A VARIATIONAL APPROACH TO GRADIENT FLOWS IN METRIC SPACES

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Abstract. In this note we report on a new variational principle for Gradient Flows in metric spaces. This new variational formulation consists in a functional defined on entire trajectories whose minimizers converge, in the case in which the energy is geodesically convex, to curves of maximal slope. The key point in the proof is a reformulation of the problem in terms of a dynamic programming principle combined with suitable a priori estimates on the minimizers. The abstract result is applicable to a large class of evolution PDEs, including Fokker Plack equation, drift diffusion and Heat flows in metric-measure spaces.

1. Introduction

In this note I will present some results regarding a new variational formulation of gradient flow equations in general metric spaces. This results have been obtained in collaboration with R. Rossi, G. Savaré and U. Stefanelli and are contained in the Comptes Rendus [21] and in the forthcoming [22]. The exposition of this paper will try to keep the informal and not-so-much technical character of the seminar held at the UMI conference in Bologna in September 2011. In particular, I will not include the full proofs of the results but rather refer to the above mentioned papers.
In order to better clarify the setting of the problem and the purpose of this note, I will first introduce and comment some known results on gradient flows in Hilbert space.
Let H be a separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and consider $\phi : H \longrightarrow (-\infty, +\infty]$ a proper, convex and lower semicontinuous function. Assume we are given $\bar{u} \in D(\phi) = \{v \in H : \phi(v) < +\infty\}$, a reference time T and consider then the following gradient flow evolution

$$\begin{cases} u' + \partial\phi(u) \ni 0, \quad \text{a.e. in } (0,T), \\ u(0) = \bar{u}, \end{cases} \tag{1.1}$$

where $\partial\phi(u)$ is the convex analysis subdifferential of ϕ. Gradient flow equations arise almost ubiquitously in dissipative evolutions. More precisely, depending on the choice of the functional framework (even restricting to the classical Hilbert space formulation (1.1)) and on the choice of the functional ϕ, they come into play in a variety of applications such as heat conduction, Hele Shaw cell problem, Stefan problem, porous media, obstacle problem and variational inequalities and the mean curvature flow for Cartesian graphs, among many others.
The mathematical analysis of evolutions of the type (1.1) dates back to the late sixties with the seminal works of Kōmura [13], Crandall & Pazy [7] and Brezis [4]. Starting from then a huge literature concerning wellposedness, approximation and long time behavior of solutions originated (see, e.g., [19] and the references therein).
There are at least two main strategies to prove existence of a solution to (1.1). In the

first approach one looks for a smooth approximation of problem (1.1). More precisely, one replaces the potential ϕ with its Moreau Yosida approximation obtaining a Gateaux differentiable potential ϕ_λ ($\lambda > 0$ being the approximation parameter intended to go to zero in the limit) and consider the (smooth) gradient flow equation driven by ϕ_λ. Once the approximate problem is solved, the task is to pass to the limit as $\lambda \searrow 0$ and consequently remove the approximation. This major step is usually based on monotonicity and convexity arguments (see [4]).
The other possible method consists in performing a time discretization of the time interval and consequently replacing the time derivative with the corresponding incremental quotients. The approximate problem is now a stationary problem which is solved relying on the maximal monotonicity of $\partial\phi$. The limit procedure is again based on monotonicity and convexity arguments. The interest for this method is twofold, since it provides a first important step towards the numerical approximation of (1.1) (see [19]) and since it is extremely flexible as the method of *Minimizing movements* for gradient flow equations in metric spaces shows (see [1]).
In this note we aim at presenting a possible third way. Again, to introduce the problem, I will start from the Hilbert space case. Thus, fixing a reference time $T > 0$ and a strictly positive ε, consider the functional $I_\varepsilon \colon H^1(0,T;H) \to (-\infty, \infty]$ defined as

$$I_\varepsilon[v] := \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\big(\frac{\varepsilon}{2}\|v'(t)\|^2 + \phi(v(t))\big)\,\mathrm{d}t. \tag{1.2}$$

Note that the functional I_ε is defined on (semi) trajectories. Moreover, it is given by a weighted sum (through the exponential weight $\frac{e^{-t/\varepsilon}}{\varepsilon}$) between a *dissipation term* (i.e. $\frac{\varepsilon}{2}\|v'\|^2$) and an *energy term* (i.e. $\phi(v)$) and the result is then integrated with respect to time. For this reason the functional I_ε is named with acronym WED functional which stands for *Weighted energy dissipation* functional. Now, if we fix some $\bar{u}$ in the domain of ϕ and suppose for a while that one is able to find a curve $u_\varepsilon \colon [0,T] \to H$ such that

$$u_\varepsilon \in \operatorname{Argmin}\left\{I_\varepsilon[v],\ v \in H^1(0,T;H),\ v(0) = \bar{u}\right\},$$

we may ask what is the behavior of u_ε as $\varepsilon \searrow$ and if we are able to characterize the dynamics of its limit in terms of $\bar{u}$ and of ϕ. To get a clue of the possible behavior, one computes the corresponding Euler Equations (see [17] and subsection 3.1) and finds that u_ε is a solution of the following elliptic (in time) evolution problem

$$\begin{cases} -\varepsilon u_\varepsilon'' + u_\varepsilon' + \partial\phi(u_\varepsilon) \ni 0, \quad \text{a.e. in } (0,T), \\ u_\varepsilon(0) = \bar{u}, \\ u_\varepsilon'(T) = 0. \end{cases} \tag{1.3}$$

Consequently, one expects that, at least formally, when $\varepsilon \searrow 0$ the sequence of minimizers u_ε approaches the solution of the gradient flow (1.1). The limit as $\varepsilon \searrow 0$ in (1.3) is named *causal limit* since it restores the causality of gradient flow evolution. More precisely, the following holds

Theorem 1.1 (Variational principle in Hilbert spaces). *As $\varepsilon \searrow 0$, $u_\varepsilon \in \operatorname{Argmin}_{v\in H^1(0,T;H), v(0)=\bar{u}} I_\varepsilon[v]$ converges in $C^0([0,T];H)$ to the solution of* (1.1).

The proof of this convergence, even for λ convex functionals, has been proved by Mielke & Stefanelli in [17]. It should be pointed out that the interest in constructing solutions of the gradient flow (1.1) as limits of minimizers of I_ε is not only limited to the question of proving existence of solutions to (1.1) but rather is related to the possibility of using

tools from Calculus of Variations (such as Γ convergence and relaxation) to study (1.1). More precisely, in situations in which the energy ϕ is not lower semicontinuous (such as in models of micro structures) one may consider the functional I_ε and then consider its relaxation hoping that it could serve as an effective macroscopic description. This is one of the motivations that led Mielke & Ortiz to introduce the WED functional (1.2) in the context of rate independent evolutions ([16]). In the same paper it is also shown, via a counterexample, that for gradient evolutions of the type (1.1) the question that the limits of minimizers of the relaxed WED functional are solutions of the gradient flow for the relaxation of ϕ has to be worked out in every situation. We refer to [6] and [24] for two examples of relaxation in the contest, respectively of micro structure evolution and the evolution of the surface area for Cartesian graphs.
It is worthwhile noting that the possibility of addressing an evolution problem via functionals defined on trajectories is not new as the De Giorgi conjecture on Semi linear Wave equations ([9], and [26], [23] for its solution), the Brezis-Ekeland and the Nayroles principles (see [5], [18], [25]) and the works of Ghoussoub [11] and Visintin [27] show. Moreover, it should be recalled that also the possibility of approximating dissipative evolutions with elliptic in time approximations is well known. For this subject we refer to the pioneering work of Lions [14], to the monograph of Lions and Magenes [15] and to Ilmanen [10] for the nonlinear case.

As already pointed out, in this note I will present the WED approach to gradient flows in metric spaces. To this end, I will first briefly recall in the next Section 2 the basic concepts and notations of gradient flows in metric space. Subsequently (see Section 3), I will sketch the proof of the convergence of the WED minimizers in the case of an Hilbert space and, moving from this I will explain the convergence in the metric context. As it will be clear later on, the Hilbert space proof will be mainly based on PDEs methods while the metric one will be more variational in nature.

2. Gradient flows in metric spaces and formulation of the problem

Suppose we are given a Riemannian manifold (M, g), a smooth function $\phi\colon M \to \mathbb{R}$ and a point $\bar{u} \in M$. We call gradient flow of ϕ on (M, g) the equation

$$\begin{cases} u' + \nabla\phi(u) = 0 \text{ in } [0,T] \times M \\ u(0) = \bar{u} \end{cases} \tag{2.1}$$

Note that the metric tensor g is an essential ingredient because it allows for the identification of the differential of ϕ (a cotangent vector) with the gradient of ϕ (a tangent vector). In particular, thanks also to the smoothness of ϕ we have the Chain rule

$$\frac{d}{dt}\phi(v(t)) = g(v'(t), \nabla\phi(v(t))). \tag{2.2}$$

On the other hand, since being a solution to (2.1) is completely equivalent to

$$\frac{1}{2} g(u' + \nabla\phi(u), u' + \nabla\phi(u)) = 0 \text{ in } [0,T], \tag{2.3}$$

by expanding we obtain the equivalent formulation

$$\frac{1}{2}\|u'(t)\|_g^2 + \frac{1}{2}\|\nabla\phi(u(t))\|_g^2 + \frac{d}{dt}\phi(u(t)) = 0, \text{ in } [0,T]. \tag{2.4}$$

Note that we have equivalently reformulated the vectorial problem (2.1) in terms of the single scalar equation (2.4). Moreover, while (2.1) could make no sense in non smooth situations, its scalar counterpart (2.4) makes sense even for evolutions $u\colon [0,T] \to X$, with (X,d) a general metric space, at the price of replacing $\|u'(t)\|_g^2$ and $\|\nabla\phi(u(t))\|_g^2$ with proper metric objects. The correct notions will be that of *metric derivative* and that of *local slope*. Consequently, following De Giorgi and coworkers [8] and [1], (2.4) will be the notion of gradient flow evolution we use.
Now, we introduce these basic concepts. Given a metric space (X,d), we say that a curve $u : [0,T] \to X$ belongs to $\mathrm{AC}^2([0,T];X)$, if there exists $m \in L^2(0,T)$ such that

$$d(u(s),u(t)) \le \int_s^t m(r)\,\mathrm{d}r \quad \text{for all } 0 < s \le t < T. \tag{2.5}$$

It was proved in [1, Sec. 1.1] that for all $u \in \mathrm{AC}^2([0,T];X)$, the limit

$$|u'|(t) = \lim_{s\to t} \frac{d(u(s),u(t))}{|t-s|}$$

exists for a.a. $t \in (0,T)$. We will refer to it as the *metric derivative* of u at t. We have that the map $t \mapsto |u'|(t)$ belongs to $L^2(0,T)$ and it is minimal within the class of functions $m \in L^2(0,T)$ fulfilling (2.5). The metric derivative is then the *metric surrogate* of the norm of time derivative.
Then, let

$$\phi : X \to (-\infty,+\infty] \quad \text{be lower semicontinuous and proper}$$

and let $D(\phi) := \{u \in X \ : \ \phi(u) < +\infty\}$ denote the effective domain of ϕ. As in the Hilbertian framework, a remarkable case is when the functional ϕ enjoys some convexity properties. In a purely metric context, the notion of (λ) convexity we will use is the (λ) *convexity along geodesic*. Thus, we say that the functional ϕ is λ-geodesically convex for some $\lambda \in \mathbb{R}$, if

$$\begin{gathered}\text{for all } v_0,\ v_1 \in D(\phi) \text{ there exists a constant-speed geodesic } \gamma : [0,1] \to X\\ \text{(i.e. satisfying } d(\gamma_s,\gamma_t) = (t-s)d(\gamma_0,\gamma_1) \text{ for all } 0 \le s \le t \le 1\text{), such that}\\ \gamma_0 = v_0, \quad \gamma_1 = v_1, \quad \text{and } \phi \text{ is } \lambda\text{-convex on } \gamma, \text{ i.e.}\\ \phi(\gamma_t) \le (1-t)\phi(\gamma_0) + t\phi(\gamma_1) - \frac{\lambda}{2}t(1-t)d^2(\gamma_0,\gamma_1) \ \text{ for all } 0 \le t \le 1.\end{gathered} \tag{2.6}$$

Then, we define the *local slope* (see [8, 1]) of ϕ at $u \in D(\phi)$ as

$$|\partial\phi|(u) := \limsup_{v\to u} \frac{(\phi(u)-\phi(v))^+}{d(u,v)}. \tag{2.7}$$

The local slope will be the *metric surrogate* of the norm of gradient. Assuming that the energy is geodesically convex, we get that the local slope is indeed a strong upper gradient (see [1]) and thus it satisfies a sort of Chain rule property, namely following [1, Def. 1.2.1], we have that for for every curve $u \in \mathrm{AC}_{\mathrm{loc}}([0,+\infty);X)$, the function $|\partial\phi|(u)$ is Borel and

$$|\phi(u(t)) - \phi(u(s))| \le \int_s^t |\partial\phi|(u(r))|u'|(r)\,\mathrm{d}r \quad \text{for all } 0 < s \le t. \tag{2.8}$$

Consequently, if $|\partial\phi|(u(\cdot))|u'|(\cdot) \in L^1$ then, $\phi\circ u$ is absolutely continuous and there holds

$$|(\phi\circ u)'(t)| \le |\partial\phi|(u(t))|u'|(t) \quad \text{for a.a. } t \in (0,+\infty). \tag{2.9}$$

Then, we say that $u \in \mathrm{AC}^2_{\mathrm{loc}}([0,+\infty);X)$ is a *curve of maximal slope* for ϕ with respect to the local slope if

$$\frac{1}{2}|u'|^2(t) + \frac{1}{2}|\partial\phi|^2(u(t)) = -\frac{d}{dt}\phi(u(t)) \qquad \text{for a.a. } t \in (0,+\infty). \tag{2.10}$$

or, equivalently, using the Chain Rule (2.9), if

$$\frac{1}{2}|u'|^2(t) + \frac{1}{2}|\partial\phi|^2(u(t)) \le -\frac{d}{dt}\phi(u(t)) \qquad \text{for a.a. } t \in (0,+\infty). \tag{2.11}$$

At least when ϕ is λ geodesically convex the notion of *curve of maximal slope* with respect to the local slope is the correct metric notion of gradient flow evolution (see [1]). The question of existence, uniqueness and approximation of curves of maximal slope is fully treated in the book [1] together with the application of the theory of gradient flows in metric spaces to the case of the (metric) space of probability measures with the Wasserstein distance. In fact starting with the pioneering work of Otto [20] and of Jordan, Kinderlehrer & Otto [12], the (metric) space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures with finite second moment, endowed with the Wasserstein 2-metric, became the natural framework to highlight the gradient flow structure of a large class of evolutionary PDEs problems with nonnegative solutions $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$, in the general form

$$\partial_t u - \nabla \cdot \left(u \frac{\delta\phi(u)}{\delta u} \right) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+. \tag{2.12}$$

In the equation above $\frac{\delta\phi(u)}{\delta u}$ is the suitably defined first variation of an integral functional, resulting from the linear combination of the terms

$$\mathcal{U}(u) = \int_{\mathbb{R}^d} U(u(x))\,\mathrm{d}x, \quad \mathcal{V}(u) = \int_{\mathbb{R}^d} V(x)u(x)\,\mathrm{d}x, \quad \mathcal{W}(u) = \int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y)u(x)u(y)\,\mathrm{d}x\,\mathrm{d}y.$$

The functionals $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are generally referred to as the *internal*, the *potential*, and the *interaction* energies, respectively. Different choices of the above functionals lead to different interesting evolutions, including Fokker-Planck and the nonlinear diffusion equations (see [1]). Transport and nonlinear drift-diffusion equations (with or without nonlocal interactions) can be considered as well. As anticipated, the PDE (2.12) is by now classically reformulated as a gradient flow equation in the metric space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures with finite second moment, endowed with the Wasserstein 2-metric. Parallel to this, another possible application of our abstract results concerns with the heat flow in a Polish metric-measure space (M,d,m) satisfying the Lott-Sturm-Villani condition (see [2]): in this case $X = \mathcal{P}_2(M)$, $\phi(\mu) = Ent_m(\mu)$ is the relative entropy functional and the family of minimizers u_ε converge to the unique solution $\mu = \rho m$, $\partial_t\rho - \Delta_{m,d}\rho = 0$ (see [2] for the definition of the operator $\Delta_{m,d}$).

At this point, we are in the position to introduce the WED functional in metric spaces and state the Variational principle corresponding to Theorem 1.1. Thus, we will suppose that the energy functional

$$\begin{gathered}\phi : X \to (-\infty,+\infty] \quad \text{is proper, lower semicontinuous, geodesically convex and such that} \\ \forall v \in X \;\; \phi(v) \ge 0 \text{ and } \forall\, r>0 \text{ the set } S_r = \{u \in X \,:\, \phi(u) \le r\} \text{ is compact.}\end{gathered} \tag{2.13}$$

Then the (metric) WED functional $I_\varepsilon : \mathrm{AC}^2(0,+\infty;X) \to (-\infty,+\infty]$ is given by

$$I_\varepsilon[v] := \int_0^\infty \frac{e^{-t/\varepsilon}}{\varepsilon}\Big(\frac{\varepsilon}{2}|v'|^2(t) + \phi(v(t))\Big)\,\mathrm{d}t. \tag{2.14}$$

Note that the metric version (2.14) is formally equivalent to the Hilbert one (1.2) (up to the obvious change of the norm of the time derivative with the metric derivative) except for the fact that (2.14) is written on the whole $(0,+\infty)$.
Under the above assumptions (2.13) on ϕ one can prove (see [22] and Theorem 3.2 in this note) that fixing $\bar u \in D(\phi)$ there exists $u_\varepsilon \in \operatorname{Argmin}\big\{I^\varepsilon[v],\, v \in AC^2(0,\infty,X),\, v(0) = \bar u\big\}$. Then we have may ask the following question:

Does u_ε converges in some suitable sense to some curve u, with u being a curve of maximal slope (for the functional ϕ, with respect to the upper gradient $|\partial\phi|$ and originating from $\bar u$)?

I will answer to the above question in the next Section 3 in the simplified case of a functional ϕ satisfying the assumptions (2.13). For the treatment of more general situations (for example λ geodesically convex potentials) and for the complete proofs, I refer to [22].

3. Sketch of the proof

3.1. **The Hilbert space setting.** The proof of Theorem 1.1 in the Hilbert space framework can be roughly divided into four main steps: *existence of minimizers, deduction of the Euler-Lagrange equation* (1.3)*, uniform a priori estimates, passage to the limit*. Due to the possibly lack of smoothness of the potential ϕ some care must be reserved to the rigorous deduction of the Euler Lagrange equation (1.3) and of the a priori estimates. To this end, a careful combination of the Yosida approximation and of a time discretization scheme is needed (see [17]). The proof given in [17] permits to treat quite general conditions on ϕ and on the initial condition $\bar u$. In particular, one can consider λ convex functionals and more general initial data (even sequences) $\bar u^\varepsilon$ than the mere $\bar u \in D(\phi)$. Here, to keep the presentation as simple as possible we will simply assume that the potential ϕ is convex, lower semicontinuous and bounded from below and that $\bar u \in D(\phi)$. As a consequence, the existence of a unique minimizer $u_\varepsilon \in \operatorname{Argmin}\big\{I_\varepsilon[v],\ v \in H^1(0,T;H),\ v(0) = \bar u\big\}$, with $\bar u \in D(\phi)$ and $\varepsilon > 0$ fixed, follows in a standard way from the Direct Method of Calculus of Variations since the WED functional I_ε is itself (strictly) convex and lower semicontinuous in $H^1(0,T;H)$. Once we have a minimizer u_ε, by smoothing the potential ϕ with its Moreau-Yosida approximation and then passing to the limit, in [17] it is shown that u_ε satisfies the Euler Lagrange equation (1.3), namely

$$\begin{cases} -\varepsilon u_\varepsilon'' + u_\varepsilon' + \xi_\varepsilon = 0, \quad \text{a.e. in } (0,T), \\ \xi_\varepsilon \in \partial\phi(u_\varepsilon), \\ u_\varepsilon(0) = \bar u, \\ u_\varepsilon'(T) = 0. \end{cases} \tag{3.1}$$

At this point, evaluating the H norm of the left hand side of (3.1) and performing some standard manipulation one can get the key estimate

$$\varepsilon\|u_\varepsilon''\|_{L^2(0,T;H)} + \varepsilon^{1/2}\|u_\varepsilon'\|_{L^\infty(0,T;H)} + \|u_\varepsilon'\|_{L^2(0,T;H)} + \|\xi_\varepsilon\|_{L^2(0,T;H)} \le C, \tag{3.2}$$

with C a positive constant depending on $\|\bar{u}\|$ but independent of ε. Note that this argument is only formal and requires some smoothness for ξ_ε. For this reason, (3.2) is first proved in a time discretization scheme and then the continuous version is obtained in the limit (see [17] for the details of the argument). Estimate (3.2) is the core of the convergence as it entails enough compactness to obtain the existence of a $u \in H^1(0,T;H)$ such that

$$u_\varepsilon \xrightarrow{\varepsilon\searrow 0} u \text{ in } C^0([0,T];H).$$

Then, passing to the limit in (3.1) and using standard monotonicity arguments we can get that u is the gradient flow for ϕ starting from $\bar{u}$, i.e. u solves (1.1). Theorem 1.1 is then proven.

3.2. **The Metric space setting.** In this subsection I will answer to the question raised at the end of Section 2 by proving (actually sketching the proof)

Theorem 3.1 (Variational principle in Metric spaces)**.** *As $\varepsilon \searrow 0$, u_ε locally uniformly converges to a curve of maximal slope for ϕ, with respect to the (local) slope $|\partial\phi|$ and starting from $\bar{u}$.*

I will prove the Theorem under the simplified assumptions (2.13). In the paper [22] more general situations are considered. First of all, we have to show that the minimization problem for the metric WED functional I_ε in (2.14) on the set of those v such that $v \in AC^2(0,T;X)$ with $v(0) = \bar{u}$ is well posed. We have the following

Theorem 3.2 (Existence of minimizers)**.** *Let (X,d) be a metric space and let $\phi : X \to (-\infty,+\infty]$ be a functional satisfying* (2.13)*.*
Then for any $\varepsilon > 0$, the set $\operatorname{Argmin}\big\{I_\varepsilon[v],\ v \in H^1(0,T;H),\ v(0) = \bar{u}\big\}$ *is not empty.*

The proof of this result is based on the Direct method of calculus of variations. Note that, even assuming that the functional ϕ is convex, the uniqueness of the minimizer of I_ε is not guaranteed by the above Theorem. In fact, contrary to the Hilbert space case, the quadratic term related to the metric derivative is not necessarily convex in the general metric setting. Now, if one wants to follow the path of the proof of the Hilbert case, the next step would be to determine some Euler Lagrange equation for the minimizers u_ε. Coming back to the Hilbertian framework, it is important to note that the Euler Lagrange equation (3.1) is fundamental for the following reason: it is the equation where both the a priori estimate (3.2) and the limit procedure are performed. Thus, the next step is to find a good metric surrogate of equation (3.1). Recall that (3.1) is obtained computing the first variation of the functional I_ε. Unfortunately, in a general metric framework the only variations we can do are inner variations with respect to the time variable. As a result, by performing these variations in the metric WED functional I_ε (2.14) we obtain (see [22] for the details of the proof) that the minimizers u_ε verify, for any $\varepsilon > 0$,

$$|u_\varepsilon'|^2(t) + \frac{d}{dt}\Big(\phi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u_\varepsilon'|^2(t)\Big) = 0 \text{ for a.e. } t \in (0,+\infty). \tag{3.3}$$

We will call (3.3) *metric inner variation equation.* Regarding (3.3), note that if we were in a Hilbert space, then the corresponding *inner variation* equation could be obtained by

simply testing (3.1) with u'_ε. Consequently, it is extremely reasonable to expect that (3.3) gives less information than (3.1), in an Hilbert space. Another indication of this fact can be inferred by considering the target equation describing a curve of maximal slope. More precisely, comparing (2.10) with (3.3), we observe that the latter contains no information on the local slope $|\partial\phi|$ which is an ingredient in the definition of a curve of maximal slope. All these reasons suggest that (3.3) is an important, although only intermediate, step in the understanding of the limit behavior of u_ε but it should be complemented with other ingredients. As it will be clear in a moment this will be also related to the possibility of obtaining the a priori estimates on u_ε needed for the limit procedure.
In a sense, the turning point is to look at the value of the minimum of (2.14) instead at the minimum point u_ε. More precisely, for any $\varepsilon > 0$ we consider the functional $V_\varepsilon \colon X \to \mathbb{R}$ defined by

$$V_\varepsilon(\bar u) := \min_{v\in AC^2(0,\infty;X),v(0)=\bar u} \int_0^\infty \frac{e^{-t/\varepsilon}}{\varepsilon}\Big(\frac{\varepsilon}{2}|v'|^2(t) + \phi(v(t))\Big)\,\mathrm{d}t. \tag{3.4}$$

As in control theory (see [3]) the functional V_ε is named *value functional.* In a moment, we will see other important analogies with the theory of optimal control. Besides control theory, it is also important to note that the *value function* related to the WED functional I_ε share some properties with the Yosida approximation. In fact, the following holds (see [22])

Proposition 3.3. *Under assumptions* (2.13) *the functional* V_ε *defined in* (3.4) *verifies*

(1) V_ε *is lower semicontinuous in* X *and is continuous on the sub levels of* ϕ
(2) $0 \le V_\varepsilon(\bar u) \le \phi(\bar u)$ *for any* $\bar u \in X$
(3) $V_{\varepsilon_1}(\bar u) \le V_{\varepsilon_0}(\bar u)$ *for any* $\bar u \in X$ *and for any* $\varepsilon_1 \ge \varepsilon_0$

The proofs of (2), of the continuity of V_ε and of (3) follow from testing the minimum problem (3.4) with suitable competitors. As an example, choosing the constant curve $v(t) = \bar u$ for any $t \ge 0$ in (3.4) and noting that the exponential weight integrates to one, we readily get $V_\varepsilon(\bar u) \le \phi(\bar u)$ for any $\bar u$. On the other hand, the proof of the semicontinuity is similar to the proof of Theorem 3.2. Note that, except for the continuity of V_ε, all the above listed properties continue to hold even for non convex energies ϕ. Moreover, using the *metric inner variation equation* (3.3), we deduce the following

Proposition 3.4. *Under assumptions* (2.13)*, for every* $u_\varepsilon \in \mathrm{Argmin}_{v\in AC^2(0,\infty;X),v(0)=\bar u} I_\varepsilon[v]$ *there holds*

$$\text{the map } t \mapsto V_\varepsilon(u_\varepsilon(t)) \text{ is absolutely continuous on } (0,+\infty)\text{, and} \tag{3.5}$$

$$\frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) + \frac{1}{2}|u'_\varepsilon|^2(t) = \frac{1}{\varepsilon}\phi(u_\varepsilon(t)) \quad \text{for all } t \in [0,+\infty). \tag{3.6}$$

Then, combining (3.6) with the metric inner variation equation (3.3), we immediately get

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(u_\varepsilon(t)) + |u'_\varepsilon|^2(t) = 0 \quad \text{for a.a. } t \in (0,+\infty),$$

which produces, upon integration,

$$V_\varepsilon(u_\varepsilon(t)) + \int_s^t |u'_\varepsilon|^2(r)\,\mathrm{d}r = V_\varepsilon(u_\varepsilon(s)) \quad \text{for all } 0 \le s \le t < +\infty. \tag{3.7}$$

Consequently, recalling that $0 \le V_\varepsilon(\bar u) \le \phi(\bar u)$ for any $\bar u \in X$, we obtain the a priori estimates

$$\begin{cases} \int_0^\infty |u_\varepsilon'|^2(t)\,\mathrm{d}t \le \phi(\bar u) + C_0, \\ \int_0^T \phi(u_\varepsilon(t))\,\mathrm{d}t \le (\phi(\bar u) + C_0)\left(T + \frac{\varepsilon}{2}\right) \end{cases} \quad \text{for all } T \ge 0 \text{ and } \varepsilon > 0. \tag{3.8}$$

As before, Proposition 3.4 and estimates (3.7)-(3.8) work also without convexity conditions.
Now, note that taking $\bar u \in X$ and $u_\varepsilon \in \mathrm{Argmin}_{v \in AC^2(0,+\infty;X), v(0)=\bar u} I_\varepsilon[v]$ we have

$$V_\varepsilon(\bar u) \ge \int_0^\infty \frac{e^{-t/\varepsilon}}{\varepsilon}\phi(u_\varepsilon(t))\,\mathrm{d}t = \int_0^\infty e^{-s}\phi(u_\varepsilon(\varepsilon s))ds. \tag{3.9}$$

Thus, since for any $s \in (0,+\infty)$ there holds

$$d(u_\varepsilon(\varepsilon s), \bar u) \le \int_0^{\varepsilon s} |u_\varepsilon'|(t)\,\mathrm{d}t \le \sqrt{\varepsilon s}\, \| |u_\varepsilon'| \|_{L^2(0,\infty)} \le \sqrt{\varepsilon s(\phi(\bar u) + C_0)}, \tag{3.10}$$

where we have used (3.8) in the latter inequality, we arrive at

$$u_\varepsilon(\varepsilon s) \xrightarrow{\varepsilon \downarrow 0} \bar u \quad \text{for any} \quad s \in (0,+\infty).$$

Hence, taking the $\liminf$ when $\varepsilon \downarrow 0$ of the two sides of (3.9), using the lower semicontinuity of ϕ and the Fatou's lemma, we obtain that

$$\liminf_{\varepsilon \downarrow 0} V_\varepsilon(\bar u) \ge \phi(\bar u),$$

which is enough to conclude the *monotone* convergence

$$V_\varepsilon(\bar u) \nearrow \phi(\bar u) \text{ as } \varepsilon \searrow 0, \text{ for every } \bar u \in X, \tag{3.11}$$

since we already know that $V_\varepsilon(\bar u) \le \phi(\bar u)$ and the monotonic character of V_ε. With a similar argument we can also prove the following

$$\left(\bar u_\varepsilon \to \bar u \text{ in } X, \text{ with } \sup_\varepsilon \phi(u_\varepsilon) \le C < \infty\right) \Rightarrow \phi(\bar u) \le \liminf_{\varepsilon \downarrow 0} V_\varepsilon(\bar u_\varepsilon). \tag{3.12}$$

The converges (3.11) and (3.12) will be fundamental in the limit procedure. To this end, note that the first of (3.8) gives some equicontinuity for the sequence u_ε. Thus, to have compactness we have to obtain a (pointwise) control on $\phi \circ u_\varepsilon$ (recall that ϕ has compact sub levels). This is the content of the following Lemma,

Lemma 3.5. *Under the assumptions* (2.13) *let* $u_\varepsilon \in \mathrm{Argmin}_{v \in AC^2(0,\infty;X), v(0)=\bar u} I_\varepsilon[v]$. *Then,*

$$t \mapsto |u_\varepsilon'|^2(t) \textit{ and } t \mapsto \phi(u_\varepsilon(t)) \textit{ are decreasing} \tag{3.13}$$

$$t \mapsto \phi(u_\varepsilon(t)) \textit{ is convex} \tag{3.14}$$

$$\phi(u_\varepsilon(t)) \le \phi(\bar u) \ \forall t \ge 0. \tag{3.15}$$

The main difficulty in the proof of the Lemma relies in showing (3.13) and the fact that the geodesic convexity of ϕ in X transfers to a convexity of $\phi \circ u_\varepsilon$ with respect to time. Once one proves (3.13) and (3.14) then (3.15) follows from a contradiction argument. It is important to observe that the above Lemma works only if we assume some geodesic convexity on the energy ϕ. It is an open problem, currently under investigation, to prove a (uniform w.r.t. ε) estimate on $\phi \circ u_\varepsilon$ in the case in which no convexity is assumed on ϕ.

Now, we come to the fundamental property of the *value function* V_ε. Guided by the analogy with infinite horizon problems we have the following

Proposition 3.6 (Dynamic programming principle)**.** *For every $T > 0$ there holds*

$$V_\varepsilon(\bar u) = \min_{v\in AC^2(0,\infty;X), v(0)=\bar u}\left(\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\left(\frac{\varepsilon}{2}|v'|^2(t)+\phi(v(t))\right)\mathrm{d}t + V_\varepsilon(v(T))e^{-T/\varepsilon}\right). \tag{3.16}$$

In particular, every $u_\varepsilon \in \operatorname{Argmin}\left\{I_\varepsilon[v], v\in AC^2(0,\infty;X), v(0)=\bar u\right\}$ is a minimizer for (3.16)*. Hence, there holds*

$$V_\varepsilon(\bar u) = \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\left(\frac{\varepsilon}{2}|u_\varepsilon'|^2(t)+\phi(u_\varepsilon(t))\right)\mathrm{d}t + V_\varepsilon(u_\varepsilon(T))e^{-T/\varepsilon} \quad \textit{for all } T>0. \tag{3.17}$$

Quoting Bardi & Capuzzo Dolcetta ([3]) we can say that to achieve the minimum $V_\varepsilon(\bar u)$ is necessary and sufficient to behave as follows:

(1) let the system evolve for a finite time T along an arbitrary trajectory.
(2) pay the corresponding cost, that is $\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\left(\frac{\varepsilon}{2}|v'|^2(t)+\phi(v(t))\right)\mathrm{d}t$
(3) pay what it remains to pay in the best possible way, that is $V_\varepsilon(v(T))e^{-T/\varepsilon}$
(4) minimize over all possible trajectories.

The proof of this crucial Proposition is, *mutatis mutandis*, analogous to the proof of the finite dimensional analogous in [3]. In particular, does not rely on any form of convexity for ϕ. The Dynamic programming principle (3.17) is the core of our limiting analysis. In fact, from (3.17) one can deduce the following chain of equalities

$$\int_0^t e^{-r/\varepsilon}\left(\frac{1}{2}|u_\varepsilon'|^2(r)+\frac{1}{\varepsilon}\phi(u_\varepsilon(r))\right)\mathrm{d}r = V_\varepsilon(u_\varepsilon(0)) - V_\varepsilon(u_\varepsilon(t))e^{-t/\varepsilon}$$
$$= -\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left(V_\varepsilon(u_\varepsilon(r))e^{-r/\varepsilon}\right)\mathrm{d}r = -\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left(V_\varepsilon(u_\varepsilon(r))\right)e^{-r/\varepsilon}\,\mathrm{d}r - \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}(e^{-r/\varepsilon})V_\varepsilon(u_\varepsilon(r))\,\mathrm{d}r,$$

form which it follows, using the Lebesque Theorem, the following

$$-\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(t) + \frac{1}{\varepsilon}\phi(u_\varepsilon(t)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a.}\, t\in(0,+\infty). \tag{3.18}$$

This relation is extremely important for the passage to the limit procedure and plays the role of the Euler equation in the Hilbert space case. To see this, note that from (3.8) and (3.15) (recall that ϕ has compact sub levels) we get that (cf. [1, Proposition 3.3.1]) there exists a limiting curve $u\colon [0,\infty)\to X$ and a not relabeled subsequence such that

$$u_\varepsilon(t)\to u(t) \text{ in } X, \text{ for any } t\in[0,+\infty). \tag{3.19}$$

On the other hand, following the same argument as in [1, Theorem 2.3.3], one obtains that $u\in AC^2(0,\infty;X)$ and that

$$\int_0^T |u'|^2(t)\,\mathrm{d}t \le \int_0^T |u_\varepsilon'|^2(t)\,\mathrm{d}t, \text{ for any } T>0. \tag{3.20}$$

Now, if we integrate (3.18) between 0 and a generic time t, we get

$$V_\varepsilon(u_\varepsilon(t)) + \frac{1}{2}\int_0^t |u_\varepsilon'|^2(s)\,\mathrm{d}s + \int_0^t \frac{1}{\varepsilon}\left(\phi(u_\varepsilon(s)) - V_\varepsilon(u_\varepsilon(s))\right)\mathrm{d}s = V_\varepsilon(\bar u), \quad \text{for any } t\ge 0. \tag{3.21}$$

Recall that from (3.12) combined with (3.15), we have that

$$\phi(u(t)) \le \liminf_{\varepsilon\searrow 0} V_\varepsilon(u_\varepsilon(t)) \text{ for any } t \ge 0 \text{ and } V_\varepsilon(\bar u) \nearrow \phi(\bar u) \text{ as } \varepsilon \searrow 0. \tag{3.22}$$

Thus, taking the limit in (3.21) we get

$$\phi(u(t)) + \frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s + \liminf_{\varepsilon\searrow 0}\int_0^t \frac{1}{\varepsilon}\left(\phi(u_\varepsilon(s)) - V_\varepsilon(u_\varepsilon(s))\right)\mathrm{d}s \le \phi(\bar u) \quad \text{for any } t \ge 0, \tag{3.23}$$

which is *almost* the integrated version of (2.11) up to the identification of the lim inf term with $\frac{1}{2}\int_0^t |\partial\phi|^2(u(s))\,\mathrm{d}s$. This is contained in the following

Proposition 3.7. *Under assumptions* (2.13)*, there holds that*

$$\frac{1}{2}\int_0^t |\partial\phi|^2(u(s))\,\mathrm{d}s \le \liminf_{\varepsilon\searrow 0}\int_0^t \frac{1}{\varepsilon}\left(\phi(u_\varepsilon(s)) - V_\varepsilon(u_\varepsilon(s))\right)\mathrm{d}s, \text{ for any } t > 0. \tag{3.24}$$

Moreover, the quantity $\frac{1}{\varepsilon}\left(\phi(u_\varepsilon(s)) - V_\varepsilon(u_\varepsilon(s))\right)$ *is related to the slope of* V_ε *in the sense that* V_ε *satisfies, for any* $v \in X$ *the Hamilton Jacobi equation*

$$\frac{1}{2}|\tilde\partial V_\varepsilon|^2(v) = \frac{1}{\varepsilon}\left(\phi(v) - V_\varepsilon(v)\right), \tag{3.25}$$

where $|\tilde\partial V_\varepsilon|(v) := \limsup_{w\to v,\,\phi(w)\to\phi(v)} \frac{(V_\varepsilon(v)-V_\varepsilon(w))^+}{d(v,w)}$, *for* $v \in X$

Note that the equation (3.25) is the metric version of the classical Hamilton-Jacobi-Bellman equation for the infinite horizon problem in finite dimensions (see [3, Chapter 3, Proposition 2.8]). The importance of this Proposition is twofold. One one side, it allows to conclude the limit procedure in (3.23) and obtain that the curve u in (3.19) satisfies

$$\phi(u(t)) + \frac{1}{2}\int_0^t |u'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial\phi|^2(u(s))\,\mathrm{d}s \le \phi(\bar u) \text{ for any } t \ge 0, \tag{3.26}$$

and hence it is a curve of maximal slope for the energy ϕ with respect to the local slope $|\partial\phi|$ and starting from $\bar u$. This fact, in particular, concludes the proof of Theorem 3.1. On the other hand, the Hamilton Jacobi equation (3.25) sheds some new light on the *value function* V_ε and on the dynamics (for $\varepsilon > 0$ fixed) of the minimizers u_ε. In fact, plugging (3.25) into (3.18) we immediately get that

$$-\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|u_\varepsilon'|^2(t) + \frac{1}{2}|\tilde\partial V_\varepsilon|^2(u_\varepsilon(t)) \text{ for a.a. } t > 0, \tag{3.27}$$

i.e., u_ε is a curve of maximal slope for V_ε with respect to the slope $|\tilde\partial V_\varepsilon|$. At this point, the definition of the slope $|\tilde\partial V_\varepsilon|$ may look a bit obscure. In particular, one may wonder under which conditions the new slope $|\tilde\partial V_\varepsilon|$ coincides with the usual slope (2.7). A sufficient condition for this is that the value funtion V_ε is geodesically convex (see [22]).

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