boldsymbol{u},\mathfrak{B}\boldsymbol{v})_{\mathbb{R}^3}\quad\text{for any }\boldsymbol{u},\boldsymbol{v}\in\mathfrak{T}(\Sigma).$$

Thus, collecting all the computations, we obtain that a minimizer $\mathbf{n}$ of $W$ is a solution of the following system of nonlinear partial differential equations

$$-\Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n} = |D\mathbf{n}|^2 \mathbf{n} + |\mathfrak{B}\mathbf{n}|^2 \mathbf{n} \quad \text{in } \Sigma. \tag{III.1.2}$$

Remark III.1. As it happens for harmonic maps, a vector field $\mathbf{n}$ solving (III.1.2) is parallel to $-\Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n}$. Viceversa, if $-\Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n}$ is parallel to $\mathbf{n}$, then there exists a function $\lambda$ on $\Sigma$ (the Lagrange multiplier) such that

$$-\Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n} = \lambda \mathbf{n},$$

from which it follows that (recall that $|\mathbf{n}| = 1$)

$$\lambda = \lambda(\mathbf{n}, \mathbf{n})_{\mathbb{R}^3} = (-\Delta_g \mathbf{n}, \mathbf{n})_{\mathbb{R}^3} + (\mathfrak{B}^2 \mathbf{n}, \mathbf{n})_{\mathbb{R}^3} = |D\mathbf{n}|^2 + |\mathfrak{B}\mathbf{n}|^2,$$

where we have used the general identity

$$0 \overset{|\mathbf{n}|=1}{=} \Delta_g |\mathbf{n}|^2 = 2\left\{|D\mathbf{n}|^2 + (\Delta_g \mathbf{n}, \mathbf{n})_{\mathbb{R}^3}\right\}, \tag{III.1.3}$$

holding for any smooth vector field $\mathbf{n}$ on $\Sigma$. Therefore, a smooth unitary vector field $\mathbf{n} \in \mathfrak{T}(\Sigma)$ is a solution of (III.1.2) if and only if it solves

$$(-\Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n}) \times \mathbf{n} = 0. \tag{III.1.4}$$

## III.2. Justification of the surface energy via dimensional reduction

In this subsection, we give a justification of the energy (III.0.7) via a two dimensional reduction starting from the corresponding Oseen Zocher Frank energy in the one constant approximation, namely the energy

$$W^{OFZ}(\mathbf{n}, V) := \frac{1}{2} \int_V |\nabla \mathbf{n}|^2 \mathrm{d}V$$

defined on a domain $V \subset \mathbb{R}^3$.

The dimensional reduction will be performed using $\Gamma$-convergence (see Section V.2 for the definition) in the spirit of [**34**]. Thus, we denote as usual with $\Sigma$ the two dimensional shell, namely a two dimensional compact surface without boundary in $\mathbb{R}$ and then we denote, for any $h > 0$ (sufficiently small), with $\Sigma_h$ the tubular neighbor of thickness $h$, namely the set

$$\Sigma_h := \left\{p \in \mathbb{R}^3 : \exists p' \in \Sigma, p = p' + \eta \boldsymbol{\nu}(p'), \text{ with } |\eta| \leq h/2\right\}. \tag{III.2.1}$$

Note that in order that $\Sigma_h$ is well defined, $h$ should be taken sufficiently small compared with the curvature of $\Sigma$. We denote with $S_{\pm h}$ the "upper" and the "lower" boundary of $\Sigma_h$, i.e.

$$S_{\pm h} := \left\{p \in \mathbb{R}^3 : \exists p' \in \Sigma, p = p' + \eta \boldsymbol{\nu}(p'), \text{ with } \eta = \pm h/2\right\}.$$

Differently from [**34**], the energy in the limit has to be defined on tangent vector fields. In order to include the tangential anchoring in the limit we have to modify the three dimensional energy by adding a penalization term penalizing the deviations (in the limit) from tangent directions. Thus, we actually consider the following functional

$$\tilde{W}^{OZF}(\mathbf{n}, \Sigma_h) = W^{OZF}(\mathbf{n}, \Sigma_h) + \int_{S_{+h}} |(\mathbf{n}, \boldsymbol{\nu})|^2 \mathrm{d}S_h + \int_{S_{-h}} |(\mathbf{n}, \boldsymbol{\nu})|^2 \mathrm{d}S_{-h}. \tag{III.2.2}$$

The last two terms in the energy (III.2.2) play the role of a boundary condition on the boundary of the domain $\Sigma_h$. This type of boundary condition is usually called *weak anchoring* (see [**55**]).

As is will be clear later on, the volume of $\Sigma_h$ scales like $h$. Thus, in order to obtain a non trivial limit we have to scale the energy with $h$, that is we will study the $\Gamma$ limit of

$$\tilde{W}_h^{OZF}(\mathbf{n},\Sigma_h) := \frac{1}{h}\tilde{W}^{OZF}(\mathbf{n},\Sigma_h). \tag{III.2.3}$$

We aim at proving the following

PROPOSITION III.2.1. *Let $\Sigma$ be a compact, two dimensional surface without boundary and such that $\chi(\Sigma) = 0$. For any $h > 0$ (sufficiently small), let $\Sigma_h$ denote the tubular neighbor of thickness $h$ defined in* (III.2.1). *Then, we have*

$$\Gamma - \lim_{h\searrow 0} \tilde{W}_h^{OFZ}(\mathbf{n},\Sigma_h) = \frac{1}{2}\int_\Sigma |D\mathbf{n}|^2 + |\mathfrak{B}\mathbf{n}|^2 \,\mathrm{dVol}, \quad \text{in } L^2(\Sigma_1;\mathbb{R}^3), \tag{III.2.4}$$

*where $\Sigma_1$ is the tubular neighbor of $\Sigma$ of thickness one.*

The proof follows the lines of LeDret and Raouolt [**34**] and it is divided in several steps. We present it just for the sake of completeness. We denote with $P$ the orthogonal projection $P : \mathbb{R}^3 \to T_p(\Sigma)$, namely the projection onto the tangent plane of $\Sigma$ at the point $p$. We have

$$P\boldsymbol{v} = \boldsymbol{v} - (\boldsymbol{\nu},\boldsymbol{v})_{\mathbb{R}^3}\boldsymbol{\nu}, \quad \forall \boldsymbol{v} \in \mathbb{R}^3. \tag{III.2.5}$$

Now, given $\mathbf{n} : \Sigma_h \to \mathbb{R}^3$ (sufficiently smooth), we have

$$\nabla\mathbf{n} = (\nabla\mathbf{n}) \circ P + (\nabla\mathbf{n}) \circ (Id - P), \tag{III.2.6}$$

namely we have decomposed (in a orthogonal way) the derivative of $\mathbf{n}$ is a derivative along direction that are tangent to $\Sigma$ and along the normal direction.

From now on, we set (see Section V.1)

$$\nabla_s\mathbf{n} := \nabla\mathbf{n} \circ P, \quad \text{that is} \quad \nabla_s\mathbf{n}[\boldsymbol{v}] = \nabla_{P\boldsymbol{v}}\mathbf{n}. \tag{III.2.7}$$

In other words, $\nabla_s$ is the restriction to directions tangent to $\Sigma$ of the derivative $\nabla$.

Now, we rescale the energy. To this end, we firs concentrate on the bulk part $W$ of the energy (III.2.2). Recalling that the decomposition (III.2.6) is indeed orthogonal, we have

$$W(\mathbf{n},\Sigma_h) := \frac{1}{2}\int_{\Sigma_h} |\nabla\mathbf{n}|^2 \mathrm{dV}_h = \frac{1}{2}\Big(\int_{\Sigma_h} |\nabla_s\mathbf{n}|^2 \mathrm{dV}_h + \int_{\Sigma_h} |(\nabla\mathbf{n}) \circ (Id - P)|^2 \mathrm{dV}_h\Big). \tag{III.2.8}$$

Now, in order to work on domain that does not depend of $h$ we introduce the following deformation that maps $\Sigma_h$ to $\Sigma_1$,

$$d_h := \Psi \circ \phi_h \circ \Psi^{-1},$$

where

$$\Psi : \Omega \times (-h/2, h/2) \to \Sigma_h, \;\; \Psi(\bar{x},\tau) := \Phi(\bar{x}) + \tau\nu(\bar{x}), \tag{III.2.9}$$

$$\phi_h : \Omega \times (-h/2, h/2) \to \Omega \times (-1/2, 1/2), \;\; \phi_h(\bar{x},\tau) = \big(\bar{x}, \frac{\tau}{h}\big). \tag{III.2.10}$$

Then, a straightforward computation gives that the determinant of the Jacobian of $d_h$ is $1/h$. Hence, we have that $\mathrm{dV}_h = h\mathrm{dV}$. Thus we rewrite the energy in $\Sigma_1$ as

$$W(\mathbf{n},\Sigma_1)=\frac{h}{2}\Big(\int_{\Sigma_1}|\nabla_s\mathbf{n}|^2\mathrm{dV}+\frac{1}{h^2}\int_{\Sigma_1}|\nabla_{\boldsymbol{\nu}}\mathbf{n}|^2\mathrm{dV}\Big). \tag{III.2.11}$$

Thus, the complete energy becomes

$$\tilde{W}^{OZF}(\mathbf{n},\Sigma_1)=W(\mathbf{n},\Sigma_1)+\int_{S_{+1/2}}|(\mathbf{n},\nu)|^2\mathrm{d}S_{1/2}+\int_{S_{-1/2}}|(\mathbf{n},\nu)|^2\mathrm{d}S_{-1/2}. \tag{III.2.12}$$

Finally, we rescale the energy with $h$. Namely we consider the the energy $\tilde{W}_h^{OZF}$ defined in (III.2.3) which becomes

$$\begin{aligned}\tilde{W}_h^{OZF}(\mathbf{n},\Sigma_1)&=\frac{1}{2}\Big(\int_{\Sigma_1}|\nabla_s\mathbf{n}|^2\mathrm{dV}+\frac{1}{h^2}\int_{\Sigma_1}|\nabla_{\boldsymbol{\nu}}\mathbf{n}|^2\mathrm{dV}\Big)\\ &\frac{1}{h}\int_{S_{+1/2}}|(\mathbf{n},\nu)|^2\mathrm{d}S_{1/2}+\frac{1}{h}\int_{S_{-1/2}}|(\mathbf{n},\nu)|^2\mathrm{d}S_{-1/2}.\end{aligned} \tag{III.2.13}$$

Note that we denote with same name symbol the extension of $\tilde{W}_h^{OZF}(\Sigma_1,\cdot)$ to $L^2(\Sigma_1;\mathbb{R}^3)$.

Now, consider a sequence $\mathbf{n}_h$ such that

$$\sup_{h>0}\tilde{W}_h^{OZF}(\mathbf{n}_h,\Sigma_1)<+\infty. \tag{III.2.14}$$

Then, we immediately have from (III.2.13) that the sequence $\mathbf{n}_h$ is bounded in $H^1(\Sigma_1;\mathbb{R}^3)$. Moreover, we have that

$$\int_{\Sigma_1}\|\nabla_{\boldsymbol{\nu}}\mathbf{n}_h\|^2\mathrm{dV}\leq h^2C\qquad\text{for any } h>0. \tag{III.2.15}$$

and

$$\int_{S_{+1/2}}|(\mathbf{n}_h,\boldsymbol{\nu})|^2\mathrm{d}S_{1/2}+\int_{S_{-1/2}}|(\mathbf{n}_h,\boldsymbol{\nu})\mathrm{d}S_{-1/2}\leq Ch\quad\text{for any } h>0. \tag{III.2.16}$$

Now, denote with $\mathbf{n}$ the weak limit in $H^1(\Sigma_1;\mathbb{R}^3)$. Since $|\mathbf{n}_h|=1$, we have that $|\mathbf{n}|=1$. Moreover (III.2.15) gives that $\nabla_{\boldsymbol{\nu}}\mathbf{n}=0$ a.e in $\Omega\times(-1/2,1/2)$. Finally, we have $\hat{\mathbf{n}}\perp\boldsymbol{\nu}$ where we have used (III.2.16). Summing up the discussion above we have found that the weak limit $\mathbf{n}$ belongs to the following set

$$\mathbf{n}\in\mathcal{V}:=\left\{\boldsymbol{v}\in H^1(\Sigma_1;\mathbb{R}^3):|\boldsymbol{v}|=1,\ \nabla_{\boldsymbol{\nu}}\boldsymbol{v}=0,\ (\boldsymbol{\nu},\boldsymbol{v})_{\mathbb{R}^3}=0\right\}, \tag{III.2.17}$$

Note that the fact that $\mathbf{n}\in\mathcal{V}$ is equivalent to the requirement that $\mathbf{n}\in H^1_{\tan}(\Sigma,\mathbb{S}^2)$.

Now, in order to prove the $\Gamma$-convergence result, we have to prove the following two statements

(1) Liminf inequality:
Given $\mathbf{n}_h\to\mathbf{n}$ strongly in $L^2(\Sigma_1;\mathbb{R}^3)$, we have to prove that

$$\liminf_{h\searrow 0}\tilde{W}_h^{OZF}(\mathbf{n}_h,\Sigma_1)\geq W(\mathbf{n}), \tag{III.2.18}$$

(2) Existence of the recovery sequence:
For any $\mathbf{n}\in\mathcal{V}$, there exists a (recovery) sequence $\mathbf{n}_h$ such that $\mathbf{n}_h\to\mathbf{n}$ strongly in $L^2(\Sigma_1;\mathbb{R}^3)$ and such that

$$\lim_{h\searrow 0}\tilde{W}_h^{OZF}(\mathbf{n}_h,\Sigma_1)=W(\mathbf{n}) \tag{III.2.19}$$

In order to prove the lim inf inequality, we we can assume that $\sup_{h>0}\tilde{W}_h^{OZF}(\mathbf{n}_h,\Sigma_1)<+\infty$. The discussion above, gives that $\mathbf{n}_h$ is bounded in $H^1(\Sigma_1;\mathbb{R}^3)$ and consequently that $\mathbf{n}_h\to\mathbf{n}$ weakly in $H^1(\Sigma_1;\mathbb{R}^3)$ with $\mathbf{n}\in\mathcal{V}$ and thus $W(\mathbf{n})$ is finite. The, we have the following

$$\liminf_{h\searrow 0}\tilde{W}_h^{OZF}(\mathbf{n}_h,\Sigma_1)\geq\liminf_{h\searrow 0}\frac{1}{2}\int_{\Sigma_1}|\nabla_s\mathbf{n}_h|^2\mathrm{dV}\geq\frac{1}{2}\int_{\Sigma_1}|\nabla_s\mathbf{n}|^2\mathrm{dV}$$

Moreover, since $\mathbf{n}$ does not depend on the thickness direction we have

$$\frac{1}{2}\int_{\Sigma_1}|\nabla_s\mathbf{n}|^2\mathrm{dV}\geq\frac{1}{2}\int_{\Sigma_1}|\nabla_s\mathbf{n}|^2\mathrm{dV}=\frac{1}{2}\int_{\Sigma}|\nabla_s\mathbf{n}|^2\mathrm{dVol}.$$

The existence of a recovery sequence is simple: consider $\mathbf{n}_h\equiv\mathbf{n}$ with $\mathbf{n}\in\mathcal{V}$, namely a vector field which does not depend on thickness direction a such that it is tangent to any inner surface of the foliation.

## III.3. Justification of the surface energy via micro-macro transition

In this Section we discuss the microscopic derivation of the surface energy (III.0.7). This amounts to starting with a microscopic energy and deriving the continuum energy (III.0.7) using $\Gamma$-convergence. The energy we want to start with is the lattice energy (I.2.5) discussed in Chapter I.

According to Theorem II.1 we consider the case of a shell $\Sigma$ with $\chi(\Sigma)=0$. For definiteness, we choose $\Sigma=\mathbb{T}^2$, the axisymmetric torus with its standard metric $g$ (see Section V.1) . We decorate the shell with a lattice obtained by first considering a square lattice $\varepsilon\mathbb{Z}^2$ with spacing $\varepsilon>0$ on $\Omega=[0,2\pi]\times[0,2\pi]$ and then mapped on the torus using the map $X:\Omega\to\mathbb{R}^3$ defined by (see Section V.1)

$$X(\theta,\phi)=\begin{pmatrix}(R+r\cos\theta)\cos\phi\\(R+r\cos\theta)\sin\phi\\r\sin\theta\end{pmatrix}.$$

We let $i\in\varepsilon\mathbb{Z}^2$. Then, denoting with $W$ the square $W=[0,1]^2$, we consider $\Omega_\varepsilon$ to be the union of all the cubes $\{i+\varepsilon W\}$ when $i$ ranges in $\mathbb{Z}^2_\varepsilon(\Omega):=\varepsilon\mathbb{Z}^2\cap\Omega$. Note that we have $\Omega_\varepsilon\subset\Omega$. Now, for any $i\in\mathbb{Z}^2_\varepsilon(\Omega)$, we set $p_i^\varepsilon:=X(i)$.

At any site $p_i^\varepsilon$ of the lattice on $\Sigma$, we consider a unit norm tangent vector $\mathbf{n}(p_i^\varepsilon)$. We denote with $\mathbf{n}$ the finite sequence $\{\mathbf{n}(p_i^\varepsilon)\}_{i\in\mathbb{Z}^2_\varepsilon(\Omega)}$. Then, we consider the following energy

$$F_\varepsilon(\mathbf{n}):=\sum_{<i,j>}\sqrt{\det g(p_i^\varepsilon)}\big(1-(\mathbf{n}(p_i^\varepsilon),\mathbf{n}(p_j^\varepsilon))\big)_{\mathbb{R}^3},\tag{III.3.1}$$

where we computed the scalar product in the "extrinsic" way (see the discussion in the Introduction).

For our purposes, it is convenient to group the interactions according to the two directions $\frac{\partial X}{\partial\theta}$ and $\frac{\partial X}{\partial\phi}$. We have

(III.3.2)
$$F_\varepsilon(\mathbf{n})=\sum_{i,i+\varepsilon v_1}\sqrt{\det g(p_i^\varepsilon)}\big(1-(\mathbf{n}(p_i^\varepsilon),\mathbf{n}(p_{i+\varepsilon v_1}^\varepsilon))\big)_{\mathbb{R}^3}+\sum_{i,i+\varepsilon v_2}\sqrt{\det g(p_i^\varepsilon)}\big(1-(\mathbf{n}(p_i^\varepsilon),\mathbf{n}(p_{i+\varepsilon v_2}^\varepsilon))\big)_{\mathbb{R}^3},$$

where $v_1$ and $v_2$ is the standard basis in $\mathbb{R}^2$.

Since $|\mathbf{n}(p_i^\varepsilon)|=1$ for any $i\in\mathbb{Z}^2_\varepsilon(\Omega)$, the energy can be written as

(III.3.3)
$$F_\varepsilon(\mathbf{n})=\frac{1}{2}\sum_{i,i+\varepsilon v_1}\varepsilon^2\sqrt{\det g(p_i^\varepsilon)}\Big|\frac{\mathbf{n}(p_{i+\varepsilon v_1}^\varepsilon)-\mathbf{n}(p_i^\varepsilon)}{\varepsilon}\Big|^2+\frac{1}{2}\sum_{i,i+\varepsilon v_2}\varepsilon^2\sqrt{\det g(p_i^\varepsilon)}\Big|\frac{\mathbf{n}(p_{i+\varepsilon v_2}^\varepsilon)-\mathbf{n}(p_i^\varepsilon)}{\varepsilon}\Big|^2$$

Now, given $\{\boldsymbol{v}(p_i^\varepsilon)\}_{i\in\mathbb{Z}^2_\varepsilon(\Omega)}$ where $\boldsymbol{v}(p_i^\varepsilon)$ are unit norm tangent vectors to $\mathbb{T}^2$, we introduce the following set of vector fields on the torus

$$C_\varepsilon(\mathbb{T}^2,\mathbb{S}^2)=\left\{\bar{\boldsymbol{v}}:\mathbb{T}^2\to\mathbb{S}^2:\ \bar{\boldsymbol{v}}(p)=\boldsymbol{v}(p_i^\varepsilon),\ \forall p\in X(\{i+\varepsilon W\})\right\}, \tag{III.3.4}$$

namely the set of the piece wise constant vector fields interpolating the values $\boldsymbol{v}(p_i^\varepsilon)$. Note that $|\bar{\boldsymbol{v}}(p)|=1$ for any $p\in\mathbb{T}^2$ but $\bar{\boldsymbol{v}}\in C_\varepsilon$ is not necessarily a tangent vector field. Then, any function $\boldsymbol{v}:\mathbb{Z}^2_\varepsilon(\Omega)\to\mathbb{R}^3$ can be regarded as an element of $C_\varepsilon(\mathbb{T}^2)$. As a consequence, we might think that the energy $F_\varepsilon$ is defined on $C_\varepsilon(\mathbb{T}^2)$ and then extended (with the same name) to $L^\infty(\mathbb{T}^2,\mathbb{S}^2)$.

We prove the following

Theorem III.2. *Let us consider the energy $F_\varepsilon: L^\infty(\mathbb{T}^2,\mathbb{S}^2)\to\mathbb{R}\cup\{+\infty\}$ defined in* (III.3.1). *Then, we have that $F_\varepsilon$ $\Gamma$-converges as $\varepsilon\searrow 0$ with respect to the weak star convergence in $L^\infty(\mathbb{T}^2,\mathbb{S}^2)$ to the functional*

$$W(\mathbf{n})=\begin{cases}\frac{1}{2}\int_{\mathbb{T}^2}|D\mathbf{n}|^2+|\mathfrak{B}\mathbf{n}|^2\mathrm{dVol} & \textit{if } \mathbf{n}\in H^1_{tan}(\mathbb{T}^2;\mathbb{S}^2)\\ +\infty & \textit{otherwise in } L^\infty(\mathbb{T}^2,\mathbb{S}^2).\end{cases} \tag{III.3.5}$$

Proof. The proof follows the lines of [**11**, Theorem 4.3].

According to the general theory of $\Gamma$-convergence (see Section V.2) we have to prove the so called lim inf inequality and the existence of a recovery sequence. We sketch the proof. The details will be given in [**18**].

Step 1: lim inf inequality

Let $\bar{\mathbf{n}}_\varepsilon$ be a sequence of piece wise constant vector fields interpolating $\mathbf{n}_\varepsilon=\{\mathbf{n}_\varepsilon(p_i^\varepsilon)\}_{i\in\mathbb{Z}^2_\varepsilon(\Omega)}$. It is not restrictive to assume that $F_\varepsilon(\mathbf{n}_\varepsilon)<+\infty$ (if not the lim inf inequality is trivially satisfied).

Now, let $\hat{\mathbf{n}}_\varepsilon$ denote the piece wise affine interpolation of $\mathbf{n}_\varepsilon(p_i^\varepsilon)$. This interpolation is constructed as in [**1**] as the piecewise affine interpolation of the (discrete) map $\mathbf{n}_\varepsilon\circ X$, namely the coordinate description of the discrete vector field $\mathbf{n}_\varepsilon$ on the curvilinear lattice. Then, we introduce the vectors $\mathbf{e}_1^\varepsilon$ and $\mathbf{e}_2^\varepsilon$ as $\mathbf{e}_1^\varepsilon:=\overrightarrow{p^\varepsilon_{i+\varepsilon v_1}\,p_i^\varepsilon}$ and $\mathbf{e}_2^\varepsilon=\overrightarrow{p^\varepsilon_{i+\varepsilon v_2}\,p_i^\varepsilon}$, respectively. Note that when $\varepsilon\searrow 0$, $\mathbf{e}_1^\varepsilon$ and $\mathbf{e}_2^\varepsilon$ are indeed approximations of the tangent vectors (see Section V.1) $\mathbf{e}_1$ and $\mathbf{e}_2$, spanning the tangent space of $\mathbb{T}^2$. More precisely, we have

$$\frac{\mathbf{e}_1^\varepsilon}{|\mathbf{e}_1^\varepsilon|}\xrightarrow{\varepsilon\searrow 0}\mathbf{e}_1\quad\text{uniformly}$$
$$\frac{\mathbf{e}_2^\varepsilon}{|\mathbf{e}_2^\varepsilon|}\xrightarrow{\varepsilon\searrow 0}\mathbf{e}_2\quad\text{uniformly}\ .$$

We denote with $P^\varepsilon$ the projection on the plane spanned by the two vectors $\mathbf{e}_1^\varepsilon$ and $\mathbf{e}_2^\varepsilon$. Then, we set $\nabla_\varepsilon:=\nabla\circ P^\varepsilon$.

Now, we rewrite the difference quotients in (III.3.3) as

$$\frac{\mathbf{n}(p^\varepsilon_{i+\varepsilon v_1})-\mathbf{n}(p_i^\varepsilon)}{\varepsilon}=\frac{\mathbf{n}(X(i+\varepsilon v_1))-\mathbf{n}(X(i))}{\varepsilon}=\frac{\mathbf{n}(X(i)+|\mathbf{e}_1^\varepsilon|\,\frac{\mathbf{e}_1^\varepsilon}{|\mathbf{e}_1^\varepsilon|})-\mathbf{n}(X(i))}{|\mathbf{e}_1^\varepsilon|}\,\frac{|\mathbf{e}_1^\varepsilon|}{\varepsilon},$$

and

$$\frac{\mathbf{n}(p^\varepsilon_{i+\varepsilon v_2})-\mathbf{n}(p_i^\varepsilon)}{\varepsilon}=\frac{\mathbf{n}(X(i+\varepsilon v_2))-\mathbf{n}(X(i))}{\varepsilon}=\frac{\mathbf{n}(X(i)+|\mathbf{e}_2^\varepsilon|\,\frac{\mathbf{e}_2^\varepsilon}{|\mathbf{e}_2^\varepsilon|})-\mathbf{n}(X(i))}{|\mathbf{e}_2^\varepsilon|}\,\frac{|\mathbf{e}_2^\varepsilon|}{\varepsilon}.$$

It is worthwhile noting that is the point where the "extrinsic" point of view comes into play: the vector $\mathbf{n}(p_i^\varepsilon)$ sees its neighbors $\mathbf{n}(p^\varepsilon_{i+\varepsilon v_1})$ and $\mathbf{n}(p^\varepsilon_{i+\varepsilon v_2})$ in $\mathbb{R}^3$, forgetting (for a while) the fact that they are tangent vectors to $\mathbb{T}^2$. The difference quotients above, can be reinterpreted in terms of the piecewise affine interpolation $\hat{\mathbf{n}}_\varepsilon$. Namely, we have

$$F_\varepsilon(\mathbf{n}_\varepsilon) \geq \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla_\varepsilon \hat{\mathbf{n}}_\varepsilon|^2 \sqrt{\det \bar{g}_\varepsilon} \mathrm{d}\theta \mathrm{d}\phi, \tag{III.3.6}$$

where $\bar{g}_\varepsilon$ is a proper piecewise constant interpolation of the values $g(p_i^\varepsilon)$. Being the left hand side bounded by assumption, we have that $\hat{\mathbf{n}}_\varepsilon$ is bounded in $H^1(\Omega;\mathbb{R}^3)$. Moreover, there holds that that $\hat{\mathbf{n}}_\varepsilon - \bar{\mathbf{n}}_\varepsilon \xrightarrow{\varepsilon \searrow 0} 0$. Finally, denoting with $\tilde{\mathbf{n}}$ the weak limit in $H^1(\Omega;\mathbb{R}^3)$ of $\hat{\mathbf{n}}_\varepsilon$, we have that the vector field $\mathbf{n}$ defined via $\tilde{\mathbf{n}} = \mathbf{n} \circ X$ belongs to $H^1_{tan}(\mathbb{T}^2;\mathbb{S}^2)$. Recall in particular that (an analogous relation also holds for the $\phi$-derivative) $\partial_\theta \tilde{\mathbf{n}} = \nabla_{\partial_\theta X}\mathbf{n}(X)$. Summing up, by semicontinuity we have

$$\liminf_{\varepsilon \searrow 0} \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla_\varepsilon \hat{\mathbf{n}}_\varepsilon|^2 \sqrt{\det \bar{g}_\varepsilon} \mathrm{d}\theta \mathrm{d}\phi \geq \frac{1}{2}\int_{\mathbb{T}^2} |\nabla_s \mathbf{n}|^2 \mathrm{dVol},$$

where we have used that $\mathbf{e}_1^\varepsilon$ and $\mathbf{e}_2^\varepsilon$ strongly converge in $L^2$ to their limits $\partial_\theta X$ and $\partial_\phi X$, respectively. We recall that $\nabla_s \mathbf{n} := \nabla \mathbf{n} \circ P$, being $P$ the projection on the tangent plane of $\mathbb{T}^2$ (see Section V.1). Thus, by taking the lim inf in (III.3.6) we conclude.

Step 2: Existence of a recovery sequence

Given a target vector field $\mathbf{n} \in H^1_{tan}(\mathbb{T}^2,\mathbb{S}^2)$ we want to find a sequence $\mathbf{n}_\varepsilon$ such that $\mathbf{n}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \mathbf{n}$ weakly star in $L^\infty(\mathbb{T}^2;\mathbb{S}^2)$ and such that

$$\lim_{\varepsilon \searrow 0} F_\varepsilon(\mathbf{n}_\varepsilon) = W(\mathbf{n}). \tag{III.3.7}$$

We can assume that $\mathbf{n}$ is a smooth vector field. Thus, we evaluate $\mathbf{n}$ at the points $p_i^\varepsilon$ of the lattice and then we consider the piecewise affine interpolation of $\mathbf{n}(p_i^\varepsilon)$: this is our recovery sequence. We have

$$\lim_{\varepsilon \searrow 0} \frac{1}{2}\int_{\mathbb{T}^2} |\nabla_s \mathbf{n}_\varepsilon|^2 \mathrm{dVol} = W(\mathbf{n}). \tag{III.3.8}$$

Moreover, there holds that (see [**11**] and [**18**])

$$\lim_{\varepsilon \searrow 0} \Big(\frac{1}{2}\int_{\mathbb{T}^2} |\nabla_s \mathbf{n}_\varepsilon|^2 \mathrm{dVol} - \sum_{<i,j>} \sqrt{\det g(p_i^\varepsilon)}\big(1 - (\mathbf{n}(p_i^\varepsilon), \mathbf{n}(p_j^\varepsilon))_{\mathbb{R}^3}\Big) = 0,$$

which together with (III.3.8) gives the desired (III.3.7).
Thus the Theorem is proved. □

Some comments are in order. We analyzed the microscopic justification on a toroidal shell only. This is clearly motivated by Theorem II.1. On the other hand, it would be extremely interesting to analyze the case of, say, a spherical shell. As already anticipated, the Poincarè-Hopf Index Theorem establishes a link between the topology of the shell $\Sigma$ and the number of singularities that a vector field with unit norm must have. In particular, the sum $M$ of the topological charges of all the defects is conserved and it is equal to the Euler Characteristic $\chi(\Sigma) = 2(1-g)$, where $g$ (the genus) is the number of handles of the surface. For example, a spherical shell has $\chi(\Sigma) = 2$ (thus $M = 2$), thus implying the necessity of having two defects. Theorem II.1 shows that these defects have infinite energy and thus the surface energy (III.0.7) can no longer describe this situation. Consequently, understand which energy describes a configuration with defects becomes extremely important. A possible strategy, could be to start with a discrete energy of the type (III.3.1), as in [**3**].

Now, as we did for the toroidal shell, we decorate a spherical shell with a lattice and we consider the corresponding energy (III.3.1). It is an interesting problem to understand the Γ-convergence of the energy (properly rescaled) (III.3.1). In particular, we expect (see [**18**]) to obtain an energy, *Renormalized Energy* (see [**7**], [**3**]), depending on the position of the singularities and coupling the degree of the defects with the curvature of the surface, as suggested in [**56**], [**58**] and in [**54**].

## III.4. Gradient flow of the energy on genus one surfaces

In this Section, we study the $L^2$ gradient flow of the energy (III.0.7), namely the following evolution

$$\partial_t \mathbf{n} - \Delta_g \mathbf{n} + \mathfrak{B}^2 \mathbf{n} = |D\mathbf{n}|^2 \mathbf{n} + |\mathfrak{B}\mathbf{n}|^2 \mathbf{n} \quad \text{a.e. in } \Sigma \times (0,+\infty), \tag{III.4.1}$$

$$\mathbf{n}(0) = \mathbf{n}_0 \quad \text{a.e. in } \Sigma. \tag{III.4.2}$$

We make precise the definition of weak solution to (III.4.1).

Definition III.4.1. $\mathbf{n}$ is a *global weak solution* to (III.4.1) if

$$\mathbf{n} \in L^\infty(0,+\infty; H^1_{\tan}(\Sigma, \mathbb{S}^2)), \quad \partial_t \mathbf{n} \in L^2(0,+\infty; L^2_{\tan}(\Sigma)),$$

$\mathbf{n}$ weakly solves (III.4.1), that is

$$\int_\Sigma (\partial_t \mathbf{n}, \phi)_{\mathbb{R}^3} \mathrm{dVol} + \int_\Sigma D\mathbf{n} : D\phi \mathrm{dVol} + \int_\Sigma (\mathfrak{B}^2 \mathbf{n} - |D\mathbf{n}|^2 \mathbf{n} - |\mathfrak{B}\mathbf{n}|^2 \mathbf{n}, \phi)_{\mathbb{R}^3} \mathrm{dVol} = 0, \tag{III.4.3}$$

for all $\phi \in H^1_{\tan}(\Sigma)$.

Here is the main result of this Section.

Theorem III.3. *Let $\Sigma$ be a two-dimensional surface satisfying* (III.1.1). *Given $\mathbf{n}_0 \in H^1_{\tan}(\Sigma, \mathbb{S}^2)$ there exists a global weak solution to* (III.4.1) *with $\mathbf{n}(\cdot,0) = \mathbf{n}_0(\cdot)$ in $\Sigma$.*

Note that equation (III.4.1) has some similarities with the heat flow for harmonic maps and it offers similar difficulties. In particular, the treatment of the quadratic terms in the right hand side requires some care. Note that these terms are related to the constraint $\mathbf{n}(x) \in \mathbb{S}^2$ for a.a. $x \in \Sigma$. As it happens in the study of the heat flow for harmonic maps (see [**21, 22**]), we relax this constraint with a Ginzburg-Landau type approximation, i.e., we allow for vectors $\mathbf{n}$ with $|\mathbf{n}| \neq 1$, but we penalise deviations from unitary length. The approximating equation is then obtained as the Euler-Lagrange equation of the *unconstrained* functional

$$\mathcal{E}_\varepsilon : H^1_{\tan}(\Sigma) \to \mathbb{R}, \qquad \mathcal{E}_\varepsilon(\boldsymbol{v}) := W(\boldsymbol{v}) + \frac{1}{4\varepsilon^2} \int_\Sigma (|\boldsymbol{v}|^2 - 1)^2. \tag{III.4.4}$$

Thus, we approximate the solutions to (III.4.1)-(III.4.2) with solutions of ($\varepsilon$ is a small parameter intended to go to zero)

$$\partial_t \mathbf{n}^\varepsilon - \Delta_g \mathbf{n}^\varepsilon + \mathfrak{B}^2 \mathbf{n}^\varepsilon + \tfrac{1}{\varepsilon^2}(|\mathbf{n}^\varepsilon|^2 - 1)\mathbf{n}^\varepsilon = 0 \quad \text{a.e. in } \Sigma \times (0,+\infty), \tag{III.4.5}$$

$$\mathbf{n}^\varepsilon(0) = \mathbf{n}_0 \quad \text{a.e. in } \Sigma. \tag{III.4.6}$$

Existence of a global solution to (III.4.5)-(III.4.6), with all the terms in $L^2(0,\infty; L^2_{\tan}(\Sigma))$ follows from a standard time discretization procedure that we skip.

Once we have constructed an approximate solution $\mathbf{n}_\varepsilon$, the main difficulty is to pass to the limit as $\varepsilon \searrow 0$. First of all, we perform some (uniform w.r.t. $\varepsilon > 0$) a priori estimates on the solutions to (III.4.5). We take the scalar product of $\mathbb{R}^3$ between the approximate equation and $\partial_t \mathbf{n}^\varepsilon$ and then we integrate over $\Sigma$. We have

$$\|\partial_t \mathbf{n}^\varepsilon(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_\varepsilon(\mathbf{n}^\varepsilon(t)) = \|\partial_t \mathbf{n}^\varepsilon(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t} W(\mathbf{n}^\varepsilon(t)) + \frac{1}{4\varepsilon^2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma (|\mathbf{n}^\varepsilon(t)|^2 - 1)^2 \mathrm{dVol} = 0. \tag{III.4.7}$$

Thus, integrating on $(0,T)$, $T>0$, and using that $\mathbf{n}_0 \in H^1_{\tan}(\Sigma, \mathbb{S}^2)$, we get the following estimate,

$$\begin{aligned} &\|\partial_t \mathbf{n}^\varepsilon\|^2_{L^2(0,T;L^2_{\tan}(\Sigma))} + \|D\mathbf{n}^\varepsilon\|^2_{L^\infty(0,T;L^2_{\tan}(\Sigma))} + \|\mathfrak{B}\mathbf{n}^\varepsilon\|^2_{L^\infty(0,T;L^2_{\tan}(\Sigma))} \\ &\qquad + \sup_{t\in(0,T)} \frac{1}{4\varepsilon^2}\int_\Sigma (|\mathbf{n}^\varepsilon(t)|^2-1)^2 \mathrm{dVol} \le 3\mathcal{E}_\varepsilon(\mathbf{n}_0) = 3W_\kappa(\mathbf{n}_0). \end{aligned} \tag{III.4.8}$$

The estimate above produces the following uniform bounds:

$$\|\mathbf{n}_\varepsilon\|_{H^1(0,T;L^2_{\tan}(\Sigma))\cap L^\infty(0,T;H^1_{\tan}(\Sigma))} \le C \tag{III.4.9}$$

$$\int_\Sigma (|\mathbf{n}^\varepsilon(t)|^2-1)^2 \mathrm{dVol} \le 12 W_\kappa(\mathbf{n}_0)\varepsilon^2 \quad \forall \varepsilon>0.\ \forall t\in(0,T), \tag{III.4.10}$$

Thus, we obtain the existence of a vector field $\mathbf{n} \in H^1(0,T;L^2_{\tan}(\Sigma)) \cap L^\infty(0,T;H^1_{\tan}(\Sigma))$ with $\mathbf{n}(0)=\mathbf{n}_0$ and of a not relabeled subsequence of $\varepsilon$ such that

$$\mathbf{n}^\varepsilon \xrightarrow{\varepsilon\searrow 0} \mathbf{n} \qquad \text{weakly star in } L^\infty(0,T;H^1_{\tan}(\Sigma)) \text{ and strongly in } L^2(0,T;L^2_{\tan}(\Sigma)), \tag{III.4.11}$$

$$\partial_t\mathbf{n}^\varepsilon \xrightarrow{\varepsilon\searrow 0} \partial_t\mathbf{n} \qquad \text{weakly in } L^2(0,T;L^2_{\tan}(\Sigma)), \tag{III.4.12}$$

$$\mathfrak{B}^2\mathbf{n}^\varepsilon \xrightarrow{\varepsilon\searrow 0} \mathfrak{B}^2\mathbf{n} \qquad \text{strongly in } L^2(0,T;L^2_{\tan}(\Sigma)), \tag{III.4.13}$$

where the last convergence follows directly from the continuity of the shape operator with respect to the strong convergence in $L^2$ and from the definition of the operator $\mathfrak{B}^2$. Moreover, (III.4.10) implies that (up to subsequences)

$$|\mathbf{n}^\varepsilon|^2 \xrightarrow{\varepsilon\searrow 0} 1 \quad \text{a.e. on } \Sigma\times(0,T). \tag{III.4.14}$$

As a consequence, we have that $|\mathbf{n}|=1$ a.e. in $\Sigma$ for any time interval $(0,T)$, and hence

$$\mathbf{n} \in L^\infty(0,+\infty; H^1_{\tan}(\Sigma, \mathbb{S}^2)).$$

Moreover, integrating (III.4.7) between 0 and $+\infty$, we have

$$\partial_t\mathbf{n} \in L^2(0,+\infty; L^2_{\tan}(\Sigma)).$$

To conclude we have to prove that $\mathbf{n}$ solves (III.4.1). It is important to observe that the bounds at our disposal on the sequence $\mathbf{n}_\varepsilon$ are too weak for directly pass to the limit in the singular term in equation (III.4.5). To this end, we follow the approach devised in [**21**]. This is based on the observation that (see Remark III.1) a smooth solution of (III.4.1) actually solves

$$(\partial_t\mathbf{n} - \Delta_g\mathbf{n} + \mathfrak{B}^2\mathbf{n})\times\mathbf{n} = 0 \tag{III.4.15}$$

and viceversa. This observation is of extreme importance in the analysis of these kind of problems and it has been noticed and used in [**21**] and [**22**], for instance.

To highlight the importance of the reformulation (III.4.15), let us consider the case of an harmonic map $u:\Omega\to\mathbb{S}^2$ with $\Omega\subset\mathbb{R}^n$ an open set. Being an harmonic map, $u$ solves the nonlinear elliptic equation

$$\Delta u + u|\nabla u|^2 = 0 \quad \text{in } \Omega. \tag{III.4.16}$$

Now, taking the vector product of the equation with $u$, one obtains that $u$ solves (III.4.16) if and only if it solves (see Remark III.1 for an idea on the computations)

$$\Delta u \times u = 0 \quad \text{in } \Omega,$$

which is equivalent to

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u \times \frac{\partial u}{\partial x_i}) = 0 \quad \text{in } \Omega. \tag{III.4.17}$$

Note that, differently from (III.4.16), the equation (III.4.17) is in divergence form and thus is it more treatable in weak regularity contexts. Moreover, it should be pointed out that this observation has a more general perspective, as observed in [**29**].

In the next Lemma III.4.1, we will prove the equivalence between (III.4.3) (namely the weak version of (III.4.1)) and a proper weak formulation of (III.4.15). In particular, this new formulation will be fundamental in the limit procedure due to its divergence-like structure. Our argument is inspired by [**21**], [**37**, Lemma 7.5.4].

LEMMA III.4.1. *A vector field* $\mathbf{n} \in H^1(0,1;L^2_{\tan}(\Sigma)) \cap L^\infty(0,T;H^1_{\tan}(\Sigma,\mathbb{S}^2))$ *solves* (III.4.3) *if and only if solves*

$$\begin{aligned}&-\int_\Sigma (\partial_t \mathbf{n} \times \mathbf{n}, \boldsymbol{\nu})_{\mathbb{R}^3} \psi \,\mathrm{dVol} + \int_\Sigma g^{ij}(D_i \mathbf{n}, \nu \times \mathbf{n})_{\mathbb{R}^3} \partial_j \psi \,\mathrm{dVol}\\ &-\int_\Sigma (\mathfrak{B}^2 \mathbf{n} \times \mathbf{n}, \boldsymbol{\nu})_{\mathbb{R}^3} \psi \,\mathrm{dVol} = 0 \ \ \forall \psi : \Sigma \to \mathbb{R} \textit{ smooth}\ .\end{aligned} \tag{III.4.18}$$

PROOF. Step 1

Let $\mathbf{n}$ be a solution of (III.4.3) with the regularity specified in the statement. Choose in the weak formulation (III.4.3) the tangent vector field $\phi = \hat{\boldsymbol{\nu}} \times \mathbf{n}$, where $\hat{\boldsymbol{\nu}} := \psi \boldsymbol{\nu}$ and $\psi : \Sigma \to \mathbb{R}$. First of all we have that the quadratic term in right hand side disappears as for any $a, b \in \mathbb{R}^3$ $(a, b \times a)_{\mathbb{R}^3} = (b, a \times a)_{\mathbb{R}^3} = 0$. Thus,

$$\int_\Sigma (|D\mathbf{n}|^2 \mathbf{n} + |\mathfrak{B}\mathbf{n}|^2 \mathbf{n}, \hat{\boldsymbol{\nu}} \times \mathbf{n})_{\mathbb{R}^3} \mathrm{dVol} = \int_\Sigma (|D\mathbf{n}|^2 \mathbf{n} + |\mathfrak{B}\mathbf{n}|^2 \mathbf{n}, \boldsymbol{\nu} \times \mathbf{n})_{\mathbb{R}^3} \psi \mathrm{dVol} = 0.$$

Then,

$$\int_\Sigma (\partial_t \mathbf{n}, \hat{\boldsymbol{\nu}} \times \mathbf{n})_{\mathbb{R}^3} \mathrm{dVol} = -\int_\Sigma (\partial_t \mathbf{n} \times \mathbf{n}, \boldsymbol{\nu})_{\mathbb{R}^3} \psi \,\mathrm{dVol} \tag{III.4.19}$$

$$\int_\Sigma (\mathfrak{B}^2 \mathbf{n}, \hat{\boldsymbol{\nu}} \times \mathbf{n})_{\mathbb{R}^3} \mathrm{dVol} = -\int_\Sigma (\mathfrak{B}^2 \mathbf{n} \times \mathbf{n}, \boldsymbol{\nu})_{\mathbb{R}^3} \psi \,\mathrm{dVol}, \tag{III.4.20}$$

for all $\psi : \Sigma \to \mathbb{R}$. Now, we come to the remaining term. To this end, we notice that there holds the following formula (see Lemma III.4.2)

$$D_j(\boldsymbol{\nu} \times \mathbf{n}) = \boldsymbol{\nu} \times D_j \mathbf{n}, \quad \forall j = 1, 2, \tag{III.4.21}$$

which implies

$$D_j(\hat{\boldsymbol{\nu}} \times \mathbf{n}) = D_j(\psi(\boldsymbol{\nu} \times \mathbf{n})) = (\partial_j \psi) \boldsymbol{\nu} \times \mathbf{n} + \psi \boldsymbol{\nu} \times D_j \mathbf{n}.$$

Thus, we get

$$\begin{aligned}\int_\Sigma D\mathbf{n} : D\phi \,\mathrm{dVol} &= \int_\Sigma g^{ij}(D_i \mathbf{n}, D_j(\hat{\boldsymbol{\nu}} \times \mathbf{n}))_{\mathbb{R}^3} \mathrm{dVol}\\ &= \int_\Sigma g^{ij}(D_i \mathbf{n}, \boldsymbol{\nu} \times D_j \mathbf{n})_{\mathbb{R}^3} \psi \mathrm{dVol} + \int_\Sigma g^{ij}(D_i \mathbf{n}, \boldsymbol{\nu} \times \mathbf{n})_{\mathbb{R}^3} \partial_j \psi \,\mathrm{dVol}\\ &= \int_\Sigma g^{ij}(D_i \mathbf{n}, \boldsymbol{\nu} \times \mathbf{n})_{\mathbb{R}^3} \partial_j \psi,\end{aligned} \tag{III.4.22}$$

where the first addendum vanishes since there holds that

$$\int_\Sigma g^{ij}(D_i\mathbf{n}, \boldsymbol{\nu}\times D_j\mathbf{n})_{\mathbb{R}^3}\psi\mathrm{dVol} = -\int_\Sigma g^{ij}(D_i\mathbf{n}\times D_j\mathbf{n}, \boldsymbol{\nu})_{\mathbb{R}^3}\psi\mathrm{dVol}$$

and since the metric tensor $g$ is symmetric and the cross product is skew symmetric.

<u>Step 2</u>

Now, suppose a vector field satisfying (III.4.18) and the regularity of the statement is given. As above, we indicate with $\hat{\nu}$ the vector field $\hat{\nu} := \psi\nu$. Thus, we have that $\mathbf{n}$ verifies

$$\int_\Sigma(\partial_t\mathbf{n}, \hat{\boldsymbol{\nu}}\times\mathbf{n})_{\mathbb{R}^3}\mathrm{dVol} + \int_\Sigma D\mathbf{n} : D(\hat{\boldsymbol{\nu}}\times\mathbf{n})\mathrm{dVol} + \int_\Sigma(\mathfrak{B}^2\mathbf{n}, \hat{\boldsymbol{\nu}}\times\mathbf{n})_{\mathbb{R}^3}\mathrm{dVol} = 0.$$

Now, choosing $\hat{\boldsymbol{\nu}}$ of the form $\hat{\boldsymbol{\nu}} = \mathbf{n}\times\phi$, and recalling that, being $|\mathbf{n}| = 1$, $(\mathbf{n}\times\phi)\times\mathbf{n} = \phi - \mathbf{n}(\mathbf{n}, \phi)$, we get

$$\begin{aligned}\int_\Sigma(\partial_t\mathbf{n}, \hat{\boldsymbol{\nu}}\times\mathbf{n})_{\mathbb{R}^3}\mathrm{dVol} &= \int_\Sigma(\partial_t\mathbf{n}, (\mathbf{n}\times\phi)\times\mathbf{n})_{\mathbb{R}^3}\mathrm{dVol}\\ &= \int_\Sigma(\partial_t\mathbf{n}, \phi)_{\mathbb{R}^3}\mathrm{dVol} - \int_\Sigma(\partial_t\mathbf{n}, \mathbf{n})_{\mathbb{R}^3}(\phi, \mathbf{n})_{\mathbb{R}^3}\mathrm{dVol}\\ &\overset{|\mathbf{n}|=1}{=} \int_\Sigma(\partial_t\mathbf{n}, \phi)_{\mathbb{R}^3}\mathrm{dVol}\end{aligned}$$

Analogously,

$$\begin{aligned}\int_\Sigma(\mathfrak{B}^2\mathbf{n}, \hat{\boldsymbol{\nu}}\times\mathbf{n})_{\mathbb{R}^3}\mathrm{dVol} &= \int_\Sigma(\mathfrak{B}^2\mathbf{n}, \phi)_{\mathbb{R}^3}\mathrm{dVol}. - \int_\Sigma(\mathfrak{B}^2\mathbf{n}, \mathbf{n})_{\mathbb{R}^3}(\mathbf{n}, \phi)_{\mathbb{R}^3}\mathrm{dVol}\\ &= \int_\Sigma(\mathfrak{B}^2\mathbf{n}, \phi)_{\mathbb{R}^3}\mathrm{dVol} - \int_\Sigma|\mathfrak{B}\mathbf{n}|^2(\mathbf{n}, \phi)_{\mathbb{R}^3}\mathrm{dVol},\end{aligned}$$

Finally, we have

$$\begin{aligned}\int_\Sigma D\mathbf{n} : D(\hat{\boldsymbol{\nu}}\times\mathbf{n})\mathrm{dVol} &= \int_\Sigma D\mathbf{n} : D((\mathbf{n}\times\phi)\times\mathbf{n})\mathrm{dVol}\\ &= \int_\Sigma D\mathbf{n} : D(\phi - \mathbf{n}(\mathbf{n}, \phi)_{\mathbb{R}^3})\mathrm{dVol}\\ &= \int_\Sigma D\mathbf{n} : D\phi\,\mathrm{dVol} - \int_\Sigma|D\mathbf{n}|^2(\mathbf{n}, \phi)_{\mathbb{R}^3}\mathrm{dVol}.\end{aligned}$$

Thus, collecting the above computations we get that $\mathbf{n}$ is a solution of (III.4.3). ☐

We are now in the position to conclude the proof of Theorem III.3. We recall that for the moment, we have proven that, up to a subsequence,

$$\mathbf{n}_\varepsilon \xrightarrow{\varepsilon\searrow 0} \mathbf{n} \quad \text{weakly in } H^1(0, T; L^2_{\tan}(\Sigma)) \text{ and weakly star in } L^\infty(0, T; H^1_{\tan}(\Sigma)),$$

where $\mathbf{n}$ is such that $|\mathbf{n}| = 1$. Moreover,

$$\mathbf{n}_\varepsilon \xrightarrow{\varepsilon\searrow 0} \mathbf{n} \quad \text{strongly in } L^2(0, T; L^2_{\tan}(\Sigma)).$$

Now, test (III.4.5) with $\phi = \hat{\nu} \times \mathbf{n}_\varepsilon$, where as before $\hat{\nu} := \psi\nu$ with $\psi : \Sigma \to \mathbb{R}$ smooth. The above computations, give

$$
\begin{aligned}
-\int_\Sigma (\partial_t \mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon, \nu)_{\mathbb{R}^3} \psi \, \mathrm{dVol} + \int_\Sigma g^{ij} (D_i \mathbf{n}_\varepsilon, \nu \times \mathbf{n}_\varepsilon)_{\mathbb{R}^3} \partial_j \psi \, \mathrm{dVol} \\
- \int_\Sigma (\mathfrak{B}^2 \mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon, \nu)_{\mathbb{R}^3} \psi \, \mathrm{dVol} = 0
\end{aligned}
\tag{III.4.23}
$$

where the penalization term has disappeared again thanks to $(a, b \times a)_{\mathbb{R}^3} = (b, a \times a)_{\mathbb{R}^3} = 0$, for $a, b \in \mathbb{R}^3$. Since (III.4.23) is in "divergence form", we can easily pass to the limit as $\varepsilon \searrow 0$ using the above proved convergences. Note indeed that all the terms easily pass to the limit as they are products of weakly and strongly convergent sequences in $L^2$. Consequently, we obtain that the limit $\mathbf{n}$ verifies (III.4.18) and thus solves (III.4.3) thanks to Lemma III.4.1.

In the next Lemma, we prove the formula (III.4.21).

LEMMA III.4.2. *Let us given $\mathbf{n} \in T_p\Sigma$. Let $\nu$ be the normal vector at the point $p$. Then, for $i = 1, 2$, there holds*

$$
D_i(\nu \times \mathbf{n}) = \nu \times D_i \mathbf{n}. \tag{III.4.24}
$$

PROOF. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a base for the tangent space at the point $p$. For $i = 1, 2$, the Gauss formula gives

$$
D_i(\nu \times \mathbf{n}) = \nabla_i(\nu \times \mathbf{n}) - h(\mathbf{e}_i, \nu \times \mathbf{n})\nu.
$$

Now,

$$
h(\mathbf{e}_i, \nu \times \mathbf{n}) = -(\nabla_i \nu, \nu \times \mathbf{n})_{\mathbb{R}^3} = (\nabla_i \nu \times \mathbf{n}, \nu)_{\mathbb{R}^3},
$$

by definition. Then, expanding $\nabla_i(\nu \times \mathbf{n}) = \nabla_i \nu \times \mathbf{n} + \nu \times \nabla_i \mathbf{n}$, we get

$$
D_i(\nu \times \mathbf{n}) = \nu \times \nabla_i \mathbf{n} + \nabla_i \nu \times \mathbf{n} - (\nabla_i \nu \times \mathbf{n}, \nu)_{\mathbb{R}^3} \nu = \nu \times \nabla_i \mathbf{n},
$$

being the last two terms normal vectors. We conclude if we prove that $\nu \times \nabla_i \mathbf{n} = \nu \times D_i \mathbf{n}$. This follows from the Gauss formula since

$$
\nu \times \nabla_i \mathbf{n} = \nu \times D_i \mathbf{n} + h(\mathbf{e}_i, \mathbf{n}) \nu \times \nu = \nu \times D_i \mathbf{n}.
$$

□

REMARK III.4. It is important to note that the above computations hold for an hypersurface of dimension $n$, upon replacing the cross product $\times$ with the wedge product $\wedge$. Thus, Theorem III.3 remains valid more geneally on an hypersurface $N$ of dimension $n$ provided the corresponding space $H^1_{tan}(N; \mathbb{S}^2)$ is well defined (see Theorem II.2).

## III.5. Gradient flow on the axisymmetric torus

In this Section, we discuss some properties of the solutions of the gradient flow (III.4.1) in the case in which $\Sigma = \mathbb{T}^2$, i.e. the axisymmetric torus. We refer to Section V.1 in Chapter V for the differential geometry of the torus. We take advantage of the representation of a unitary vector field in terms of the deviation angle $\alpha$ that $\mathbf{n}$ forms with the base vector $\mathbf{e}_1$. More precisely, $\mathbf{n} \in H^1_{tan}(\mathbb{T}, \mathbb{S}^2)$ will be represented as

$$
\mathbf{n} = \cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2. \tag{III.5.1}
$$

with $\alpha : \mathbb{R}^2 \to \mathbb{R}$ taking into account the possibly non zero *winding number of* $\mathbf{n}$ via the space $\mathscr{A}$ in Proposition II.1.1.

We recall that given $h = (h_\theta, h_\phi) \in \mathbb{Z}^2$, we have defined

$$\mathscr{A}_h := \left\{ \alpha \in H^1(Q) : \alpha_{|\{x_j=2\pi\}} = \alpha_{|\{x_j=0\}} + 2\pi h_{x_j},\ \text{for } x_j = \theta, \phi \right\}, \qquad \mathscr{A} := \bigcup_{h\in\mathbb{Z}^2} \mathscr{A}_h, \tag{III.5.2}$$

where the equality is in the sense of traces of $H^1$-regular functions. Note that $\mathscr{A}_0$ and $\mathscr{A}$ are linear vector spaces, while each $\mathscr{A}_h$ is an affine space. Indeed, for $h = (h_\theta, h_\phi),\ m = (m_\theta, m_\phi) \in \mathbb{Z}^2$, $\alpha \in \mathscr{A}_h$ and $\beta \in \mathscr{A}_m$, the function $u(x) := \alpha(x) + \beta(x) \in H^1(Q)$ satisfies

$$\begin{aligned} u_{|\{x_j=2\pi\}} &= \alpha_{|\{x_j=2\pi\}} + \beta_{|\{x_j=2\pi\}} \\ &\overset{(\text{II.1.2})}{=} \alpha_{|\{x_j=0\}} + 2\pi h_{x_j} + \beta_{|\{x_j=0\}} + 2\pi m_{x_j} \\ &= u_{|\{x_j=0\}} + 2\pi(h_{x_j} + m_{h_j}) \end{aligned}$$

in the sense of traces, which implies that $u = \alpha + \beta \in \mathscr{A}_{h+m}$, for $h + m = (h_\theta + m_\theta, h_\phi + m_\phi)$. As norm we choose

$$\|\alpha\|_{\mathscr{A}} := \left( \int_Q \left\{ |\nabla_s \alpha|^2 + \alpha^2 \right\} \mathrm{dVol} \right)^{\frac{1}{2}}, \tag{III.5.3}$$

where $\mathrm{dVol} = \sqrt{g}\,\mathrm{d}\theta\mathrm{d}\phi = r(R + r\cos\theta)\mathrm{d}\theta\mathrm{d}\phi$ is the area element induced by the metric $g$ (see Section V.1 in Chapter V). Owing to definition (II.1.2), this choice of norm yields $(\mathscr{A}_0, \|\cdot\|_{\mathscr{A}}) = H^1_{per}(Q; \mathrm{Vol})$. In the remainder of this section, we will alternate between the notations $\mathscr{A}_0$ and $H^1_{per}(Q)$, depending on the context.

When $\Sigma = \mathbb{T}^2$ the energy in Lemma II.1.3 becomes

$$W_\kappa(\alpha) = \frac{1}{2} \int_Q \left\{ \kappa |\nabla_s \alpha|^2 + \eta \cos(2\alpha) \right\} \mathrm{dVol} + \kappa\pi^2 \left( \frac{2 - b^2}{\sqrt{b^2 - 1}} + 2b \right), \tag{III.5.4}$$

and its Euler Lagrange equation is

$$-\kappa \Delta_s \alpha = \frac{\kappa}{2}(c_1^2 - c_2^2)\sin(2\alpha). \tag{III.5.5}$$

Hence, the $L^2$ gradient flow of the energy (III.0.7) written in terms of the deviation angle $\alpha$ is the following evolution

$$\partial_t \alpha = \kappa \Delta_s \alpha + \frac{\kappa}{2}(c_1^2 - c_2^2)\sin(2\alpha), \quad \text{on } \mathbb{R}^2 \times (0, +\infty) \tag{III.5.6}$$

with suitable initial data $\alpha_0 \in \mathscr{A}$. As above, denote $\Phi : \alpha \mapsto \mathbf{n} = \mathbf{e}_1 \cos\alpha + \mathbf{e}_2 \sin\alpha$. Since the *winding number* of a vector field $h(\Phi[\alpha])$ is invariant under homotopy, if $\alpha_0 \in \mathscr{A}_h$, then $\alpha(t) \in \mathscr{A}_h$ for all $t > 0$. The spaces $\mathscr{A}_h$ (see (II.1.2)) are constructed to take care of the correct boundary conditions, which require some attention, since in general we cannot expect a periodic solution.

Exploiting the affine structure of $\mathscr{A}$, for any $h \in \mathbb{Z}^2$, for any fixed $\psi_h \in \mathscr{A}_h$, it holds $\mathscr{A}_h = \mathscr{A}_0 + \psi_h$, i.e., any $\alpha \in \mathscr{A}_h$ can be decomposed as

$$\alpha(x) = u(x) + \psi_h(x), \qquad \text{with } u \in \mathscr{A}_0.$$

Using the decomposition $\alpha(t, x) = u(t, x) + \psi_h(x)$, we see that problem (III.5.6) is equivalent to finding $u \in C^0([0, +\infty); \mathscr{A}_0)$ such that

$$\partial_t u - \kappa \Delta_s u = \kappa \Delta_s \psi_h + \frac{\kappa}{2}(c_1^2 - c_2^2)\sin(2u + 2\psi_h) \text{ on } Q \times (0, +\infty), \tag{III.5.7}$$

with initial condition $u_0 \in \mathscr{A}_0$ and where $h \in \mathbb{Z}^2$ is the constant degree of the mappings $\Phi[\alpha(t)]$. Equation (III.5.7) can be further simplified by choosing a $\Delta_s$-harmonic function $\psi_h$, so that the term $\kappa\Delta_s\psi_h$ vanishes.

LEMMA III.5.1. *Let $h := (h_\theta, h_\phi) \in \mathbb{Z}^2$, and let $b = R/r$, where $R > r > 0$ are the radii of the torus. Define*

$$\psi(\theta,\phi) := h_\theta\sqrt{b^2-1}\int_0^\theta \frac{1}{b+\cos(s)}\mathrm{d}s + h_\phi\phi. \tag{III.5.8}$$

*Then $\psi \in C^\infty(\mathbb{R}^2)$, $\psi_{|Q} \in \mathscr{A}_h$, and $\Delta_s\psi = 0$.*

PROOF. Since $b > 1$, $\psi \in C^\infty(\mathbb{R}^2)$ and a simple check, using the explicit expression of the Laplace-Beltrami operator on the torus (V.1.11) shows that $\Delta_s\psi = 0$. In order to check that $\psi \in \mathscr{A}_h$, according to definition (II.1.2), we use the $2\pi$-periodicity of $1/(b+\cos(s))$ and the explicit integration $\sqrt{b^2-1} = \int_0^{2\pi} 1/(b+\cos(s))$ to compute

$$\begin{aligned}
\psi(\theta+2\pi,\phi+2\pi) &= h_\theta\sqrt{b^2-1}\int_0^{\theta+2\pi}\frac{1}{b+\cos(s)}\mathrm{d}s + h_\phi(\phi+2\pi)\\
&= h_\theta\sqrt{b^2-1}\int_0^{2\pi}\frac{1}{b+\cos(s)}\mathrm{d}s + h_\phi 2\pi + h_\theta\sqrt{b^2-1}\int_{2\pi}^{2\pi+\theta}\frac{1}{b+\cos(s)}\mathrm{d}s + h_\phi\phi\\
&= h_\theta 2\pi + h_\phi 2\pi + h_\theta\sqrt{b^2-1}\int_0^{\theta}\frac{1}{b+\cos(s)}\mathrm{d}s + h_\phi\phi\\
&= h_\theta 2\pi + h_\phi 2\pi + \psi(\theta,\phi).
\end{aligned}$$

$\square$

We now have all the ingredients to state and prove the result regarding solutions to the $L^2$-gradient flow of the one-constant approximation of the surface elastic energy $W_\kappa$.

THEOREM III.5. *Let $X$ be the standard parametrization of the torus with radii $R, r$, embedded in $\mathbb{R}^3$. Let $\mathscr{A}_h, \mathscr{A}$ be the spaces defined in* (II.1.2)*, endowed with the norm* (III.5.3)*. Then*

*(0) For all $h \in \mathbb{Z}^2$ there exists a classical solution $\alpha \in \mathscr{A}_h \cap C^\infty(Q)$ to the stationary problem*

$$-\kappa\Delta_s\alpha = \frac{\kappa}{2}(c_1^2 - c_2^2)\sin(2\alpha). \tag{III.5.9}$$

*Moreover, $\alpha$ is odd on any line passing through the origin.*

*(i)* (Weak well-posedness) *For any $\alpha_0 \in \mathscr{A}$, for all $T > 0$, there exists a unique mild solution $\alpha$ to* (III.5.6) *and it satisfies*

$$\alpha \in C^0([0,T];\mathscr{A}).$$

*Moreover, if $\alpha_0 \in \mathscr{A}_h$, then $\alpha(t) \in \mathscr{A}_h$ for all $t > 0$.*

*(ii)* (Strong well-posedness) *For any $m \in \mathbb{N}$, for any $\alpha_0 \in H^{2m}(Q) \cap \mathscr{A}$, for all $T > 0$, the unique solution $\alpha$ to* (III.5.6) *satisfies*

$$\alpha \in \bigcap_{k=0,\dots,m} C^k([0,T];H^{2m-2k}(Q)). \tag{III.5.10}$$

*In particular, if $\alpha_0 \in C^\infty(Q) \cap \mathscr{A}$, then $\alpha \in C^\infty([0,T]\times Q)$.*

*(iii)* (A maximum principle) *Under the hypothesis of step (ii),*

$$\alpha \in L^\infty(0,+\infty;\mathscr{A}) \quad \text{and} \quad \partial_t\alpha \in L^2(0,+\infty;L^2(Q)). \tag{III.5.11}$$

*(iv)* (Long-time behaviour) *Define the omega-limit set of a solution $\alpha$ to* (III.5.6) *by*

$$\omega(\alpha) := \left\{ \alpha_\infty \in \mathscr{A} : \text{ there exists } t_n \nearrow +\infty \text{ with } \alpha(t_n) \to \alpha_\infty \text{ in } L^2(Q) \right\} .$$

*Under the hypothesis of Step (ii), the omega-limit set is nonempty and it is contained in the set of solutions to* (III.5.5)*, namely if $\alpha_\infty \in \omega(\alpha)$ then $\alpha_\infty$ is a solution of* (III.5.5)*.*

The proof of the Theorem is detailed in [**49**] uses classical PDEs tools, such as fixed point theorems, a priori estimates. An interesting feature of the result is that as soon as the initial condition $\alpha_0$ is smooth the evolution $\alpha$ is smooth as well, actually smoother according to parabolic regularity. Correspondingly, once we fix a smooth initial configuration, say $\mathbf{n}_0$, for the vector field $\mathbf{n}$ with some *winding number $h$*, this is represented via a smooth $\alpha_0 \in \mathscr{A}_h$. Thus, the evolution $\alpha$ is smooth as well and it belongs to $\mathscr{A}_h$. As a result, we obtain that $\mathbf{n} = \cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2$ is smooth, actually smoother than $\mathbf{n}_0$ and has the same winding number of $\mathbf{n}_0$. Note that the regularity result and the existence of evolutions with fixed winding number seem difficult to be obtained using the vectorial formulation and the gradient flow (III.4.1). In the next Chapter IV we will use the gradient flow in order to numerically find the minimizers of the energy (III.0.7).

CHAPTER IV

# Qualitative analysis on the axisymmetric torus

In this Chapter, we persecute with the analysis on the axisymmetric torus and we present some qualitative result on the structure of the minimizers for the energy (III.0.6). These results are complemented with some numerical simulation and highlight the role of extrinsic terms in the energy. Moreover, they permit a comparison with the results obtained with the classical intrinsic energy.

In the first section we will discuss a toy model: the case in which the deviation angle $\alpha$ is assumed to be independent of the position. Clearly, this is a drastic simplification. However, it has the advantage of showing the dependence of the energy, and hence of the minimizers, on the constants $K_1, K_2$ and $K_3$.

In the second section, we will analyse the energy on the one constant approximation. We will present some numerical simulation on the gradient flow (III.5.6) that show how the behavior of minimizers depends on the aspect ratio of the torus. In particular, we will show the emergence of a new solution. We anticipate that these are all effects related to the extrinsic terms in the energy that are not shared by the classical intrinsic energy.

## IV.1. A toy model: constant $\alpha$ configurations

In this section, with a slight abuse of notation, we let $W(\alpha) := W(\mathbf{n})$, for $\mathbf{n} = \cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2$. We study the simpler case of $\alpha \equiv const$, where the energy $W(\alpha)$ in (II.1.15) reduces to

$$W(\alpha) = \frac{1}{2}\int_Q \Big\{ K_1\cos^2\alpha\,(\kappa_2)^2 + K_2(c_1 - c_2)^2\sin^2\alpha\,\cos^2\alpha \\ + K_3\sin^2\alpha\,(\kappa_2)^2 + K_3(c_1\cos^2\alpha + c_2\sin^2\alpha)^2 \Big\}\,\mathrm{dVol}.$$

Here $K_1, K_2, K_3$ are positive constants and (see Appendix A)

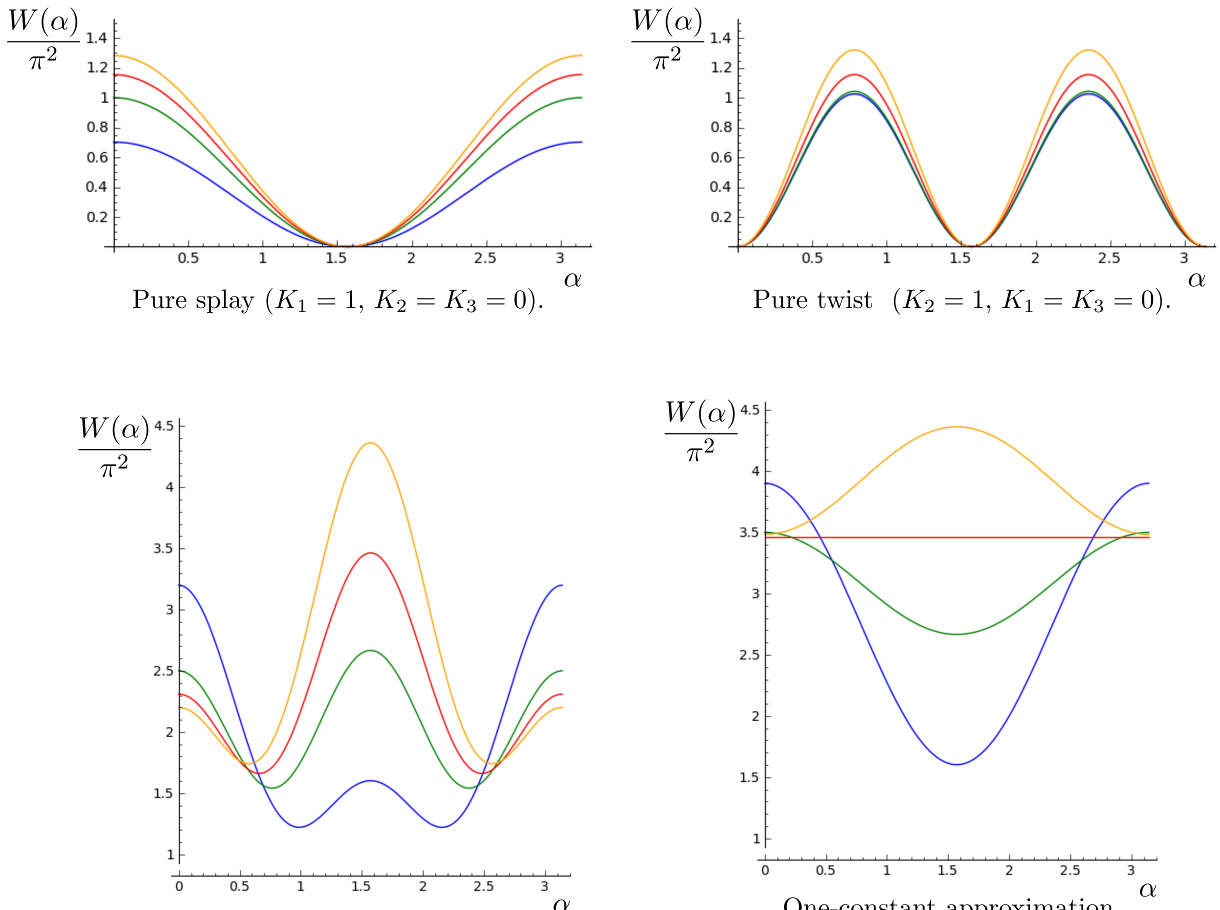
$$c_1 = \frac{1}{r^2}, \qquad c_2 = \frac{\cos\theta}{R + r\cos\theta}, \qquad \kappa_2 = -\frac{\sin\theta}{R + r\cos\theta}, \qquad \mathrm{dVol} = r(R + r\cos\theta)\mathrm{d}\theta\mathrm{d}\phi.$$

Lemma IV.1.1. *Let $b := R/r$. In the case of constant deviation $\alpha$, the energy $W$ has the explicit expression*

$$W(\alpha) = \pi^2\left[(K_1 + K_3)\left(b - \sqrt{b^2 - 1}\right) + \frac{K_2 + K_3}{2}\left(\frac{b^2}{\sqrt{b^2 - 1}}\right)\right] \\ + \pi^2\cos(2\alpha)\left[(K_1 - K_3)\left(b - \sqrt{b^2 - 1}\right) + K_3\left(2b - \frac{b}{\sqrt{b^2 - 1}}\right)\right] \\ + \pi^2\cos^2(2\alpha)\left[\frac{K_3 - K_2}{2}\left(\frac{b^2}{\sqrt{b^2 - 1}}\right)\right].$$

The proof relies on algebraic manipulations and integration of trigonometric functions, which are detailed in [**49**]. There are four parameters which influence the minimizers of $W$, that is $R/r$,

$K_1, K_2, K_3$. In Figure 1 we plot the graph $\{(\alpha, W(\alpha)/\pi^2\}$ for some especially meaningful choices of these parameters. The rescaling by $\pi^2$ is just for plotting purposes.




Pure splay ($K_1 = 1$, $K_2 = K_3 = 0$). Pure twist ($K_2 = 1$, $K_1 = K_3 = 0$).

Pure bend ($K_3 = 1$, $K_1 = K_2 = 0$). One-constant approximation ($K_1 = K_2 = K_3 = 1$).

FIGURE 1. Frank energy $W$ (rescaled by $\pi^2$) as a function of deviation $\alpha$ from $\mathbf{e}_1$, for different choices of the parameters $K_i$. The four colours represent four different choices of the ratio $R/r$, namely: $R/r = 1.1$ (orange), $R/r = 2/\sqrt{3}$ (red), $R/r = 1.25$ (green), $R/r = 1.6$ (blue).

Since we are assuming that $\alpha = const$, instead of the first variation of $W$ we can just take the first derivative with respect to $\alpha$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}W(\alpha) &= 2\pi^2\sin(2\alpha)\big[A(K_3 - K_1) - CK_3\big] + 2B(K_2 - K_3)\cos(2\alpha)\sin(2\alpha)\\ &= 2\sin(2\alpha)\Big(A(K_3 - K_1) + B\cos(2\alpha)(K_2 - K_3) - CK_3\Big),\end{aligned}$$

where

$$A := b - \sqrt{b^2 - 1}, \qquad B := \frac{b^2}{\sqrt{b^2-1}}, \qquad C := 2b - \frac{b^2}{\sqrt{b^2-1}}.$$

Therefore, $W'(\alpha) = 0$ if and only if

$$\sin(2\alpha) = 0 \qquad \text{or} \qquad \cos(2\alpha) = \frac{CK_3 - A(K_3 - K_1)}{B(K_2 - K_3)},$$

i.e.

$$\alpha = m\frac{\pi}{2} \qquad \text{or} \qquad \alpha = \pm\frac{1}{2}\arccos\left(\frac{CK_3 - A(K_3 - K_1)}{B(K_2 - K_3)}\right) + m\pi,$$

for $m \in \mathbb{Z}$, provided the argument of the arccos function is in $[-1, 1]$. For short, we refer to the critical points obtained via the arccos function as to points of the *second type*.

To check stability, we compute the second derivative of $W$

$$\begin{aligned}\frac{1}{\pi^2}\frac{d^2}{d\alpha^2}W(\alpha) &= 4\cos(2\alpha)\Big(A(K_3 - K_1) + B\cos(2\alpha)(K_2 - K_3) - CK_3\Big) - 4B\sin^2(2\alpha)(K_2 - K_3) \\ &= 4A(K_3 - K_1)\cos(2\alpha) + 4B(K_2 - K_3)\cos(4\alpha) - 4CK_3\cos(2\alpha).\end{aligned}$$

Therefore,

- critical points of type $\alpha = m\pi$ are stable local minimizers if

$$A(K_3 - K_1) + B(K_2 - K_3) - CK_3 > 0$$

  i.e. if

$$K_1(\sqrt{b^2 - 1} - b) + K_2\frac{b^2}{\sqrt{b^2 - 1}} - K_3(\sqrt{b^2 - 1} + b) > 0,$$

- critical points of type $\alpha = (2m + 1)\frac{\pi}{2}$ are stable local minimizers if

$$-A(K_3 - K_1) + B(K_2 - K_3) + CK_3 > 0,$$

- critical points of the second type are (stable local) minimizers if $K_3 > K_2$.

We make now a special choice of the parameters, in order to be able to plot a stability diagram for the minimizers. Namely, we assume that $K_1 = K_3$, $K_2 \neq 0$, and we introduce the variables

$$\lambda := \frac{K_3}{K_2}, \qquad \eta := \frac{C}{B} = 2\frac{\sqrt{b^2 - 1}}{b} - 1,$$

so that second type minimizers take the form

$$\alpha = \pm\frac{1}{2}\arccos\left(\frac{CK_3}{B(K_2 - K_3)}\right) = \pm\frac{1}{2}\arccos\left(\eta\frac{\lambda}{1 - \lambda}\right).$$

Note that $\lambda \geq 0$ and, since $b = R/r > 1$, then $\eta \in (-1, 1)$ and $\eta = 0$ if and only if $R/r = 2/\sqrt{3}$. A necessary condition for $\alpha = m\pi$ to be a stable local minimum for $W$ is then

$$\frac{B}{B + C} = \frac{2b}{\sqrt{b^2 - 1}} = \frac{1}{1 + \eta} > \lambda.$$

A necessary condition for a second type $\alpha$ to be a critical point of $W$ is that

$$\left|\frac{\lambda}{1 - \lambda}\right| \leq \frac{1}{|\eta|},$$

while a sufficient condition for a critical point to be a stable local minimum is that $\lambda > 1$. Finally, $\lambda_1 := \frac{1}{1+\eta}$ is a bifurcation point for the unstable critical points of $W$, while $\lambda_2 := \frac{1}{1-\eta}$ is a bifurcation point for the stable global minimizers of $W$.

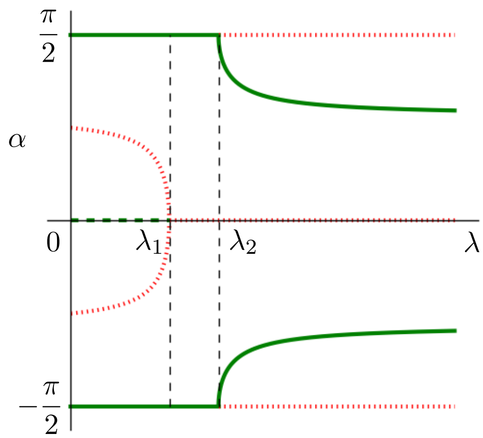



FIGURE 2. Bifurcation diagram for minimizers $\alpha$ of $W$ as a function of $\lambda = K_3/K_2$, for $\lambda \in (0, 3.25)$. The other parameters are chosen as $K_1 = K_3$, $R/r = 1.25$. The diagram shows the stable global minimizer (green continuous line), the stable local minimizer (green dashed line) and the unstable critical points (red dotted lines).

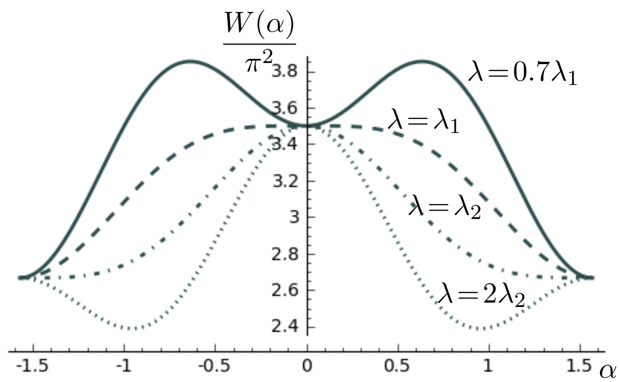



FIGURE 3. Graphs of the energy $W$ (rescaled by $\pi^2$) as a function of $\alpha$, for $R/r = 1.25$, $K_1 = K_3 = 1$, and different choices of $\lambda = K_3/K_2$.

## IV.2. One constant approximation

We recall from Chapter III (see (III.5.4) and (III.5.5)) that the energy $W$, $\Sigma = \mathbb{T}$, has the explicit representation

$$W(\alpha) = \frac{1}{2}\int_Q \left\{\kappa|\nabla_s\alpha|^2 + \eta\cos(2\alpha)\right\} \mathrm{dVol} + \kappa\pi^2\left(\frac{2-b^2}{\sqrt{b^2-1}} + 2b\right), \tag{IV.2.1}$$

where $\eta(\theta,\phi) := \kappa\frac{c_1^2 - c_2^2(\theta,\phi)}{2} = \kappa\frac{R^2+2Rr\cos\theta}{2r^2(R+r\cos\theta)^2}$, and $b := \frac{R}{r}$. Moreover, its Euler-Lagrange equation is

$$-\kappa\Delta_s\alpha = \frac{\kappa}{2}(c_1^2 - c_2^2)\sin(2\alpha). \tag{IV.2.2}$$

In the next Proposition we discuss the dependence of minimizers on the aspect ratio of the torus. In particular, we discuss the stability of the minimizers. To this end, we first compute the second variation of $W_\kappa$ in the direction $\omega$.

$$\text{(IV.2.3)} \qquad \frac{d^2}{dt^2} W_\kappa(\alpha + t\omega)\Big|_{t=0} = \kappa \int_Q |\nabla_s \omega|^2 - (c_1^2 - c_2^2)\cos(2\alpha)\omega^2 \,\mathrm{dVol}.$$

Proposition IV.2.1. *Let $b := R/r$. There exists $b^* \in (2/\sqrt{3}, 2]$ such that the constant values $\alpha = \pi/2 + m\pi$, $m \in \mathbb{Z}$, are local minimizers for $W_\kappa$ in $\mathscr{A}_0$ if and only if $b \geq b^*$. Moreover, if $b \geq 2$, there exists no non-constant solution $w$ to* (IV.2.2) *such that*

$$\text{(IV.2.4)} \qquad \frac{\pi}{2} + m\pi \leq w \leq \frac{\pi}{2} + (m+1)\pi.$$

Proof. Owing to the periodicity of the functions involved, it is not restrictive to assume $m = -1$. By (IV.2.3), the second variation of $W_\kappa$, in $\alpha = \pi/2$, in the direction $\omega \in \mathscr{A}_0$, is positive if and only if

$$\text{(IV.2.5)} \qquad \int_Q |\nabla_s \omega|^2 + (c_1^2 - c_2^2)\omega^2 \,\mathrm{dVol} > 0.$$

Let $b = R/r > 1$, since

$$\text{(IV.2.6)} \qquad c_1^2 - c_2^2 \overset{\text{(V.1.9)}}{=} \left(\frac{1}{r^2} - \frac{\cos^2\theta}{(R + r\cos\theta)^2}\right) = \frac{b}{r^2(b + \cos\theta)^2}(b + 2\cos\theta),$$

we see immediately that if $b \geq 2$ then $c_1^2 - c_2^2 \geq 0$ everywhere in $Q$, and $c_1^2 - c_2^2 = 0$ if and only if $b = 2$ and $\theta = \pi$. Therefore, if $b \geq 2$, the integral in (IV.2.5) is nonnegative for all $\omega \in \mathscr{A}_0$ (equal to zero if and only if $\omega = 0$) and we can conclude that the stationary point $\alpha = \pi/2$ is a local minimum. Restricting to constant variations $\omega$, (IV.2.5) is satisfied if and only if

$$0 < \int_Q (c_1^2 - c_2^2)\,\mathrm{dVol} = 2\pi b \int_0^{2\pi} \frac{b + 2\cos\theta}{b + \cos\theta}\, d\theta = 4\pi^2 b \left(2 - \frac{b}{\sqrt{b^2 - 1}}\right)$$

(see Section V.1 for the integration formula), that is, if and only if $b > 2/\sqrt{3}$. If $b = 2/\sqrt{3}$, then all configurations with constant angle $\alpha(x) = \bar{\alpha}$ have the same energy, while for $b < 2/\sqrt{3}$, $W_\kappa(\alpha \equiv 0) < W_\kappa(\alpha \equiv \pi/2)$. The uniqueness of the bifurcation point $b^*$ follows from the monotonicity of $(c_1^2 - c_2^2)\mathrm{Vol}$ with respect to $b$:

$$\frac{\partial}{\partial b}(c_1^2 - c_2^2)\mathrm{Vol} = \frac{\partial}{\partial b}\left\{\frac{b^2 + 2b\cos\theta}{b + \cos\theta}\right\} = 1 + \frac{\cos^2\theta}{(b + \cos\theta)^2} > 0, \qquad \forall\, \theta \in [0, 2\pi], \quad \forall\, b > 1.$$

The proof of the last step of the statement of Proposition IV.2.1 is inspired by [**20**, Theorem 2.4]. Assume that $b \geq 0$ and let $w$ be a solution to (IV.2.2), satisfying (IV.2.4) for $m = -1$. Then $v(x) := \pi/2 - w(x)$ satisfies

$$\text{(IV.2.7)} \qquad \Delta_s v = -\Delta_s w = \frac{1}{2}(c_1^2 - c_2^2)\sin(2w) = \frac{1}{2}(c_1^2 - c_2^2)\sin(\pi - 2v) = \frac{1}{2}(c_1^2 - c_2^2)\sin(2v).$$

Multiplying the first and the last member of (IV.2.7) by $v$, and integrating on $Q$ with respect to dVol, after integration by parts we obtain

$$-\int_Q |\nabla_s v|^2 \,\mathrm{dVol} = \int_Q \frac{1}{2}(c_1^2 - c_2^2)\sin(2v)v \,\mathrm{dVol} \overset{\text{(IV.2.6)}}{\geq} \frac{b(b-2)}{2(b+1)} \int_Q \sin(2v)v \,\mathrm{d}\theta\,\mathrm{d}\phi \overset{\text{(IV.2.4)}}{\geq} 0.$$

Thus,

$$\int_Q |\nabla_s v|^2 \,\mathrm{dVol} = 0, \quad \text{and} \quad \int_Q \sin(2v)v \,\mathrm{d}\theta\,\mathrm{d}\phi = 0,$$

implying $v \equiv 0$, $v \equiv -\pi/2$ or $v \equiv \pi/2$, as we wanted to prove. ☐

It is worthwhile noting that it is an interesting open problem to analytically determine the exact value of the critical threshold $b^*$. Its exact determination depends on some sharp estimates on the first eigenvalue of the Laplace Beltrami operator on the torus. Numerics indicates that $b^* \approx 1.52$.

Proposition (IV.2.1) is important since it describes how the Napoli-Vergori energy (IV.2.1) acts. In particular, it shows the differences-for a toroidal shell-with the classical intrinsic energy. It turns out that the presence of the extrinsic term related to the shape operator acts as a selection principle for equilibrium configurations. More precisely, when $\mu := R/r$ is sufficiently large then (see Proposition IV.2.1) the only constant solution is $\alpha = \pi/2 + m\pi$ ($m \in \mathbb{Z}$). Moreover, when $R/r < b^*$ a new class of non constant solution appears (see Figures 4 and 5). With respect to the heuristic principle expressed in [**42**], that "the nematic elastic energy promotes the alignment of the flux lines of the nematic director towards geodesics and/or lines of curvature of the surface", we make the following observation: This new solution tries to minimize the effect of the curvature by orienting the director field along the meridian lines ($\alpha = 0$), which are geodesics on the torus, near the hole of the torus, while near the external equator the director is oriented along the parallel lines $\alpha = \pi/2$, which are lines of curvature. The fact that the solution $\alpha = \alpha_p$ is no longer stable for sufficiently small $\mu$ is due to the high bending energy associated to $\alpha = \alpha_p$ in the internal hole of the torus. In fact, in a small strip close to the internal equator of the torus, we can approximate (see V.1)

$$c_1^2 - c_2^2 \approx \frac{1}{r^2} - \frac{1}{(R-r)^2}, \quad \mathrm{d}A \approx r(R-r)\mathrm{d}\theta\,\mathrm{d}\phi,$$

and therefore

$$(c_1^2 - c_2^2)\cos(2\alpha_p)\mathrm{dVol} \approx \mu\frac{2-\mu}{\mu-1}\mathrm{d}\theta\,\mathrm{d}\phi,$$

which tends to $+\infty$ as $\mu \to 1$.

Due to its "double well" like structure, the energy (IV.2.1) favors a smooth transition occurs $\alpha = \pi/2$ and $\alpha = 0$. In this sense, the new solution can be understood as an interpolation between $\alpha = \pi/2$ and $\alpha = 0$, which are the two constant stationary solutions of the system.

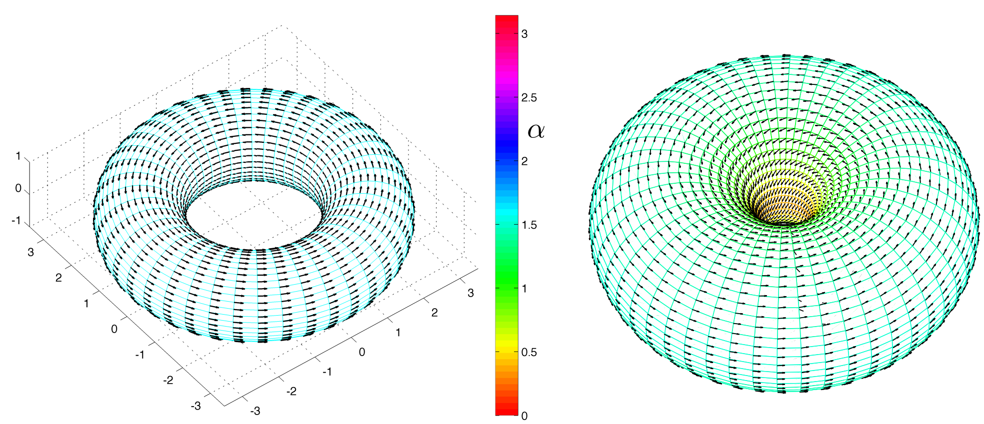



FIGURE 4. Configuration of a numerical solution $\alpha$ of the gradient flow. If $R/r = 2.5$, then $\alpha = \pi/2$, $W(\alpha) = 11.61 \cdot \pi^2$ (left). When $R/r = 1.33$, $W(\alpha) = 9.95 \cdot \pi^2 < W(\pi/2) = 10.22 \cdot \pi^2$ (right). The colour represents the angle $\alpha \in [0, \pi]$, the arrows represent the corresponding vector field $\mathbf{n}$.

It is interesting to compare these configurations with the equilibrium ones of the intrinsic surface energy [**38**] and see why we say that the extrinsic term in the energy (IV.2.1) is responsible for

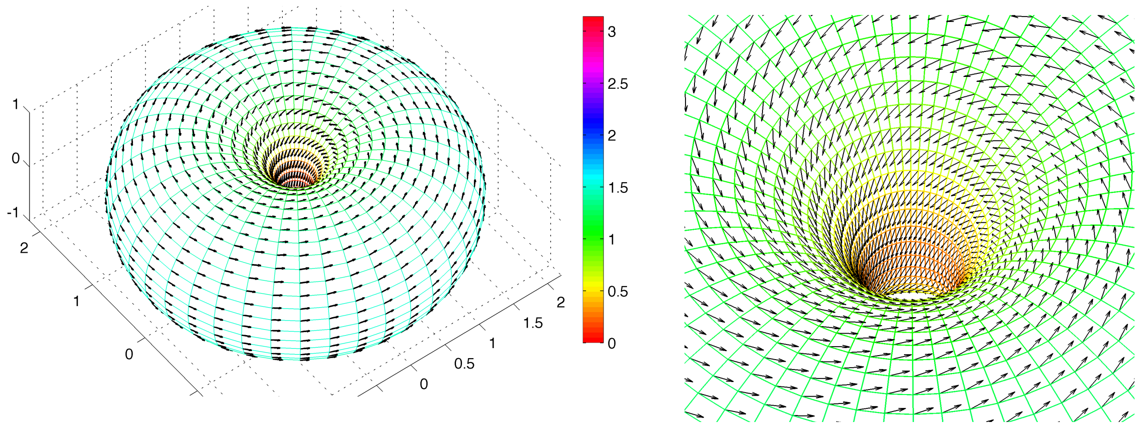



FIGURE 5. Configuration of the scalar field $\alpha$ and of the vector field $\mathbf{n}$ of a numerical solution to the the gradient flow (III.5.6), in the case $R/r = 1.2$ (left). Zoom-in of the central region of the same fields (right). The colour represents the angle $\alpha \in [0, \pi]$, the arrows represent the corresponding vector field $\mathbf{n}$.

selecting the minimizer. In the one-constant approximation, the (intrinsic) energy on a torus is (see II.1.2 and recall that $\mu := \frac{R}{r}$)

$$
\begin{aligned}
W_{\mathrm{Cl}}(\alpha) &= \frac{k}{2}\int_{\mathbb{T}} |\nabla_s\alpha - \mathbb{A}|^2 \,\mathrm{dVol} \\
&= \frac{k}{2}\int_{\mathbb{T}} \left[|\nabla_s\alpha|^2 - 2\nabla_s\alpha\cdot\mathbb{A} + |\mathbb{A}|^2\right] \mathrm{dVol} \\
&= \frac{k}{2}\int_{\mathbb{T}} |\nabla_s\alpha|^2 \,\mathrm{dVol} + 2k\pi^2(\mu - \sqrt{\mu^2-1})
\end{aligned}
$$

and the corresponding equilibrium equation is $\Delta_s\alpha = 0$. (Note that in [**38**] $\mu = r/R$.) Therefore, when using the intrinsic energy, every field $\mathbf{n} = \mathbf{e}_1\cos\bar{\alpha} + \mathbf{e}_2\sin\bar{\alpha}$ with a constant $\bar{\alpha}$, is an equilibrium state with the same energy independently of $\bar{\alpha}$.

CHAPTER V

# Mathematical Tools

## V.1. Differential Geometry tools

In this Appendix we collect some Differential Geometry tools that are used throughout the notes. We refer the reader to, e.g., [**36**], for all the material regarding Riemannian geometry.

Let $\Sigma \subset \mathbb{R}^3$ be an embedded regular surface of $\mathbb{R}^3$. We assume that $\Sigma$ is compact, connected and smooth. For any point $x \in \Sigma$, let $T_x\Sigma$ and $N_x\Sigma$ denote the tangent and the normal space to $\Sigma$ in the point $x$, respectively. Let $T\Sigma$ denote the tangent bundle of $\Sigma$, i.e. the (disjoint) union over $\Sigma$ of the tangent planes $T_x\Sigma$.

Let $\pi : T\Sigma \to \Sigma$ be the (smooth) map that assigns to any tangent vector its application point on $\Sigma$.

A vector field $\mathbf{n}$ on a open neighbourhood $A \subset \Sigma$, is a section of $T\Sigma$, i.e. a map $\mathbf{n} : A \to T\Sigma$ for which $\pi \circ \mathbf{n}$ is the identity on $\Sigma$. We denote by $\mathfrak{T}(\Sigma)$ the space of all the smooth sections of $T\Sigma$. For any point $x \in \Sigma$ let $T^*_x\Sigma = (T_x\Sigma)^*$ be the dual space of $T_x\Sigma$, also named cotangent space. Its elements are called covectors. The disjoint union over $\Sigma$ of the cotangent spaces $T^*_x\Sigma$ is $T^*\Sigma$. As we did for vector fields, we introduce the space of smooth sections of $T^*\Sigma$. We denote this space by $\mathfrak{T}^*(\Sigma)$, its elements are the covector fields.

We denote by $g$ the metric induced on $\Sigma$ by the embedding, i.e. the restriction of the metric of $\mathbb{R}^3$ to tangent vectors to $\Sigma$. As a consequence, we can unambiguously use the inner product notation $(\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^3}$ instead of $g(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in T_x\Sigma$, $x \in \Sigma$. Similarly, we write $|\boldsymbol{u}| = \sqrt{(\boldsymbol{u}, \boldsymbol{u})_{\mathbb{R}^3}}$ to denote the norm of a tangent vector $\boldsymbol{u}$ to $\Sigma$. For a two-tensor $\mathbb{A} = \{a_i^j\}$ we adopt the norm $|\mathbb{A}|^2 := \mathrm{tr}(\mathbb{A}^T\mathbb{A}) = \sum_{ij}(a_i^j)^2$, which is invariant under change of coordinates. We denote with $A : B$ the corresponding scalar product between the tensors $A$ and $B$. If $\{\mathbf{e}_1, \mathbf{e}_2\}$ is any local frame for $T\Sigma$, we denote by $g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{e}_i, \mathbf{e}_j)_{\mathbb{R}^3}$ the components of the metric tensor with respect to $\{\mathbf{e}_1, \mathbf{e}_2\}$. By $g^{ij}$ and $\bar{g}$ we denote the components of the inverse $g^{-1}$ and the determinant of $g$, respectively. As it is customary, if $(x^1, x^2)$ is a coordinate system for $\Sigma$, then $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$ is the corresponding local basis for $T\Sigma$ and $(\mathrm{d}x^1, \mathrm{d}x^2)$ is the dual basis. Given a vector $X$, we denote by $X^\flat$ the covector such that $X^\flat(\boldsymbol{v}) = g(X, \boldsymbol{v})$. In coordinates,

$$X^\flat = X^\flat_i \mathrm{d}x^i, \quad \text{with} \quad X^\flat_i = g_{ij}X^j.$$

Being the flat $^\flat$ operator invertible, we denote by the sharp $^\sharp$ symbol its inverse, which acts in the following way: Given a covector $\omega$, let $\omega^\sharp$ be the vector such that $\omega(\boldsymbol{v}) = g(\omega^\sharp, \boldsymbol{v})$. In coordinates, we have

$$\omega^\sharp = (\omega^\sharp)^i \frac{\partial}{\partial x^i}, \quad \text{with} \quad (\omega^\sharp)^i = g^{ij}\omega_j.$$

Notice that we use Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise specified. In particular, indices with greek letters are summed from 1 to 3, while latin ones are summed from 1 to 2.

**Differential Operators**. Let $\nabla$ be the connection with respect to the standard metric of $\mathbb{R}^3$, i.e., given two smooth vector fields $Y$ and $X$ in $\mathbb{R}^3$ (identified with its tangent space), the vector field $\nabla_X Y$ is the vector field whose components are the directional derivatives of the components of $Y$ in the direction $X$. When $\mathbf{e}_\alpha$ ($\alpha = 1, \dots, 3$) is a basis of $\mathbb{R}^3$ we will set $\nabla_\alpha Y := \nabla_{\mathbf{e}_\alpha} Y$. Given $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathfrak{T}(\Sigma)$, we denote with $D_{\boldsymbol{v}}\boldsymbol{u}$ the covariant derivative of $\boldsymbol{u}$ in the direction $\boldsymbol{v}$, with respect to the Levi Civita (or Riemannian) connection $D$ of the metric $g$ on $\Sigma$. Now, if $\boldsymbol{u}$ and $\boldsymbol{v}$ are extended arbitrarily to smooth vector fields on $\mathbb{R}^N$, we have the Gauss Formula along $\Sigma$:

$$\nabla_{\boldsymbol{v}}\boldsymbol{u} = D_{\boldsymbol{v}}\boldsymbol{u} + h(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{\nu}. \tag{V.1.1}$$

In the relation above, the symmetric bilinear form $h : \mathfrak{T}(\Sigma) \times \mathfrak{T}(\Sigma) \longrightarrow \mathbb{R}$ is the *scalar second fundamental form* of $\Sigma$. Associated to $h$, there is a linear self adjoint operator, called *shape operator* and denoted with $\mathfrak{B} : \mathfrak{T}(\Sigma) \longrightarrow \mathfrak{T}(\Sigma)$, such that $\mathfrak{B}\boldsymbol{v} = -\nabla_{\boldsymbol{v}}\boldsymbol{\nu}$ for any $\boldsymbol{v} \in \mathfrak{T}(\Sigma)$. We recall that the operator $\mathfrak{B}$ satisfies the Weingarten relation

$$(\mathfrak{B}\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^3} = h(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathfrak{T}(\Sigma).$$

Beside the covariant derivative, we introduce another differential operator for vector fields on $\Sigma$, which takes into account also the way that $\Sigma$ embeds in $\mathbb{R}^3$. Let $\boldsymbol{u} \in \mathfrak{T}(\Sigma)$ and extend it smoothly to a vector field $\tilde{\boldsymbol{u}}$ on $\mathbb{R}^3$; denote its standard gradient by $\nabla\tilde{\boldsymbol{u}}$ on $\mathbb{R}^3$. For $x \in \Sigma$, define the *surface gradient* of $\boldsymbol{u}$

$$\nabla_s \boldsymbol{u}(x) := \nabla\tilde{\boldsymbol{u}}(x)P(x), \tag{V.1.2}$$

where $P(x) := (Id - \boldsymbol{\nu} \otimes \boldsymbol{\nu})(x)$ is the orthogonal projection on $T_x\Sigma$. Note that $\nabla_s\boldsymbol{u}$ is well-defined, as it does not depend on the particular extension $\tilde{\boldsymbol{u}}$. The object just defined is a smooth mapping $\nabla_s\boldsymbol{u} : \Sigma \to \mathbb{R}^{3\times3}$, or equivalently $\nabla_s\boldsymbol{u} : \Sigma \to \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ (the space of linear continuous operators on $\mathbb{R}^3$), such that $\ker \nabla_s\boldsymbol{u}(x) = N_x\Sigma$, for all $x \in \Sigma$. In general, $\nabla_s\boldsymbol{u} \neq D\boldsymbol{u} = P(\nabla\boldsymbol{u})$ since the matrix product is non commutative. Using the decomposition (V.1.1), it is immediate to get

$$\nabla_s\boldsymbol{u}[\boldsymbol{v}] = \nabla_{\boldsymbol{v}}\boldsymbol{u} = D_{\boldsymbol{v}}\boldsymbol{u} + h(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{\nu}, \quad \forall \boldsymbol{v} \in T_x\Sigma, \forall x \in \Sigma,$$

which gives, recalling that the decomposition is orthogonal,

$$|\nabla_s\boldsymbol{u}|^2 = |D\boldsymbol{u}|^2 + |\mathfrak{B}\boldsymbol{u}|^2, \quad \forall \boldsymbol{u} \in T_x\Sigma, \forall x \in \Sigma. \tag{V.1.3}$$

Having defined $\nabla_s\boldsymbol{u}$, we can introduce the related notions of divergence and curl

$$\mathrm{tr}_g \nabla_s\boldsymbol{u} = \mathrm{tr}_g D\boldsymbol{u} =: \mathrm{div_s}\,\boldsymbol{u}, \quad \text{in coordinates, } \mathrm{div_s}\,\boldsymbol{u} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,\boldsymbol{u}^i\right)$$

and $\mathrm{curl_s}\,\boldsymbol{u} := -\epsilon\nabla_s\boldsymbol{u}$, where $\epsilon$ is the Ricci alternator:

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} 0 & \text{if any of } \alpha, \beta, \gamma \text{ are the same,} \\ +1 & \text{if } (\alpha, \beta, \gamma) \text{ is a cyclic permutation of (1,2,3),} \\ -1 & \text{otherwise.} \end{cases}$$

Note that the trace operator in the definition of the divergence acts only on tangential directions. Moreover, note that, contrary to the so-called covariant curl (denoted with $\mathrm{curl}_\Sigma$, see [**36**]) the surface $\mathrm{curl_s}$ defined above has, unless the surface $\Sigma$ is a plane, also in-plane components. To see this, we introduce the Darboux orthonormal frame (or Darboux trihedron) $(\mathbf{n}, \mathbf{t}, \boldsymbol{\nu})$, where $\mathbf{t} = \boldsymbol{\nu} \times \mathbf{n}$. Let $\kappa_{\mathbf{n}}, \kappa_{\mathbf{t}}$ be the geodesic curvatures of the flux lines of $\mathbf{n}$ and $\mathbf{t}$, defined as $\kappa_{\mathbf{n}} := (D_{\mathbf{n}}\mathbf{n}, \mathbf{t})_{\mathbb{R}^3}$, $\kappa_{\mathbf{t}} := -(D_{\mathbf{t}}\mathbf{t}, \mathbf{n})_{\mathbb{R}^3}$, respectively; let $c_{\mathbf{n}} := (\mathfrak{B}\mathbf{n}, \mathbf{n})_{\mathbb{R}^3}$ be the normal curvature and let $\tau_{\mathbf{n}} = -(\mathfrak{B}\mathbf{n}, \mathbf{t})_{\mathbb{R}^3}$

be the geodesic torsion of the flux lines of $\mathbf{n}$ (see, e.g., [**25**]). The surface gradient of $\mathbf{n}$, with respect to the Darboux frame, has the simple expression (see, e.g., [**46**])

$$\nabla_s \mathbf{n} = \begin{pmatrix} 0 & 0 & 0 \\ \kappa_{\mathbf{n}} & \kappa_{\mathbf{t}} & 0 \\ c_{\mathbf{n}} & -\tau_{\mathbf{n}} & 0 \end{pmatrix},$$

from which we read

$$\operatorname{div_s} \mathbf{n} = \kappa_{\mathbf{t}} \quad \text{and} \quad \operatorname{curl_s} \mathbf{n} = -\tau_{\mathbf{n}} \mathbf{n} - c_{\mathbf{n}} \mathbf{t} + \kappa_{\mathbf{n}} \boldsymbol{\nu}. \tag{V.1.4}$$

On the other hand, also the norm of the covariant derivative $D\mathbf{n}$ can be expressed in terms of the geodesic curvatures $\kappa_{\mathbf{t}}$ and $\kappa_{\mathbf{n}}$ as $|D\mathbf{n}|^2 = \kappa_{\mathbf{t}}^2 + \kappa_{\mathbf{n}}^2$. As a result, we have the following useful expression

$$(\operatorname{div_s} \mathbf{n})^2 + (\mathbf{n} \cdot \operatorname{curl_s} \mathbf{n})^2 + |\mathbf{n} \times \operatorname{curl_s} \mathbf{n}|^2 = (\operatorname{div_s} \mathbf{n})^2 + |\operatorname{curl_s} \mathbf{n}|^2 = \kappa_{\mathbf{t}}^2 + \kappa_{\mathbf{n}}^2 + \tau_{\mathbf{n}}^2 + c_{\mathbf{n}}^2 = |\nabla_s \mathbf{n}|^2. \tag{V.1.5}$$

For a smooth scalar function $f : \Sigma \to \mathbb{R}$, with differential application $\mathrm{d}f_x : T_x\Sigma \to T_{f(x)}\mathbb{R} \simeq \mathbb{R}$, we introduce its gradient as $\operatorname{grad_s} f = \mathrm{d}f^\sharp$, that is, the vector field such that

$$\mathrm{d}f(X) = g(\operatorname{grad_s} f, X) \quad \text{for all } X \in T\Sigma.$$

Since for scalar functions the expressions of $\operatorname{grad_s} f$ and $\nabla_s f$ coincide, in what follows we replace $\operatorname{grad_s}$ with the more common notation $\nabla_s f$. In coordinates, denoting $X = X^i \frac{\partial}{\partial x^i}$, the above relation means

$$\nabla_s f := g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}.$$

The Laplace Beltrami operator on $\Sigma$ is given by

$$\Delta_s := \operatorname{div_s} \circ \nabla_s = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right).$$

We denote with dVol the volume form of $\Sigma$ (see, e.g., [**36**]). We recall the following integration by parts formula ($f$ and $h$ are smooth functions on $\Sigma$)

$$-\int_\Sigma \Delta_s f \, h \, \mathrm{dVol} = \int_\Sigma g(\nabla_s f, \nabla_s h) \, \mathrm{dVol} - \int_{\partial\Sigma} h \, \mathrm{d}f(N) \, \mathrm{d}S', \tag{V.1.6}$$

where $f$ and $h$ are smooth functions on $\Sigma$ and $\mathrm{d}S'$ is the element of length of the induced metric on $\partial\Sigma$. For a smooth vector field $\mathbf{n} \in \mathfrak{T}(\Sigma)$, we denote with $D^2\mathbf{n}$ the double covariant derivative of $\mathbf{n}$, i.e. the following tensor field

$$D^2\mathbf{n}(X, Y) := D_X(D_Y \mathbf{n}) - D_{D_X Y} \mathbf{n} \quad \text{for } X, Y \in \mathfrak{T}(\Sigma).$$

If $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$, we set $D^2_{ij}\mathbf{n} := D^2\mathbf{n}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. Then, we denote with $\Delta_g \mathbf{n}$ the rough laplacian of $\mathbf{n}$, namely the vector field defined as

$$\Delta_g \mathbf{n} := g^{ij}(D^2_{ij}\mathbf{n}) = g^{ij} D_i(D_j \mathbf{n}) - g^{ij} D_{D_i \frac{\partial}{\partial x_j}} \mathbf{n}.$$

In particular, in a local orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2\}$, we have that

$$\Delta_g \mathbf{n} = \delta^{ij} D_i(D_j \mathbf{n}) - \delta^{ij} D_{D_i \mathbf{e}_j} \mathbf{n}.$$

Note that $\Delta_g$ can be expressed in divergence form as $\Delta_g \mathbf{n} = \operatorname{div}_s D\mathbf{n}$. In the flat case, the rough laplacian reduces to the componentwise laplacian of $\mathbf{n}$.

**Differential Geometry on the Torus**. Let $Q := [0, 2\pi] \times [0, 2\pi] \subset \mathbb{R}^2$, and let $X : Q \to \mathbb{R}^3$ be the following parametrization of an embedded axisymmetric torus $\mathbb{T}$

$$X(\theta, \phi) = \begin{pmatrix} (R + r\cos\theta)\cos\phi \\ (R + r\cos\theta)\sin\phi \\ r\sin\theta \end{pmatrix}. \tag{V.1.7}$$

Using parametrization (V.1.7), we derive the main geometrical quantities, like tangent and normal vectors, first and second fundamental form, in order to obtain an explicit expression for the metric and the curvatures of $\mathbb{T}$ and for $\nabla_s \mathbf{n}$.

Letting

$$X_\theta := \frac{\partial}{\partial\theta} X, \qquad X_\phi := \frac{\partial}{\partial\phi} X, \qquad \boldsymbol{\nu} := \frac{X_\theta \wedge X_\phi}{|X_\theta \wedge X_\phi|},$$

we have

$$X_\theta = \begin{pmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}, \qquad X_\phi = \begin{pmatrix} -(R + r\cos\theta)\sin\phi \\ (R + r\cos\theta)\cos\phi \\ 0 \end{pmatrix}, \qquad \boldsymbol{\nu} = -\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix},$$

$$X_{\theta\theta} = \begin{pmatrix} -r\cos\theta\cos\phi \\ -r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix}, \qquad X_{\phi\phi} = \begin{pmatrix} -(R + r\cos\theta)\cos\phi \\ -(R + r\cos\theta)\sin\phi \\ 0 \end{pmatrix}, \qquad X_{\theta\phi} = \begin{pmatrix} r\sin\theta\sin\phi \\ -r\sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

The unit tangent vectors are

$$\mathbf{e}_1(\theta, \phi) := \frac{X_\theta}{|X_\theta|} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \mathbf{e}_2(\theta, \phi) := \frac{X_\phi}{|X_\phi|} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

Note that this choice of tangent vectors yields an *inner* unit normal $\boldsymbol{\nu}$. The first and second fundamental forms are

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos\theta)^2 \end{pmatrix}, \qquad II = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{\cos\theta}{R + r\cos\theta} \end{pmatrix}.$$

We have $\sqrt{g} = r(R + r\cos\theta)$, $g^{ii} := (g_{ii})^{-1}$. Thus, the shape operator $\mathfrak{B}$ has the form

$$\begin{cases} \mathfrak{B}\mathbf{e}_1 = \frac{1}{r}\mathbf{e}_1 \\ \mathfrak{B}\mathbf{e}_2 = \frac{\cos\theta}{R + r\cos\theta}\mathbf{e}_2 \end{cases} \tag{V.1.8}$$

from which we have that $\mathbf{e}_1$ and $\mathbf{e}_2$ are the principal directions. Then, the principal curvatures are

$$c_1 = \frac{1}{r}, \qquad c_2 = \frac{\cos\theta}{R + r\cos\theta}. \tag{V.1.9}$$

Now, we compute $(\nabla\mathbf{e}_i)\mathbf{e}_j$. Deriving the relation $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ we see that

$$(\nabla\mathbf{e}_1)^T\mathbf{e}_1 = (\nabla\mathbf{e}_2)^T\mathbf{e}_2 = 0 \qquad \text{and} \qquad (\nabla\mathbf{e}_1)^T\mathbf{e}_2 = -(\nabla\mathbf{e}_2)^T\mathbf{e}_1. \tag{V.1.10}$$

To differentiate along $\mathbf{e}_1$, let

$$\begin{cases} \theta(t) = \frac{t}{r} + \theta_0 \\ \phi(t) = \phi_0 \end{cases},$$

and set $\gamma(t) = X(\theta(t), \phi(t))$. We have $\gamma(0) = X(\theta_0, \phi_0)$ and $\gamma'(0) = \frac{1}{r}X_\theta(\theta_0, \phi_0) = \mathbf{e}_1(\theta_0, \phi_0)$. Thus, the directional derivatives of $\mathbf{e}_1$ and $\mathbf{e}_2$ along $\mathbf{e}_1$ are given by

$$(\nabla\mathbf{e}_1)\mathbf{e}_1 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{1}{r}X_\theta(\theta(t), \phi(t)) = \frac{1}{r^2}X_{\theta\theta}, \qquad (\nabla\mathbf{e}_2)\mathbf{e}_1 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{e}_2(\theta(t), \phi(t)) = \mathbf{0}.$$

To differentiate along $\mathbf{e}_2$, we set

$$\begin{cases}\theta(t)=\theta_0\\ \phi(t)=\frac{t}{R+r\cos\theta_0}+\phi_0\end{cases},$$

and take $\gamma(t)=X(\theta(t),\phi(t))$, so that $\gamma(0)=X(\theta_0,\phi_0)$ and $\gamma'(0)=\frac{1}{R+r\cos\theta_0}X_\phi(\theta_0,\phi_0)=\mathbf{e}_2(\theta_0,\phi_0)$. Thus,

$$\begin{aligned}(\nabla\mathbf{e}_1)\mathbf{e}_2&=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\frac{1}{r}X_\theta(\theta(t),\phi(t))=\frac{1}{r(R+r\cos\theta_0)}X_{\theta\phi},\\ (\nabla\mathbf{e}_2)\mathbf{e}_2&=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\frac{1}{R+r\cos\theta(t)}X_\phi(\theta(t),\phi(t))=\frac{1}{(R+r\cos\theta_0)^2}X_{\phi\phi}.\end{aligned}$$

The geodesic curvatures $\kappa_1$ and $\kappa_2$ of the principal lines of curvature can thus be obtained by

$$\begin{aligned}\kappa_1&=\mathbf{e}_2(\nabla\mathbf{e}_1)\mathbf{e}_1=\frac{1}{R+r\cos\phi}X_\phi\cdot\frac{1}{r^2}X_{\theta\theta}=0,\\ \kappa_2&=\mathbf{e}_2(\nabla\mathbf{e}_1)\mathbf{e}_2=\frac{1}{r(R+r\cos\theta)^2}X_\phi\cdot X_{\theta\phi}=\frac{-\sin\theta}{R+r\cos\theta}.\end{aligned}$$

By the definition of spin connection $\mathbb{A}$ in subsection II.1.1, we also read

$$\mathbb{A}^1=(\mathbf{e}_1,D_{\mathbf{e}_1}\mathbf{e}_2)_{\mathbb{R}^3}\overset{\text{(V.1.10)}}{=}-\kappa_1=0,\qquad \mathbb{A}^2=(\mathbf{e}_1,D_{\mathbf{e}_2}\mathbf{e}_2)_{\mathbb{R}^3}\overset{\text{(V.1.10)}}{=}-\kappa_2=\frac{\sin\theta}{R+r\cos\theta}.$$

The explicit forms of the surface differential operators on the torus are

$$\begin{aligned}\nabla_s\alpha=g^{ii}\partial_i\alpha&=\frac{1}{r^2}(\partial_\theta\alpha)X_\theta+\frac{1}{(R+r\cos\theta)^2}(\partial_\phi\alpha)X_\phi\\ &=\frac{1}{r}(\partial_\theta\alpha)\mathbf{e}_1+\frac{1}{R+r\cos\theta}(\partial_\phi\alpha)\mathbf{e}_2,\\ \Delta_s=\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)&=\frac{1}{\sqrt{g}}\left(\partial_\theta\left(\sqrt{g}\frac{1}{r^2}\partial_\theta\right)+\partial_\phi\left(\sqrt{g}\frac{1}{(R+r\cos\theta)^2}\partial_\phi\right)\right)\\ &=\frac{1}{r^2}\partial^2_{\theta\theta}-\frac{\sin\theta}{r(R+r\cos\theta)}\partial_\theta+\frac{1}{(R+r\cos\theta)^2}\partial^2_{\phi\phi}.\end{aligned}\tag{V.1.11}$$

For $\mathbf{n}=\cos\alpha\mathbf{e}_1+\sin\alpha\mathbf{e}_2$, the explicit expression of the surface gradient $\nabla_s\mathbf{n}$ in terms of the deviation angle $\alpha$, with respect to the Darboux frame $(\mathbf{n},\mathbf{t},\boldsymbol{\nu})$ is

$$\nabla_s\mathbf{n}=\begin{pmatrix}0&0&0\\ \frac{\alpha_\theta}{r}\cos\alpha+\left(\frac{\alpha_\phi}{R+r\cos\theta}-\frac{\sin\theta}{R+r\cos\theta}\right)\sin\alpha & -\frac{\alpha_\theta}{r}\sin\alpha+\left(\frac{\alpha_\phi}{R+r\cos\theta}-\frac{\sin\theta}{R+r\cos\theta}\right)\cos\alpha & 0\\ \frac{1}{r}\cos^2\alpha+\frac{\cos\theta}{R+r\cos\theta}\sin^2\alpha & \left(\frac{\cos\theta}{R+r\cos\theta}-\frac{1}{r}\right)\sin\alpha\,\cos\alpha & 0\end{pmatrix}.$$

## V.2. Gamma-convergence: basic definitions

In this section we recall some basic results on Gamma-convergence that we used in the notes. We present only the basic definitions that we need and we refer to the books [**23**] and [**10**] for a detailed presentation of the theory.

Definition V.2.1 (De Giorgi). Let us given $(X,d)$ a metric space, a sequence $F_n$ of functions $F_n:X\to(-\infty;+\infty]$ and $F:X\to(-\infty;+\infty]$. We say that $F_n$ $\Gamma$-converges to $F$ is $(X,d)$ and write

$$F(u)=\Gamma-\lim_{n\nearrow+\infty}F_n(u)\ \forall u\in X\qquad\text{or}\qquad F_n\xrightarrow{\Gamma}F,$$

if and only if

$$\forall \{u_n\} \in X \quad \text{with } u_n \to u \quad \text{then } \liminf_{n \nearrow +\infty} F_n(u_n) \geq F(u) \tag{V.2.1}$$

$$\forall u \in X, \ \exists \{u_n\} \ \text{with } u_n \to u \ \text{ and } F_n(u_n) \to F(u). \tag{V.2.2}$$

The sequence $u_n$ in (V.2.2) is usually named *recovery sequence*.

The importance of this notion of convergence is that it entails, under reasonable assumptions, the convergence of the infima of $F_n$ to the minima of $F$. We have the following

Theorem V.1 (Fundamental Theorem of $\Gamma$ convergence). *Let $(X, d)$ a metric space and let $F_n$ be e equi-coercive sequence (see below) of functions on $X$. Let $F = \Gamma - \lim_{n \nearrow +\infty} F_n$. Then*

$$\exists \min_X F = \lim_{n \nearrow +\infty} \inf_X F_n. \tag{V.2.3}$$

*Moreover, is $u_n$ is a precompact sequence such that $\lim_{n \nearrow +\infty} F_n(u_n) = \lim_{n \nearrow +\infty} \inf_X F_n$, then every limit of a subsequence of $\{u_n\}$ is a minimum point for $F$.*

In the Theorem, equicoercive sequence means that for any $t \in \mathbb{R}$ there exists a compact set $K_t$ such that $\{F_n \leq t\} \subset K_t$ for any $n$.

# Bibliography


[1] Alicandro, R., and Cicalese, M. Variational analysis of the asymptotics of the $XY$ model. *Arch. Ration. Mech. Anal. 192*, 3 (2009), 501–536.

[2] Alicandro, R., Cicalese, M., and Gloria, A. Variational description of bulk energies for bounded and unbounded spin systems. *Nonlinearity 21*, 8 (2008), 1881–1910.

[3] Alicandro, R., De Luca, L., Garroni, A., and Ponsiglione, M. Metastability and dynamics of discrete topological singularities in two dimensions: a Γ-convergence approach. *Arch. Ration. Mech. Anal. 214*, 1 (2014), 269–330.

[4] Ball, J. M., and Zarnescu, A. Orientability and energy minimization in liquid crystal models. *Arch. Ration. Mech. Anal. 202*, 2 (2011), 493–535.

[5] Bethuel, F. Variational methods for Ginzburg-Landau equations. In *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, vol. 1713 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 1–43.

[6] Bethuel, F., Brezis, H., Coleman, B. D., and Hélein, F. Bifurcation analysis of minimizing harmonic maps describing the equilibrium of nematic phases between cylinders. *Arch. Rational Mech. Anal. 118*, 2 (1992), 149–168.

[7] Bethuel, F., Brezis, H., and Hélein, F. *Ginzburg-Landau vortices.* Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston, Inc., Boston, MA, 1994.

[8] Bethuel, F., and Zheng, X. Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal. 80*, 1 (1988), 60–75.

[9] Bowick, M. J., and Giomi, L. Two-dimensional matter: order, curvature and defects. *Adv. Phys. 58*, 5 (2009), 449–563.

[10] Braides, A. *Γ-convergence for beginners*, vol. 22 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2002.

[11] Braides, A., Cicalese, M., and Solombrino, F. $Q$-Tensor Continuum Energies as Limits of Head-to-Tail Symmetric Spin Systems. *SIAM J. Math. Anal. 47*, 4 (2015), 2832–2867.

[12] Braides, A., and Gelli, M. S. From discrete systems to continuous variational problems: an introduction. In *Topics on concentration phenomena and problems with multiple scales*, vol. 2 of *Lect. Notes Unione Mat. Ital.* Springer, Berlin, 2006, pp. 3–77.

[13] Brezis, H. New energies for harmonic maps and liquid crystals. In *Functional analysis and related topics, 1991 (Kyoto)*, vol. 1540 of *Lecture Notes in Math.* Springer, Berlin, 1993, pp. 11–24.

[14] Brezis, H. Mathematical problems related to liquid crystals, superconductors and superfluids. In *Physics on manifolds (Paris, 1992)*, vol. 15 of *Math. Phys. Stud.* Kluwer Acad. Publ., Dordrecht, 1994, pp. 11–21.

[15] Brezis, H., Coron, J., and Lieb, E. H. Harmonic maps with defects. *Comm. Math. Phys. 107*, 4 (1986), 649–705.

[16] Brezis, H., and Nirenberg, L. Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math. (N.S.) 1*, 2 (1995), 197–263.

[17] Brezis, H., and Nirenberg, L. Degree theory and BMO. II. Compact manifolds with boundaries. *Selecta Math. (N.S.) 2*, 3 (1996), 309–368. With an appendix by the authors and Petru Mironescu.

[18] Canevari, G., Segatti, A., and Veneroni, M. in preparation. 2015.

[19] Canevari, G., Segatti, A., and Veneroni, M. Morse's index formula in VMO for compact manifolds with boundary. *J. Funct. Anal. 269*, 10 (2015), 3043–3082.

[20] Chen, G., Ding, Z., Hu, C.-R., Ni, W.-M., and Zhou, J. A note on the elliptic sine-Gordon equation. *Contemp. Math. 357* (2004), 49–68.

[21] Chen, Y. M. The weak solutions to the evolution problems of harmonic maps. *Math. Z. 201*, 1 (1989), 69–74.

[22] Chen, Y. M., and Struwe, M. Existence and partial regularity results for the heat flow for harmonic maps. *Math. Z. 201*, 1 (1989), 83–103.

[23] Dal Maso, G. *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.

[24] De Gennes, P. G., and Prost, J. *The Physics of Liquid Crystals*, vol. 83 of *International Series of Monographs on Physics*. Oxford Science Publications, Oxford, 1995.

[25] do Carmo, M. P. *Differential geometry of curves and surfaces*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1976. Translated from the Portuguese.

[26] Guillemin, V., and Pollack, A. *Differential topology*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.

[27] Hardt, R., Kinderlehrer, D., and Lin, F.-H. Existence and partial regularity of static liquid crystal configurations. *Comm. Math. Phys. 105*, 4 (1986), 547–570.

[28] Hatcher, A. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[29] Hélein, F. *Harmonic maps, conservation laws and moving frames*, second ed., vol. 150 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the 1996 French original, With a foreword by James Eells.

[30] Ignat, R., Nguyen, L., Slastikov, V., and Zarnescu, A. Stability of the melting hedgehog in the landau de gennes theory of nematic liquid crystals. *Arch. Rational Mech. Anal. 215*, 2 (2015), 633–673.

[31] John, F., and Nirenberg, L. On functions of bounded mean oscillation. *Comm. Pure Appl. Math. 14* (1961), 415–426.

[32] Kralj, S., Rosso, R., and Virga, E. G. Curvature control of valence on nematic shells. *Soft Matter 7* (2011), 670–683.

[33] Kramien, R. D. The geometry of soft materials: a primer. *Rev. Mod. Phys. 74* (2002), 953.

[34] Le Dret, H., and Raoult, A. The membrane shell model in nonlinear elasticity: a variational asymptotic derivation. *J. Nonlinear Sci. 6*, 1 (1996), 59–84.

[35] Lebwohl, P. A., and Lasher, G. Nematic-liquid-crystal order¯a monte carlo calculation. *Phys. Rev. A 6* (Jul 1972), 426–429.

[36] Lee, J. M. *Introduction to smooth manifolds*, second ed., vol. 218 of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[37] Lin, F., and Wang, C. *The analysis of harmonic maps and their heat flows*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[38] Lubensky, T. C., and Prost, J. Orientational order and vesicle shape. *J. Phys. II France 2*, 3 (1992), 371–382.

[39] Majumdar, A., and Zarnescu, A. Landau-De Gennes theory of nematic liquid crystals: the Oseen-Frank limit and beyond. *Arch. Ration. Mech. Anal. 196*, 1 (2010), 227–280.

[40] Milnor, J. W. *Topology from the differentiable viewpoint*. Based on notes by David W. Weaver. The University Press of Virginia, Charlottesville, Va., 1965.

[41] Morse, M. Singular points of vector fields under general boundary conditions. *Amer. J. Math. 51*, 2 (1929), 165–178.

[42] Napoli, G., and Vergori, L. Extrinsic curvature effects on nematic shells. *Phys. Rev. Lett. 108*, 20 (2012), 207803.

[43] Napoli, G., and Vergori, L. Surface free energies for nematic shells. *Phys. Rev. E 85*, 6 (2012), 061701.

[44] Nelson, D. R. Toward a tetravalent chemistry of colloids. *Nano Lett. 2*, 10 (2002), 1125–1129.

[45] Nguyen, T.-S., Geng, J., Selinger, R. L. E., and Selinger, J. V. Nematic order on deformable vesicle:theory and simulation. *Soft Matter 9* (2013), 8314–8326.

[46] Rosso, R. Curvature effects in vesicle-particle interactions. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 459*, 2032 (2003), 829–852.

[47] Rosso, R., Virga, E. G., and Kralj, S. Parallel transport and defects on nematic shells. *Contin. Mech. Thermodyn. 24*, 4-6 (2012), 643–664.

[48] Sarason, D. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc. 207* (1975), 391–405.

[49] Segatti, A., Snarski, M., and Veneroni, M. Analysis of a variational model for nematic shells. Isaac Newton Institute Preprint, NI14037-FRB, 2014.

[50] Segatti, A., Snarski, M., and Veneroni, M. Equilibrium configurations of nematic liquid crystals on a torus. *Phys. Rev. E 90*, 1 (2014), 012501.

[51] Selinger, R. L., Konya, A., Travesset, A., and Selinger, J. V. Monte carlo studies of the xy model on two-dimensional curved surfaces. *J.Phys.Chem B 48* (2011), 12989–13993.

[52] Shkoller, S. Well-posedness and global attractors for liquid crystals on Riemannian manifolds. *Comm. Partial Differential Equations 27*, 5-6 (2002), 1103–1137.

[53] Straley, J. Liquid crystals in two dimensions. *Phys. Rev. A 4*, 2 (1971), 675–681.

[54] Turner, A. M., Vitelli, V., and D., N. Vortices on curved surfaces. *Rev. Mod. Phys. 82*, 2 (2010), 1301–1348.

[55] Virga, E. G. *Variational theories for liquid crystals*, vol. 8 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1994.

[56] Vitelli, V., and Nelson, D. Defect generation and deconfinement on corrugated topographies. *Phys. Rev. E 70*, 5 (2004), 051105.

[57] Vitelli, V., and Nelson, D. Nematic textures in spherical shells. *Phys. Rev. E 74*, 2 (2006), 021711.

[58] Vitelli, V., and Turner, M. A. Anomalous coupling between topological defects and curvature. *Phys. Rev. Lett 93*, 21 (2004), 215301.

[59] Yi, G.-R., Pine, D. J., and Sacanna, S. Recent progress on patchy colloids and their self-assembly. *J. Phys.: Condens. Matter 25*, 19 (2013), 193101.

[60] Zarnescu, A. Topics in the Q-tensor theory of liquid crystals. In *Topics in mathematical modeling and analysis*, vol. 7 of *Jindřich Nečas Cent. Math. Model. Lect. Notes*. Matfyzpress, Prague, 2012, pp. 187–252.