Analysis of a solid-solid phase change model coupling hyperbolic momentum balance and diffusive phase dynamics *

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Abstract

This work deals with a nonlinear system modelling solid-solid phase transitions and mechanical deformations in shape memory alloys. The model is studied in the non-stationary case and accounts for local microscopic interactions between the different phases introducing the gradients of the phase parameters as state variables. By using an approximation - a priori estimates - passage to the limit procedure we prove existence and uniqueness of a weak solution to the resulting initial-boundary value problem and we give some regularity results. Moreover, continuous dependence on data of the solutions is proved under stronger regularity assumptions on data.

1 Introduction

This paper deals with a nonlinear system describing the behaviour of shape memory alloys subjected to mechanical treatments when the temperature field is known in the space-time domain. Shape memory alloys are metallic alloys which can be permanently deformed by mechanical actions and then recover their original shape just by thermal means. The phenomenon has been interpreted (see, e.g., [1, 16]), at a microscopic scale, as the effect of a structural phase transition between two different configurations of the metallic lattice, the austenite (prevailing at high temperature) and its shared counterpart, termed martensite (prevailing at low temperature). Here, we investigate the phenomenon at the macroscopic scale and assume the phases to coexist at each point with appropriate proportions. Furthermore, we suppose that just two martensitic variants are present together with one austenite (actually, in three dimensions, 24 different martensitic variants have been detected). In the literature, several models have been introduced to achieve an efficient and predictive description of the shape memory effect (we refer, for an exhaustive presentation of these models, to [14] and references therein). In this work we follow the approach proposed by Frémond [13, 14], deriving the model by continuum mechanics laws. In this direction, we take as state variables the absolute temperature ϑ (which is assumed to be known), the

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linearized strain tensor $\varepsilon(\boldsymbol{u})$ ($\boldsymbol{u}$ is the vector of small displacements), and the volumetric ratios of the two martensitic (β_1, β_2) and of the austenite (β_3) variants. In particular, we ask these last quantities to fulfill the constraint

$$\beta_1 + \beta_2 + \beta_3 = 1, \qquad 0 \le \beta_i \le 1 \quad \text{for} \quad i = 1, 2, 3. \tag{1.1}$$

From a physical point of view, (1.1) means that we are requiring no void nor overlapping between the phases. Then, we introduce two linearly independent variables (χ_1, χ_2), related to the phase proportions as follows

$$\chi_1 := \beta_1 + \beta_2, \quad \chi_2 := \beta_2 - \beta_1,$$

so that (1.1) implies

$$(\chi_1, \chi_2) \in C := \left\{ (\gamma_1, \gamma_2) \in \mathbb{R}^2 \text{ such that } |\gamma_2| \le \gamma_1 \le 1 \right\}. \tag{1.2}$$

Thus, assuming that the temperature is known in the space-time domain, by the principle of virtual power written for microscopic and macroscopic forces (cf. [14]), we get the equation governing the evolution of the unknowns. In particular, it is worthwhile to note that we write the momentum balance accounting also for macroscopic accelerations. Concerning a complete thermomechanical derivation of the model, the reader can refer to [5, 14]. We point out that, as the temperature is known, we address only the equations governing the evolution of the phase parameters and the macroscopic displacements. See, e.g., [14] for the expression of the energy balance and for the physical meaning of the constants $\nu, \lambda, \mu, k, \eta, \ell, \vartheta^*$ involved in the equations. Here, we let these constants be strictly positive, except for ν, that could be eventually equal to zero. The momentum balance and the equation governing the phase dynamics read, respectively,

$$\boldsymbol{u}_{tt} - \operatorname{div}\big(-\nu\Delta(\operatorname{div}\boldsymbol{u})\,\boldsymbol{I} + \lambda \operatorname{div}\boldsymbol{u}\,\boldsymbol{I} + 2\mu\varepsilon(\boldsymbol{u}) + \alpha(\vartheta)\chi_2\,\boldsymbol{I}\big) = \boldsymbol{G}, \tag{1.3}$$

$$k \begin{pmatrix} \chi_{1t} \\ \chi_{2t} \end{pmatrix} - \eta \begin{pmatrix} \Delta\chi_1 \\ \Delta\chi_2 \end{pmatrix} + \begin{pmatrix} \frac{\ell}{\vartheta^*}(\vartheta - \vartheta^*) \\ \alpha(\vartheta)\operatorname{div}\boldsymbol{u} \end{pmatrix} + \partial I_C(\chi_1, \chi_2) \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.4}$$

The latter relations are intended to be fulfilled a.e. in $Q = \Omega \times (0, T)$, where Ω is an open and bounded closed set of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and T stands for some final time. The symbol I_C in (1.4) represents the indicator function of C (namely $I_C = 0$ if $(\chi_1, \chi_2) \in C$, $I_C = +\infty$ otherwise) and $\partial I_C : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ stands for the subdifferential of I_C, that turns out to be a maximal monotone operator (we refer to [6] for the details). We recall that ∂I_C accounts for the constraint (1.2) as ∂I_C is defined only for $(\chi_1, \chi_2) \in C$ and it is $\partial I_C(\chi_1, \chi_2) = (0, 0)$ if (χ_1, χ_2) belongs to the interior of C and coincides with the cone of the normal vectors to the boundary at the point (χ_1, χ_2), if (χ_1, χ_2) lies on the boundary of C. In addition, $\boldsymbol{I}$ stands for the identity matrix in $\mathbb{R}^3$. As usual, the linearized strain tensor ε is given by,

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for} \quad i = 1, 2, 3.$$

Furthermore, $\alpha : \mathbb{R} \to \mathbb{R}$, is a given function of the temperature which accounts for the energy associated to the termal expansion during the phase transition, while $\boldsymbol{G} : Q \to \mathbb{R}^3$ stands for an applied volume force. The system (1.3 - 1.4) has to be supplied by suitable initial and boundary conditions. Thus, we prescribe

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}^0 \quad \boldsymbol{u}_t(\cdot, 0) = \boldsymbol{w}^0 \quad \chi_1(\cdot, 0) = \chi_1^0 \quad \chi_2(\cdot, 0) = \chi_2^0. \tag{1.5}$$

Denoting by ∂_n the outward normal derivative to the boundary $\partial\Omega$, we fix

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T) \tag{1.6}$$

$$\partial_n(\nu \mathrm{div}\boldsymbol{u}) = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{1.7}$$

$$\partial_n \chi_j = 0 \quad \text{on } \partial\Omega \times (0,T) \;\; j = 1,2. \tag{1.8}$$

We point out that (1.6) corresponds to assume that the body is fixed on its boundary, while (1.7) means that no double forces are applied on the surface. Hence, (1.8) corresponds to the fact that no surface forces involving microscopic motions are applied on $\partial\Omega$.

Let us point out that the main novelty, if we compare this paper with literature, is the presence of the inertial term $\boldsymbol{u}_{tt}$ in the macroscopic balance equation (1.3). Indeed, in related works the term $\boldsymbol{u}_{tt}$ is omitted (cf. [3] for an analogous problem in the quasi-stationary case). Furthermore, we include a diffusion term in the variational inequality governing the kinetics of phase proportions. This position, very useful from a mathematical point of view, is also physically justified as it corresponds to assume that the microstructure of the material at one point is influenced by its neighborhood [5, 14]. We note that this term has already been included in [4], where the model with thermal memory in the heat flux law is studied.

In this paper, we will prove an existence result for the system (1.3 - 1.4) under suitable initial and boundary conditions for data, as well as a regularity result (Theorem **2.2**). Moreover, under stronger assumptions, we prove the continuous dependence on data (Theorem **2.4**) from which it follows uniqueness of the solution. We point out that the above results work also if we omit the regularizing fourth order term from the momentum equation (1.3). This term comes form the second gradient theory and accounts for mechanical actions exerted on internal surfaces. Nonetheless, our result is interesting both from a mathematical and mechanical point of view as this term provides a regularity for the solutions we can avoid in our analysis. Moreover, this term is usually neglected in the first gradient theory, which is the usual framework for applications.

Although we deal with the case of known temperature, which is an interesting situation from a physical and an experimental point of view, we point out that Frémond's model is complemented by the energy equation (cf. [14]). The resulting system is highly nonlinear, especially due to nonlinearities in the energy balance coupling temperature, displacements and phase fractions. In the last years, several works have been dedicated to the study of this problem. For the sake of simplicity, we mention the ones strictly connected with our paper and refer to them for the general theory (we point out that the existence of a solution to the three dimensional problem with full momentum, i.e including the term $\boldsymbol{u}_{tt}$, and nonlinearities in the energy balance, as far as we know, is still an open problem). Among the others, let us recall [8], where the author finds an existence result for the n-dimensional problem with the full momentum and the linearized energy balance equation, while in [12] and in [19] the authors give an existence and uniqueness result for the full model in the one-dimensional setting. In [10], a first existence and uniqueness result for the quasi-static situation was given in the case when all the nonlinearities in the energy balance equation are neglected. In [9], the author shows the existence of a unique solution to the full quasi-static three dimensional model under suitable regularity and compatibility assumptions on the thermal expansion coefficient, which are coherent with realistic data. Finally, for the sake of completeness, we should quote some numerical results concerning the error control of the time discretization scheme, with variable time-step, approximating the system (1.3-1.4) (cf. [18]). We remind that the investigation of the error control for problems related to Frémond model has recently received a good deal of interest, as the papers [20], [21] show.

Here is the outline of the paper. In section 2, we present system (1.3 - 1.8) as an abstract Cauchy problem and formulate the main results of the paper. Section 3 brings the proof of the existence and regularity of a solution to the problem. Finally, section 4 contains the proof of the continuous dependence on data along with the uniqueness result.

2 Continuous problem and main results

In this section, we face the system (1.3 - 1.4) in an abstract setting as a Cauchy problem for two coupled evolution equations and we state the main results of the paper.

Let us introduce four Hilbert spaces $H, V, \boldsymbol{H}$, and $\boldsymbol{V}$ specified as follows

$$\begin{aligned} &H := L^2(\Omega), \quad V := H^1(\Omega), \\ &\boldsymbol{H} := (L^2(\Omega))^3, \quad \boldsymbol{V} := \left\{ \boldsymbol{v} \in (H_0^1(\Omega))^3 : \nu \mathrm{div} \boldsymbol{v} \in V \right\}. \end{aligned}$$

As usual, we identify H and $\boldsymbol{H}$ with their respective duals H' and $\boldsymbol{H}'$ so that $V \subset H \subset V'$ and $\boldsymbol{V} \subset \boldsymbol{H} \subset \boldsymbol{V}'$ turn out to be standard Hilbert triplets. The symbol $(\cdot,\cdot)_\Omega$ denotes the scalar product in H or in $\boldsymbol{H}$, while $< \cdot, \cdot >$ indicates the duality pairing between V' and V and between $\boldsymbol{V}'$ and $\boldsymbol{V}$. The symbol $\| \cdot \|_E$, will indicate the norm in the generic normed vector space E. The space $\boldsymbol{V}$ is naturally endowed with the hilbertian norm

$$\| \boldsymbol{v} \|_{\boldsymbol{V}}^2 := \sum_{i=1}^{3} \| \nabla v_i \|_{\boldsymbol{H}}^2 + \nu \| \nabla(\mathrm{div} \boldsymbol{v}) \|_{\boldsymbol{H}}^2 . \tag{2.1}$$

We point out that the factor ν in (2.1), allows us to consider in the analysis at the same time both the case $\nu = 0$ (i.e. the fourth order term in (1.3) is omitted) and the case $\nu > 0$. We observe that if $\nu = 0$, the space $\boldsymbol{V}$ simply reduces to $(H_0^1(\Omega))^3$.
Next, we introduce a continuous and symmetric bilinear form defined, for all $\boldsymbol{v}_1, \boldsymbol{v}_2$ in $\boldsymbol{V}$, as follows

$$a(\boldsymbol{v}_1, \boldsymbol{v}_2) := \nu \int_\Omega \nabla(\mathrm{div}\, \boldsymbol{v}_1) \cdot \nabla(\mathrm{div}\, \boldsymbol{v}_2) + \lambda \int_\Omega \mathrm{div}\, \boldsymbol{v}_1 \, \mathrm{div}\, \boldsymbol{v}_2 + 2\mu \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\boldsymbol{v}_1)\, \varepsilon_{ij}(\boldsymbol{v}_2). \tag{2.2}$$

Recalling the Korn inequality (see, e.g. [7, pag. 291]), we infer that there exists a positive constant $c_{\mathbf{v}}$ such that,

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq c_{\mathbf{v}} \| \boldsymbol{v} \|_{\mathbf{V}}^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3}$$

Moreover, it is a standard matter to verify that

$$\| \mathrm{div} \boldsymbol{v} \|_H^2 \leq 3 \| \boldsymbol{v} \|_{\boldsymbol{V}}^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.4}$$

Now, we generalize the set C introduced in (1.2) and consider a bounded convex set $\mathcal{K} \subset \mathbb{R}^2$ such that $0 \in \mathcal{K}$. Then, we introduce the corresponding convex in H^2

$$K := \left\{ (\gamma_1, \gamma_2) \in H^2 : (\gamma_1, \gamma_2) \in \mathcal{K} \text{ a.e. in } \Omega \right\}, \tag{2.5}$$

and observe that there exists a positive constant c_κ, which depends only on $\mathcal{K}$, such that

$$\{|\gamma_1(x)|^2 + |\gamma_2(x)|^2\}^{1/2} \leq c_\kappa, \quad \text{for a.e. } x \in \Omega, \ \forall\, (\gamma_1, \gamma_2) \in K. \tag{2.6}$$

Now, to put the problem in the abstract setting of the above Hilbert spaces, we introduce the operators

$$\mathcal{H} : \boldsymbol{V} \to \boldsymbol{V}', \tag{2.7}$$
$$\mathcal{B} : H \to \boldsymbol{V}', \tag{2.8}$$
$$\mathcal{A} : V \to V', \tag{2.9}$$

specified by

$$< \mathcal{H}\boldsymbol{v}, \boldsymbol{w} > = a(\boldsymbol{v}, \boldsymbol{w}), \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}, \tag{2.10}$$
$$< \mathcal{B}u, \boldsymbol{v} > = (u, \mathrm{div}\boldsymbol{v})_\Omega, \quad \forall u \in H, \ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.11}$$
$$< \mathcal{A}u, v > = (\nabla u, \nabla v)_\Omega, \quad \forall u, v \in V. \tag{2.12}$$

Finally, we prescribe the regularity assumptions on data

$$\alpha \in W^{1,\infty}(\mathbb{R}), \tag{2.13}$$
$$\boldsymbol{G} \in L^2(0,T;\boldsymbol{H}), \tag{2.14}$$
$$\vartheta \in L^2(0,T;V), \tag{2.15}$$
$$\boldsymbol{u}^0 \in \boldsymbol{V}, \ \boldsymbol{w}^0 \in \boldsymbol{H}, \tag{2.16}$$
$$(\chi_1^0, \chi_2^0) \in K. \tag{2.17}$$

Remark 2.1. We note that some properties of the function α, such as monotonicity (in the sense that α is a decreasing function) and positiveness, although physically motivated (see, e.g., [14]), are not required by our analysis.

Now, we can state the precise formulation of the problem.

PROBLEM (P): Find

$$\boldsymbol{u} \in H^2(0,T;\boldsymbol{V'}) \cap C^1([0,T];\boldsymbol{H}) \cap C^0([0,T];\boldsymbol{V}), \tag{2.18}$$
$$\chi_1, \chi_2 \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^\infty(Q) \cap L^2(0,T;V), \tag{2.19}$$

such that the conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}^0 \ \text{ in } \ \boldsymbol{V}, \quad \boldsymbol{u}_t(0) = \boldsymbol{w}^0 \ \text{ in } \ \boldsymbol{H}, \tag{2.20}$$
$$\chi_1(0) = \chi_1^0, \ \ \chi_2(0) = \chi_2^0 \ \text{ in } \ H. \tag{2.21}$$

are fulfilled, and such that $\boldsymbol{u}, \chi_1, \chi_2$ solve almost everywhere in $(0,T)$ the abstract equations

$$\boldsymbol{u}_{tt} + \mathcal{H}\boldsymbol{u} + \mathcal{B}(\alpha(\vartheta)\chi_2) = \boldsymbol{G} \quad \text{in} \quad \boldsymbol{V}', \tag{2.22}$$
$$k \begin{pmatrix} \chi_{1t} \\ \chi_{2t} \end{pmatrix} + \eta \begin{pmatrix} \mathcal{A}\chi_1 \\ \mathcal{A}\chi_2 \end{pmatrix} + \begin{pmatrix} \frac{\ell}{\vartheta^*}(\vartheta - \vartheta^*) \\ \alpha(\vartheta)\,\mathrm{div}\boldsymbol{u} \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in} \quad V', \tag{2.23}$$

for some $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in L^2(0,T;H^2)$ with

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \partial I_{K,V}(\chi_1, \chi_2), \quad \text{a.e. in } (0,T), \tag{2.24}$$

where $\partial I_{K,V}(\chi_1,\chi_2) : V^2 \to V'^2$ denotes the subdifferential of the indicator function of $K \cap (V)^2$. Using (2.12) and the definition of subdifferential, one can easily check that (2.23 - 2.24) are equivalent to the variational formulation

$$(\chi_1,\chi_2) \in K \cap (V)^2, \tag{2.25}$$

$$\begin{aligned}&\sum_{j=1}^{2} k < \chi_{jt}, \chi_j - \gamma_j > + \eta\,(\nabla\chi_j, \nabla(\chi_j - \gamma_j))_\Omega \\ &+ \frac{\ell}{\vartheta^*} < \vartheta - \vartheta^*, \chi_1 - \gamma_1 > + < \alpha(\vartheta)\mathrm{div}\boldsymbol{u}, \chi_2 - \gamma_2 > \leq 0 \\ &\forall(\gamma_1,\gamma_2) \in K \cap (V)^2 \quad \text{a.e. in } (0,T).\end{aligned} \tag{2.26}$$

For Problem (**P**), we can prove the following existence and regularity result.

Theorem 2.2. *Under assumptions (2.6-2.17), there exists a solution* $(\chi_1,\chi_2,\boldsymbol{u})$ *to* (**P**). *If in addition,* $(\chi_1^0,\chi_2^0) \in V^2$, *then* $\chi_1,\chi_2 \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega))$.

Remark 2.3. Let us observe that Theorem **2.2**, gives $(h_1,h_2) \in L^2(0,T;H^2)$, thus the subdifferential $\partial I_{K,V}$ makes sense almost everywhere in Ω, while the other terms in (2.23) make sense only in the duality between V' and V. This fact is relevant also from a mechanical point of view, because the subdifferential $\partial I_{K,V}$ represents the thermodynamical reaction of the system to the internal constraint (1.2). Indeed as already specified, $\partial I_{K,V}(\chi_1,\chi_2)$ is not an empty set if and only if $(\chi_1,\chi_2) \in K \cap (V)^2$ which ensures that (1.2) holds. Moreover, the thermodynamical reaction is zero if (χ_1,χ_2) belongs to the interior of $K \cap (V)^2$, while a normal reaction force appears if (χ_1,χ_2) lies on the boundary of K, as $\partial I_{K,V}(\chi_1,\chi_2)$ coincides with the cone of the normal vectors to the boundary at the point (χ_1,χ_2).

Now, we establish some further results of continuous dependence on data of the solutions to (**P**). To this aim, we consider two families of data $\mathcal{F}_i$, $i = 1,2$,

$$\mathcal{F}_i = \left\{\boldsymbol{u}_i^0, \boldsymbol{w}_i^0, (\chi_{1i}^0, \chi_{2i}^0), \vartheta_i, \mathbf{G}_i\right\} \tag{2.27}$$

satisfying conditions (2.14), (2.16 - 2.17) and in addition

$$\vartheta_1, \vartheta_2 \in L^2(0,T;W^{1,3}(\Omega)). \tag{2.28}$$

We denote by $(\boldsymbol{u}_i, \chi_{1i}, \chi_{2i})$ the corresponding solution of (**P**) related to the set $\mathcal{F}_i$, $i = 1,2.$, whose existence is established by Theorem **2.2**.
The following statement holds.

Theorem 2.4. *Let* $\mathcal{F}_1, \mathcal{F}_2$ *be as in (2.27-2.28) and let* $(\boldsymbol{u}_1,\chi_{11},\chi_{21}), (\boldsymbol{u}_2,\chi_{12},\chi_{22})$, *represent the corresponding solutions to* (**P**). *Then, there exists a positive constant* $\mathcal{C}_1$ *depending only on* $k,\eta,\ell,\vartheta^*,c_\kappa,c_{\mathbf{v}},T,\Omega,\| \alpha \|_{W^{1,\infty}(\mathbb{R})}, \max_{i=1,2}\left\{\| \nabla\vartheta_i(t) \|_{L^2(0,T;(L^3(\Omega))^3)}\right\}$, *and* c_Ω *where* c_Ω *is a positive constant given by the Sobolev immersions* $H^1(\Omega) \hookrightarrow L^6(\Omega)$ *and* $(H^1(\Omega))^3 \hookrightarrow (L^6(\Omega))^3$, *such*

that

$$
\begin{aligned}
&\|(\boldsymbol{u}_1-\boldsymbol{u}_2)\|^2_{C^1([0,T];\boldsymbol{V'})\cap C^0([0,T];\boldsymbol{H})}\\
&+\sup_{t\in[0,T]}\left\|\int_0^t(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds\right\|^2_{\mathbf{V}}+\sum_{j=1}^{2}\|\chi_{j1}-\chi_{j2}\|^2_{C^0([0,T];H)\cap L^2(0,T;V)}\\
&\leq \mathcal{C}_1\Bigg(\left\|\boldsymbol{w}_1^0-\boldsymbol{w}_2^0\right\|^2_{\boldsymbol{V'}}+\|\,\boldsymbol{u}_1^0-\boldsymbol{u}_2^0\,\|^2_{\boldsymbol{H}}+\sum_{j=1}^{2}\left\|\chi_{j1}^0-\chi_{j2}^0\right\|^2_{H}\\
&+\|\boldsymbol{G}_1-\boldsymbol{G}_2\|^2_{L^2(0,T;\boldsymbol{H})}+\|\vartheta_1-\vartheta_2\|^2_{L^2(0,T;H)}+\|\alpha(\vartheta_1)-\alpha(\vartheta_2)\|^2_{L^2(0,T;L^3(\Omega))}\Bigg).
\end{aligned}
\tag{2.29}
$$

In particular, it follows that Problem (**P**) *admits a unique solution. If, in addition* $(\chi_{1i}^0,\chi_{2i}^0)\in K\cap(V)^2$, *then*

$$
\begin{aligned}
&\|\,\boldsymbol{u}_1-\boldsymbol{u}_2\,\|^2_{C^1([0,T];\boldsymbol{H})\cap C^0([0,T];\boldsymbol{V})}+\sum_{j=1}^{2}\|\,\chi_{j1}-\chi_{j2}\,\|^2_{C^0([0,T];H)\cap L^2(0,T;V)}\\
&\leq \mathcal{C}_2\Bigg(\|\,\boldsymbol{w}_1^0-\boldsymbol{w}_2^0\,\|^2_{\boldsymbol{H}}+\|\,\boldsymbol{u}_1^0-\boldsymbol{u}_2^0\,\|^2_{\mathbf{V}}+\sum_{j=1}^{2}\|\,\chi_{j1}^0-\chi_{j2}^0\,\|^2_{H}\\
&+\|\,\boldsymbol{G}_1-\boldsymbol{G}_2\,\|^2_{L^2(0,T;\boldsymbol{H})}+\|\,\vartheta_1-\vartheta_2\,\|^2_{L^2(0,T;H)}+\|\,\alpha(\vartheta_1)-\alpha(\vartheta_2)\,\|^2_{L^2(0,T;V)}\Bigg),
\end{aligned}
\tag{2.30}
$$

for a positive constant $\mathcal{C}_2$ *depending on the same constants as* $\mathcal{C}_1$.

3 Existence

In this section we detail the proof of Theorem **2.2**, which will be split into three parts. First, we approximate Problem (**P**) by means of an implicit time discretization scheme, then we derive some uniform a priori estimates on the solutions of the discretized system, and finally we pass to the limit by compactness and monotonicity arguments.

3.1 Approximation

Letting N be an arbitrary positive integer, we denote by $\mathcal{P}$ a partition of the time interval $[0,T]$, namely

$$
\mathcal{P}:=\{0=t^0<t^1<\ldots<t^{N-1}<t^N=T\},
\tag{3.1}
$$

defined by an uniform time step $\tau=\dfrac{T}{N}$. By virtue of (2.14) and (2.15), we are allowed to set for $i=1,\ldots,N$

$$
\boldsymbol{G}^i:=\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}\boldsymbol{G}(\cdot,t)dt\ \in\ \boldsymbol{H},\quad \Theta^i:=\frac{1}{\tau}\int_{(i-1)\tau}^{\tau}\vartheta(\cdot,t)dt\ \in V.
\tag{3.2}
$$

Moreover, we introduce a suitable approximation of (χ_1^0, χ_2^0) (for further details see, e.g., [11, Appendix]), satisfying

$$\{\chi_{j\tau}^0\} \in H^2(\Omega), \qquad \partial_n \chi_{j\tau}^0 = 0 \quad \text{on } \Gamma = \partial\Omega, \;\; j = 1, 2, \tag{3.3}$$

$$\left\{(\chi_{1\tau}^0, \chi_{2\tau}^0)\right\} \in K \;\text{ a.e. in }\; \Omega, \;\; \left\{(\chi_{1\tau}^0, \chi_{2\tau}^0)\right\} \to (\chi_1^0, \chi_2^0) \quad \text{in} \quad H^2, \;\text{ as }\; \tau \searrow 0, \tag{3.4}$$

$$\tau^{1/2} \sum_{j=1}^{2} \| \chi_{j\tau}^0 \|_V \leq c \quad \text{with } c \text{ independent of } \tau. \tag{3.5}$$

For any fixed τ, the approximating problem is stated as follows

PROBLEM $(\mathbf{P}_\tau)$.
Find vectors $(\boldsymbol{U}^0, \ldots, \boldsymbol{U}^N) \in \boldsymbol{V}^{N+1}$, $(\boldsymbol{W}^0, \ldots, \boldsymbol{W}^N) \in \boldsymbol{H}^{N+1}$, $(\mathcal{X}_j^0, \ldots, \mathcal{X}_j^N) \in V^{N+1}$, $j = 1, 2$, such that

$$\mathcal{X}_1^0 = \chi_{1\tau}^0, \;\; \mathcal{X}_2^0 = \chi_{2\tau}^0, \;\; \boldsymbol{U}^0 = \boldsymbol{u}^0, \;\; \boldsymbol{W}^0 = \boldsymbol{w}^0, \tag{3.6}$$

and fulfilling, for $i = 1, ..., N$,

$$\frac{\boldsymbol{W}^i - \boldsymbol{W}^{i-1}}{\tau} + \mathcal{H}\boldsymbol{U}^i + \mathcal{B}(\alpha(\vartheta^i)\mathcal{X}_2^i) = \boldsymbol{G}^i, \tag{3.7}$$

$$\boldsymbol{W}^i = \frac{\boldsymbol{U}^i - \boldsymbol{U}^{i-1}}{\tau}, \tag{3.8}$$

$$\frac{k}{\tau}\begin{pmatrix} \mathcal{X}_1^i - \mathcal{X}_1^{i-1} \\ \mathcal{X}_2^i - \mathcal{X}_2^{i-1} \end{pmatrix} + \eta \begin{pmatrix} \mathcal{A}\mathcal{X}_1^i \\ \mathcal{A}\mathcal{X}_2^i \end{pmatrix} + \begin{pmatrix} \frac{\ell}{\vartheta^*}(\Theta^i - \vartheta^*) \\ \alpha(\Theta^i)\,\mathrm{div}\boldsymbol{U}^i \end{pmatrix} + \begin{pmatrix} h_1^i \\ h_2^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.9}$$

for some $\begin{pmatrix} h_1^i \\ h_2^i \end{pmatrix} \in H^2$ with

$$\begin{pmatrix} h_1^i \\ h_2^i \end{pmatrix} \in \partial I_{K,V}(\mathcal{X}_1^i, \mathcal{X}_2^i). \tag{3.10}$$

Under the same assumptions of Theorem **2.2**, for sufficiently small τ, Problem $(\mathbf{P}_\tau)$ has a unique solution, as it is stated by the following Lemma.

Lemma 3.1. *There exists a positive constant δ depending only on $k, \eta, \ell, \vartheta^*, c_{\mathbf{v}}, c_\kappa, \| \alpha \|_{L^\infty(\mathbb{R})}$ such that for any time step $\tau < \delta$, Problem $(\mathbf{P}_\tau)$ admits one and only one solution.*

Proof. First of all, thanks to (3.6) and (2.16 - 2.17), it suffices to prove that for $i \geq 1$ the system (3.7 - 3.8) has a unique solution, which will be found by exploiting an iterative procedure. At first step, we fix an arbitrary pair $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2) \in K \cap (V)^2$ and substitute $\mathcal{X}_2^i$ by $\tilde{\mathcal{X}}_2$ in (3.7). By (2.6) and (2.13), we have that $\alpha(\Theta^i)\tilde{\mathcal{X}}_2 \in L^2(\Omega)$. Thus, recalling (2.2) and (2.3), an application of the Lax-Milgram Lemma yields existence and uniqueness of a solution $\tilde{\boldsymbol{U}} = L(\tilde{\mathcal{X}}_2) \in \boldsymbol{V}$ fulfilling

$$\tau^{-2}\tilde{\boldsymbol{U}} + \mathcal{H}\tilde{\boldsymbol{U}} = -\mathcal{B}(\alpha(\Theta^i)\tilde{\mathcal{X}}_2) + \boldsymbol{G}^i + \tau^{-2}\boldsymbol{U}^{i-1} + \tau^{-1}\boldsymbol{W}^{i-1}. \tag{3.11}$$

At the second step, we take $\tilde{\boldsymbol{U}}$ instead of $\boldsymbol{U}^i$ in (3.9) (cf. 3.10) and denote by $(\mathcal{X}_1, \mathcal{X}_2) = S(\tilde{\boldsymbol{U}})$ the unique solution of the resulting variational inequality. Thus, we can define an operator $E : K \cap (V)^2 \to K \cap (V)^2$ such that

$$E(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2) := S(L(\tilde{\mathcal{X}}_2)) = (\mathcal{X}_1, \mathcal{X}_2). \tag{3.12}$$

Now, we aim to apply the Banach Theorem to show that E admits a fixed point. In order to prove that E is a contracting map, we will show some Lipschitz continuity estimates for the operators L and S. To this aim, we denote by $\{(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2)\}$ and $\{(\hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2)\}$ two sets of data both belonging to $K \cap V^2$. Then, we write (3.7) for the two pairs, take the difference and test by the difference of the corresponding solutions. Accounting for (2.3) and (2.13) and by applying the elementary inequality

$$2ab \leq \frac{1}{\varepsilon} a^2 + \varepsilon b^2 \quad \forall a, b \in \mathbf{R}, \ \ \forall \varepsilon > 0, \tag{3.13}$$

we find a constant c_1 such that

$$\| \tilde{\boldsymbol{U}} - \hat{\boldsymbol{U}} \|_{\boldsymbol{V}}^2 \leq c_1 \sum_{j=1}^{2} \| \tilde{\mathcal{X}}_j - \hat{\mathcal{X}}_j \|_V^2, \tag{3.14}$$

where, for instance, $c_1 = 3 \| \alpha \|_{L^\infty(\mathbb{R})}^2 / c_{\mathbf{v}}^2$. Concerning (3.9), we substitute $\boldsymbol{U}^i$ by $\tilde{\boldsymbol{U}}$ and $\hat{\boldsymbol{U}}$ respectively, denote by $(\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2)$ and $(\hat{\mathcal{Z}}_1, \hat{\mathcal{Z}}_2)$ the corresponding solutions, take the difference and test by the difference of the solutions. For $\tau < \frac{k}{2\eta}$, one obtains

$$\sum_{j=1}^{2} \| \tilde{\mathcal{Z}}_j - \hat{\mathcal{Z}}_j \|_V^2 \leq c_2 \tau \| \tilde{\boldsymbol{U}} - \hat{\boldsymbol{U}} \|_{\boldsymbol{V}}^2, \tag{3.15}$$

where, for instance, $c_2 = 3 \| \alpha \|_{L^\infty(\mathbb{R})} / 2\eta k$. Recalling now (3.12), for $\tau < \frac{k}{2\eta}$, (3.14) and (3.15), imply that there exists a constant c_3, independent on τ, such that

$$\left\| E(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2) - E(\hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2) \right\|_{V^2}^2 \leq c_3 \tau \left\| (\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2) - (\hat{\mathcal{X}}_1, \hat{\mathcal{X}}_2) \right\|_{V^2}^2. \tag{3.16}$$

Thus, taking $\delta = \min\left\{1, \dfrac{k}{2\eta}, \dfrac{1}{c_3}\right\}$, for $\tau < \delta$, the operator E is a contracting map in V^2 and consequently there exists one and only one fixed point $(\mathcal{X}_1^i, \mathcal{X}_2^i)$ of E. As a consequence, for $i = 1, \ldots, N$, $(\mathcal{X}_1^i, \mathcal{X}_2^i, L(\mathcal{X}_2^i))$ will be the unique solution to (3.7 - 3.9). □

3.2 A Priori Estimates

First of all, we introduce some convenient notations. Given a vector $\{Z^i\}_{i=1}^N$ in the linear space $\mathcal{Z}$, we denote by $\bar{Z}_\tau$ the piecewise constant and by $\hat{Z}_\tau$ the piecewise linear interpolating functions, i.e.

$$\begin{aligned} \bar{Z}_\tau(t) \ := \ Z^i, \quad \hat{Z}_\tau(t) \ := \ Z^i + \frac{Z^i - Z^{i-1}}{\tau}(t - i\tau) \\ \text{for } \ t \ \in \ ](i-1)\tau, i\tau], \ \ i = 1, \ldots, N \end{aligned} \tag{3.17}$$

Thus, owing to (3.17), we may conveniently rewrite relations (3.7 - 3.10) as follows

$$\hat{\boldsymbol{w}}_{\tau t} + \mathcal{H}\bar{\boldsymbol{u}}_\tau + \mathcal{B}(\alpha(\bar{\Theta}_\tau)\bar{\chi}_{2\tau}) = \bar{\boldsymbol{G}}_\tau, \tag{3.18}$$

$$\bar{\boldsymbol{w}}_\tau = \hat{\boldsymbol{u}}_{\tau t}, \tag{3.19}$$

$$k \begin{pmatrix} \hat{\chi}_{1\tau t} \\ \hat{\chi}_{2\tau t} \end{pmatrix} + \eta \begin{pmatrix} \mathcal{A}\bar{\chi}_{1\tau} \\ \mathcal{A}\bar{\chi}_{2\tau} \end{pmatrix} + \begin{pmatrix} \frac{\ell}{\vartheta^*}\left(\bar{\Theta}_\tau - \vartheta^*\right) \\ \alpha\left(\bar{\Theta}_\tau\right) \operatorname{div}\bar{\boldsymbol{u}}_\tau \end{pmatrix} + \begin{pmatrix} \bar{h}_{1\tau} \\ \bar{h}_{2\tau} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.20}$$

$$\text{where} \begin{pmatrix} \bar{h}_{1\tau} \\ \bar{h}_{2\tau} \end{pmatrix} \in \partial I_{K,V}\left(\bar{\chi}_{1\tau}, \bar{\chi}_{2\tau}\right). \tag{3.21}$$

The remaining part of the section is devoted to prove some a priori estimates, independent of τ, for the functions defined above. Letting δ be defined as in Lemma **3.1**, we can state the following result

Lemma 3.2. *There exist two constants $\tau^* \in (0,\delta)$ and $\mathcal{C}_3$ such that for any time step $\tau \leq \tau^*$, one has*

$$\begin{aligned}&\| \hat{\boldsymbol{u}}_\tau \|_{W^{1,\infty}(0,T;\boldsymbol{H})\cap L^\infty(0,T;\boldsymbol{V})} + \| \hat{\boldsymbol{w}}_\tau \|_{H^1(0,T;\boldsymbol{V}')\cap L^\infty(0,T;\boldsymbol{H})}\\ &+ \| \bar{\boldsymbol{u}}_\tau \|_{L^\infty(0,T;\boldsymbol{V})} + \sum_{j=1}^{2} \| \hat{\chi}_{j\tau} \|_{H^1(0,T;V')\cap L^\infty(Q)}\\ &+ \sum_{j=1}^{2} \| \bar{\chi}_{j\tau} \|_{L^\infty(Q)\cap L^2(0,T;V)} + \sum_{j=1}^{2} \| \bar{h}_{j\tau} \|_{L^2(0,T;H)} \leq \mathcal{C}_3. \end{aligned} \tag{3.22}$$

If in addition $(\chi_1^0, \chi_2^0) \in V^2$, there exists a constant $\mathcal{C}_4$ such that, for any time step $\tau < \tau^*$, one has

$$\| \hat{\chi}_{j\tau} \|_{H^1(0,T;H)\cap C^0([0,T];V)} + \| \bar{\chi}_{j\tau} \|_{L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} \leq \mathcal{C}_4, \quad j = 1, 2. \tag{3.23}$$

Henceforth, we let c denote any constant which may depend on $k, \eta, \ell, \vartheta^*, c_\kappa, c_{\mathbf{v}}, \| \alpha \|_{W^{1,\infty}(\mathbb{R})}, T, |\Omega|$ ($|\Omega|$ stands for the Lebesque measure of Ω). Obviously, c is always independent of τ and may vary from line to line.

Proof. First, we test (3.9) by $\tau \begin{pmatrix} \mathcal{X}_1^i \\ \mathcal{X}_2^i \end{pmatrix}$. Since we have $2A(A-B) = A^2 + (A-B)^2 - B^2$, $\forall\, A, B \in \mathbb{R}$, and being $0 \in K$, we can infer that

$$\begin{aligned}\sum_{j=1}^{2} \Big(&\frac{k}{2} \| \mathcal{X}_j^i \|_H^2 + \frac{k}{2} \| \mathcal{X}_j^i - \mathcal{X}_j^{i-1} \|_H^2 - \frac{k}{2} \| \mathcal{X}_j^{i-1} \|_H^2 + \eta\tau \| \nabla \mathcal{X}_j^i \|_{\mathbf{H}}^2 \Big)\\ &\leq \frac{\ell\tau}{2\vartheta^*} \| \Theta^i - \vartheta^* \|_H^2 + \frac{\ell\tau}{2\vartheta^*} \| \mathcal{X}_1^i \|_H^2 - \tau(\alpha(\Theta^i)\mathrm{div}\boldsymbol{U}^i, \mathcal{X}_2^i)_\Omega, \end{aligned} \tag{3.24}$$

for $i = 1....N$. Next, testing (3.7) by $\boldsymbol{W}^i = \boldsymbol{U}^i - \boldsymbol{U}^{i-1}$, similarly we obtain

$$\begin{aligned}&\frac{1}{2} \big(\| \boldsymbol{W}^i \|_{\boldsymbol{H}}^2 + \| \boldsymbol{W}^i - \boldsymbol{W}^{i-1} \|_{\boldsymbol{H}}^2 - \| \boldsymbol{W}^{i-1} \|_{\boldsymbol{H}}^2\\ &+ a(\boldsymbol{U}^i, \boldsymbol{U}^i) + a(\boldsymbol{U}^i - \boldsymbol{U}^{i-1}, \boldsymbol{U}^i - \boldsymbol{U}^{i-1}) - a(\boldsymbol{U}^{i-1}, \boldsymbol{U}^{i-1}) \big)\\ &\leq \frac{\tau}{2} \| \boldsymbol{G}^i \|_{\boldsymbol{H}}^2 + \frac{\tau}{2} \| \boldsymbol{W}^i \|_{\boldsymbol{H}}^2 - \tau(\alpha(\Theta^i)\mathcal{X}_2^i, \mathrm{div}\boldsymbol{W}^i)_\Omega, \end{aligned} \tag{3.25}$$

for $i = 1....N$. Adding (3.25) to (3.24), and summing up for $i = 1, \ldots, m$, where $m \leq N = T/\tau$,

with the help of (2.3), (2.16-2.17), (3.2), (3.6), we have that

$$\begin{aligned} S_m &:= \sum_{j=1}^{2} \frac{k}{2} \parallel \mathcal{X}_j^m \parallel_H^2 + \frac{k}{2}\tau \sum_{i=1}^{m} \tau \sum_{j=1}^{2} \left\| \frac{\mathcal{X}_j^i - \mathcal{X}_j^{i-1}}{\tau} \right\|_H^2 + \eta \sum_{i=1}^{m} \tau \sum_{j=1}^{2} \parallel \nabla \mathcal{X}_j^i \parallel_{\boldsymbol{H}}^2 \\ &+ \frac{1}{2} \left(\parallel \boldsymbol{W}^m \parallel_{\boldsymbol{H}}^2 + \tau \sum_{i=1}^{m} \tau \left\| \frac{\boldsymbol{W}^i - \boldsymbol{W}^{i-1}}{\tau} \right\|_{\boldsymbol{H}}^2 + c_{\mathbf{v}} \parallel \boldsymbol{U}^m \parallel_{\boldsymbol{V}}^2 \right. \\ &+ \left. \tau c_{\mathbf{v}} \sum_{i=1}^{m} \tau \left\| \frac{\boldsymbol{U}^i - \boldsymbol{U}^{i-1}}{\tau} \right\|_{\boldsymbol{V}}^2 \right) \leq \mathcal{D} + \sum_{j=1}^{3} T_m^j, \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} \mathcal{D} &:= \frac{1}{2} \parallel \boldsymbol{W}^0 \parallel_{\boldsymbol{H}}^2 + a(\boldsymbol{U}^0, \boldsymbol{U}^0) + \frac{k}{2} \sum_{j=1}^{2} \parallel \chi_{j\tau}^0 \parallel_H^2, \\ T_m^1 &:= \frac{1}{2} \sum_{i=1}^{m} \tau \parallel \boldsymbol{G}^i \parallel_{\boldsymbol{H}}^2 + \frac{\ell}{\vartheta^*} \sum_{i=1}^{m} \tau \left(\parallel \Theta^i \parallel_H^2 + \parallel \vartheta^* \parallel_H^2 \right), \\ T_m^2 &:= \frac{1}{2} \sum_{i=1}^{m} \tau \parallel \boldsymbol{W}^i \parallel_{\boldsymbol{H}}^2 + \frac{\ell}{2\vartheta^*} \sum_{i=1}^{m} \tau \parallel \mathcal{X}_1^i \parallel_H^2 - \sum_{i=1}^{m} \tau \left(\alpha(\Theta^i) \mathrm{div} \boldsymbol{U}^i, \mathcal{X}_2^i \right)_\Omega, \\ T_m^3 &:= -\sum_{i=1}^{m} \tau \left(\alpha(\Theta^i) \mathcal{X}_2^i, \mathrm{div} \boldsymbol{W}^i \right)_\Omega, \end{aligned}$$

for $i = 1, \ldots, m$. Now we are going to handle the quantities T_m^1, T_m^2, T_m^3. It is easy to check that the following estimates hold (cf. (3.2))

$$T_m^1 \leq \frac{1}{2} \parallel \boldsymbol{G} \parallel_{L^2(0,T;\boldsymbol{H})}^2 + \ell T \vartheta^* |\Omega| + \frac{\ell}{\vartheta^*} \parallel \vartheta \parallel_{L^2(0,T;H)}^2, \tag{3.27}$$

and (cf. (2.3-2.4))

$$\begin{aligned} T_m^2 &\leq \frac{1}{2} \sum_{i=1}^{m} \tau \parallel \boldsymbol{W}^i \parallel_{\boldsymbol{H}}^2 + \frac{3 \parallel \alpha \parallel_{L^\infty(\mathbb{R})}^2}{2} \sum_{i=1}^{m} \tau \parallel \boldsymbol{U}^i \parallel_{\boldsymbol{V}}^2 \\ &+ \frac{\ell}{2\vartheta^*} \sum_{i=1}^{m} \tau \parallel \mathcal{X}_1^i \parallel_H^2 + \frac{1}{2} \sum_{i=1}^{m} \tau \parallel \mathcal{X}_2^i \parallel_H^2 . \end{aligned} \tag{3.28}$$

Finally, integrating by parts with respect to space variables, for $\varepsilon > 0$ to be chosen later, one has

$$\begin{aligned} T_m^3 &\leq \frac{1}{2}(c_\kappa^2 + \varepsilon^{-1}) \sum_{i=1}^{m} \tau \parallel \boldsymbol{W}^i \parallel_{\boldsymbol{H}}^2 + \frac{\varepsilon}{2} \parallel \alpha' \parallel_{L^\infty(\mathbb{R})}^2 \sum_{i=1}^{m} \tau \parallel \nabla \Theta^i \parallel_{\boldsymbol{H}}^2 \\ &+ \frac{1}{2} \parallel \alpha \parallel_{L^\infty(\mathbb{R})}^2 \varepsilon \sum_{i=1}^{m} \tau \parallel \nabla \mathcal{X}_2^i \parallel_{\boldsymbol{H}}^2 . \end{aligned} \tag{3.29}$$

Fixing ε such that $\varepsilon < \dfrac{2\eta}{\parallel \alpha \parallel_{L^\infty(\mathbb{R})}^2}$ and combining $(3.27 - 3.29)$, it is straightforward to conclude that there is a positive constant c_4 such that

$$S_m \leq c_4 \left(1 + \sum_{i=1}^{m} \tau S_i \right) \tag{3.30}$$

for any m satisfying $1 \leq m \leq N$. Now, choosing $\tau^* := \min\left\{\delta, \dfrac{1}{2c_4}\right\}$ and letting $S_0 := 0$, we have that

$$S_m \leq 2c_4 \left(1 + \sum_{i=0}^{m-1} \tau S_i\right) \tag{3.31}$$

for $m = 1, \ldots, N$ and for any time step $\tau \leq \tau^*$. Finally, by applying the discrete Gronwall Lemma (see e.g. [17, pag. 14]) to the finite sequence (3.31), we obtain (cf. also (3.26), (3.17) and (2.6))

$$\begin{aligned} \| \bar{\boldsymbol{u}}_\tau \|_{L^\infty(0,T;\boldsymbol{V})} + \| \bar{\boldsymbol{w}}_\tau \|_{L^\infty(0,T;\boldsymbol{H})} + \| \hat{\boldsymbol{u}}_\tau \|_{W^{1,\infty}(0,T;\boldsymbol{H})\cap L^\infty(0,T;\boldsymbol{V})} \\ + \| \hat{\boldsymbol{w}}_\tau \|_{L^\infty(0,T;\boldsymbol{H})} + \sum_{j=1}^{2} \| \bar{\chi}_{j\tau} \|_{L^\infty(Q)\cap L^2(0,T;V)} \leq c. \end{aligned} \tag{3.32}$$

Moreover, it is a standard matter to check that

$$\sum_{j=1}^{2} \| \hat{\chi}_{j\tau} \|_{L^\infty(Q)} \leq c. \tag{3.33}$$

Now, a comparison in (3.18) yields

$$\| \hat{\boldsymbol{w}}_\tau \|_{H^1(0,T;\boldsymbol{V}')} \leq c. \tag{3.34}$$

Now, we show that $\| \hat{\chi}_{j\tau} \|_{H^1(0,T;V')}$ is bounded independently of τ. To this end, let us test (3.20) by (h_1^i, h_2^i) and then sum up for $i = 1, \ldots, m$. All this is formal, but it can be justified using the procedure presented in, e.g., [4, Appendix]. We first observe that the assumption $(\chi_{1\tau}^0, \chi_{2\tau}^0) \in K$ and the definition of subdifferential yield

$$\sum_{j=1}^{2} \left(\frac{\mathcal{X}_j^i - \mathcal{X}_j^{i-1}}{\tau}, h_j^i\right) \geq \frac{1}{\tau}\left(I_K(\mathcal{X}_1^i, \mathcal{X}_2^i) - I_K(\mathcal{X}_1^{i-1}, \mathcal{X}_2^{i-1})\right) = 0. \tag{3.35}$$

Besides, being $\sum_{j=1}^{2}\left(\nabla\mathcal{X}_j^i, \nabla h_j^i\right) \geq 0$ because of the monotonicity of $\partial I_{K,V}$, we easily obtain

$$\sum_{j=1}^{2}\sum_{i=1}^{m} \tau \| h^i \|_H^2 \leq c\left(\sum_{i=1}^{m} \tau \| \Theta^i - \vartheta^* \|_H^2 + \sum_{i=1}^{m} \tau \| \mathrm{div}\boldsymbol{U}^i \|_H^2\right). \tag{3.36}$$

Thus, owing to (3.32), one has

$$\sum_{j=1}^{2} \| \bar{h}_{j\tau} \|_{L^2(0,T;H)} \leq c. \tag{3.37}$$

Finally, a comparison in (3.20), yields

$$\sum_{j=1}^{2} \| \hat{\chi}_{j\tau} \|_{H^1(0,T;V')} \leq c. \tag{3.38}$$

Now, combining (3.38) with (3.32 - 3.34), (3.22) easily follows. Let us remark that a consequence of (3.32) and (3.26) is the following estimate

$$\tau\left(\| \hat{\boldsymbol{u}}_\tau \|_{H^1(0,T;\mathbf{V})}^2 + \| \hat{\boldsymbol{w}}_\tau \|_{H^1(0,T;\boldsymbol{H})}^2 + \| \hat{\chi}_{j\tau} \|_{H^1(0,T;H)}^2\right) \leq c. \tag{3.39}$$

Furthermore, we present here a stability estimate for the functions $\hat{\chi}_{j\tau}$ $j = 1, 2$ in the norm of $L^2(0, T; V)$, that can not be directly deduced by (3.22). More precisely, recalling (3.6) we have

$$\sum_{j=1}^{2} \| \hat{\chi}_{j\tau} \|^2_{L^2(0,T;V)} = \sum_{j=1}^{2} \sum_{i=2}^{N} \int_{(i-1)\tau}^{i\tau} \left\| \mathcal{X}_j^i + \frac{\mathcal{X}_j^i - \mathcal{X}_j^{i-1}}{\tau}(t - i\tau) \right\|_V^2$$
$$+ \sum_{j=1}^{2} \int_0^{\tau} \left\| \mathcal{X}_j^1 + \frac{\mathcal{X}_j^1 - \chi_{j\tau}^0}{\tau}(t - \tau) \right\|_V^2 \leq c \sum_{j=1}^{2} \|\bar{\chi}_{j\tau}\|^2_{L^2(0,T;V)} + c\tau \sum_{j=1}^{2} \left\|\chi_{j\tau}^0\right\|_V^2 . \tag{3.40}$$

Thus, (3.5) and (3.22), lead to

$$\sum_{j=1}^{2} \| \hat{\chi}_{j\tau} \|^2_{L^2(0,T;V)} \leq c, \tag{3.41}$$

independently of τ.

Now, we want to deduce the second part of the Lemma. In order to the get (3.23), we assume that $(\chi_1^0, \chi_2^0) \in K \cap V^2$ and consequently we replace the convergence (3.4) by

$$\{(\chi_{1\tau}^0, \chi_{2\tau}^0)\} \to (\chi_1^0, \chi_2^0) \quad \text{in} \quad V^2 \tag{3.42}$$

as $\tau \searrow 0$. Testing (3.9) by $\begin{pmatrix} \mathcal{X}_1^i - \mathcal{X}_1^{i-1} \\ \mathcal{X}_2^i - \mathcal{X}_2^{i-1} \end{pmatrix}$ and summing up for $i = 1, \ldots, N$, by reasoning as in (3.35), we infer that

$$\sum_{j=1}^{2} \left(\sum_{i=1}^{m} \tau \left\| \frac{\mathcal{X}_j^i - \mathcal{X}_j^{i-1}}{\tau} \right\|_H^2 + \| \nabla \mathcal{X}_j^m \|^2_{\mathbf{H}} + \tau \sum_{i=1}^{m} \tau \left\| \frac{\nabla(\mathcal{X}_j^i - \mathcal{X}_j^{i-1})}{\tau} \right\|_{\mathbf{H}}^2 \right)$$
$$\leq c \left(\sum_{j=1}^{2} \| \nabla \mathcal{X}_{j\tau}^0 \|^2_{\mathbf{H}} + \sum_{i=1}^{m} \tau \| \Theta^i - \vartheta^* \|^2_H + \sum_{i=1}^{m} \tau \| \mathbf{U}^i \|^2_{\mathbf{V}} \right), \tag{3.43}$$

and consequently, due to (3.22) and (3.42), one has

$$\sum_{j=1}^{2} \| \hat{\chi}_{j\tau} \|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \sum_{j=1}^{2} \| \bar{\chi}_{j\tau} \|_{L^\infty(0,T;V)} \leq c, \tag{3.44}$$

and

$$\tau \sum_{j=1}^{2} \| \hat{\chi}_{j\tau} \|^2_{H^1(0,T;V)} \leq c \tag{3.45}$$

We conclude by noting that a comparison of (3.20) and (3.37), yields

$$\sum_{j=1}^{2} \| \bar{\chi}_{j\tau} \|_{L^2(0,T;H^2(\Omega))} \leq c \tag{3.46}$$

so that Lemma **3.2** is proved. □

3.3 Passage to the Limit

In this part, we take the limit of Problem $(\mathbf{P}_\tau)$ as τ goes to 0, and show that Problem $(\mathbf{P})$ has at least one solution, so that Theorem **2.2** will be proved. First of all, owing to (3.17), (3.2), (2.15), (2.14), it is a standard matter to deduce that as $\tau \searrow 0$

$$\bar{\boldsymbol{G}}_\tau \to \boldsymbol{G} \quad \text{in} \quad L^2(0,T;\boldsymbol{H}), \quad \bar{\Theta}_\tau \to \vartheta \quad \text{in} \quad L^2(0,T;V). \tag{3.47}$$

Next, due to (3.22) and (3.41), well known results on weak and weak star compactness ensure that there exist $\boldsymbol{u}, \boldsymbol{w}, \chi_j, h_j, j = 1,2$ such that, possibly taking suitable subsequences (not relabeled), the following convergences hold

$$\hat{\boldsymbol{u}}_\tau \overset{*}{\rightharpoonup} \boldsymbol{u} \ \text{ in } \ W^{1,\infty}(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}), \tag{3.48}$$

$$\hat{\boldsymbol{w}}_\tau \overset{*}{\rightharpoonup} \boldsymbol{w} \ \text{ in } \ H^1(0,T;\boldsymbol{V}') \cap L^\infty(0,T;\boldsymbol{H}), \tag{3.49}$$

$$\hat{\chi}_{j\tau} \overset{*}{\rightharpoonup} \chi_j \ \text{ in } \ H^1(0,T;V') \cap L^\infty(Q) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \quad j = 1,2, \tag{3.50}$$

$$\bar{h}_{j\tau} \rightharpoonup h_j \ \text{ in } \ L^2(0,T;H), \quad j = 1,2. \tag{3.51}$$

On account of (3.50), one has , thanks to the Aubin-Lions Lemma (see, e.g.,[15, pag. 58]),

$$\hat{\chi}_{j\tau} \to \chi_j \ \text{ in } \ L^2(0,T;H), \tag{3.52}$$

and, by interpolation,

$$\chi_j \in C^0([0,T];H) \ \text{ for } \ j = 1,2. \tag{3.53}$$

Now, since (cf. (3.17), (3.22), (3.39))

$$\|\hat{\boldsymbol{u}}_\tau - \bar{\boldsymbol{u}}_\tau\|^2_{L^2(0,T;\boldsymbol{V})} = \frac{\tau^2}{3} \left\| \frac{\partial \hat{\boldsymbol{u}}_\tau}{\partial t} \right\|^2_{L^2(0,T;\boldsymbol{V})} \le c\tau, \tag{3.54}$$

and analogous estimates hold for $\| \hat{\boldsymbol{w}}_\tau - \bar{\boldsymbol{w}}_\tau \|^2_{L^2(0,T;\boldsymbol{H})}$, $\| \hat{\chi}_{j\tau} - \bar{\chi}_{j\tau} \|^2_{L^2(0,T;H)}$ for $j = 1,2$, by using (3.22), (3.48 - 3.52), and (3.19), it is a standard matter to deduce that

$$\bar{\boldsymbol{u}}_\tau \overset{*}{\rightharpoonup} \boldsymbol{u} \ \text{ in } \ L^\infty(0,T;\boldsymbol{V}), \tag{3.55}$$

$$\bar{\boldsymbol{w}}_\tau \overset{*}{\rightharpoonup} \boldsymbol{w} \ \text{ in } \ L^\infty(0,T;\boldsymbol{H}), \tag{3.56}$$

$$\bar{\chi}_{j\tau} \overset{*}{\rightharpoonup} \chi_j \ \text{ in } \ L^\infty(Q) \cap L^\infty(0,T;H) \cap L^2(0,T;V) \quad j = 1,2, \tag{3.57}$$

$$\bar{\chi}_{j\tau} \to \chi_j \quad \text{in} \ L^2(0,T;H), \ j = 1,2, \tag{3.58}$$

as $\tau \searrow 0$. In particular, (3.55) implies

$$\mathrm{div}\bar{\boldsymbol{u}}_\tau \overset{*}{\rightharpoonup} \mathrm{div}\boldsymbol{u} \ \text{ in } \ L^\infty(0,T;H). \tag{3.59}$$

Then, owing to (3.48 - 3.50), (3.17), (3.6), (3.4), and (3.33), accounting for $\boldsymbol{u}_t = \boldsymbol{w}$, it is straightforward to check that the functions $\boldsymbol{u}, \chi_1, \chi_2$, satisfy the initial conditions (2.20 - 2.21) and the restriction (2.25). In fact K, being convex, is closed also for the weak topology of H^2. Moreover, by the convergence result (3.47) and the regularity of α, the Lebesgue Theorem implies

$$\alpha(\bar{\Theta}_\tau) \to \alpha(\vartheta) \ \text{ strongly in } \ L^p(Q) \ \forall \ p \in [1,+\infty), \tag{3.60}$$

and consequently, as (3.57) holds, one has $\alpha(\bar{\Theta}_\tau)\bar{\chi}_{2\tau} \rightharpoonup \alpha(\vartheta)\chi_2$ weakly in $L^2(Q)$ as $\tau \searrow 0$. Hence, recalling (3.47), (3.49), (3.55), (3.19), one easily verifies that (2.22) holds. The last step consists in proving that $\chi_1, \chi_2, \mathrm{div}\boldsymbol{u}$, satisfy (2.23). To this aim we start to note that by (3.59) and (3.60), we are allowed to deduce

$$\alpha(\bar{\Theta}_\tau)\mathrm{div}\bar{\boldsymbol{u}}_\tau \rightharpoonup \alpha(\vartheta)\mathrm{div}\boldsymbol{u} \quad \text{in} \quad L^2(0,T;L^{2-\varepsilon}(\Omega)) \quad \text{for any} \ \varepsilon > 0,$$

and thus the convergence could be easily extended to $L^2(0,T;V')$. Hence, taking the limit in (3.20) as $\tau \searrow 0$ and owing to (3.47), (3.50),(3.51), (3.57 - 3.58) and to the continuity of $\mathcal{A}$, one has

$$k\frac{d}{dt}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \eta \begin{pmatrix} \mathcal{A}\chi_1 \\ \mathcal{A}\chi_2 \end{pmatrix} + \begin{pmatrix} \frac{\ell}{\vartheta^*}(\vartheta - \vartheta^*) \\ \alpha(\vartheta)\,\mathrm{div}\boldsymbol{u} \end{pmatrix} = -\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{3.61}$$

and

$$\lim_{\tau\searrow 0}\sum_{j=1}^{2}(\bar{h}_{j\tau}, \bar{\chi}_{j\tau})_\Omega = \sum_{j=1}^{2}(h_j, \chi_j)_\Omega. \tag{3.62}$$

Thus, thanks to [6, Prop. 2.5, pag. 27], we have that $(h_1, h_2) \in \partial I_{K,V}(\chi_1, \chi_2)$ a.e. in $(0,T)$ and so $\chi_1, \chi_2, \mathrm{div}\boldsymbol{u}$, satisfy (2.23).

Finally, we note that $\boldsymbol{u}$ is more regular than $W^{1,\infty}(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V})$; more precisely it belongs to $C^1([0,T];\boldsymbol{H}) \cap C^0([0,T];\boldsymbol{V})$. In fact, being $\alpha(\vartheta)$ and $\chi_2 \in L^2(0,T;V)$ and noting that

$$(\alpha(\vartheta(t))\chi_2(t), \mathrm{div}\boldsymbol{v}) = -(\alpha'(\vartheta(t))\chi_2(t)\nabla\vartheta(t), \boldsymbol{v}) - (\alpha(\vartheta(t))\nabla\chi_2(t), \boldsymbol{v}) \ \ \forall \boldsymbol{v} \in \boldsymbol{V}$$

we have that $\boldsymbol{u}$ satisfies a linear hyperbolic problem with data in $L^2(0,T;\boldsymbol{H})$ and initial conditions for $\boldsymbol{u}$ (resp. $\boldsymbol{u}_t$) in $\boldsymbol{V}$ (resp. $\boldsymbol{H}$). Thus, an application of ([2, Th 8.1, pag. 295]), gives the regularity required for $\boldsymbol{u}$. The existence of a solution to (**P**) is thus proved. To conclude the proof of Theorem **2.2**, it remains to show the regularity result. To this purpose, we first note that, thanks to (3.45), there holds an analogous of (3.54) for $\| \hat{\chi}_j - \bar{\chi}_j \|_{L^2(0,T;V)}$ $j = 1,2$. Hence, (3.23), (3.50), and (3.57), give that for $j = 1,2$, $\hat{\chi}_{j\tau}, \bar{\chi}_{j\tau}$, converge weakly star also in $H^1(0,T;H) \cap L^\infty(0,T;V)$ and in $L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$, respectively. Thus, for $j = 1,2$, we have that

$$\chi_j \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)),$$

and, by interpolation,

$$\chi_j \in C^0([0,T],V).$$

4 Continuous dependence and uniqueness

First we aim to estimate the norm of $(\chi_{j1} - \chi_{j2})$ for $j = 1,2.$ in $C^0([0,T];H) \cap L^2(0,T;V)$. To this purpose, we write relation (2.26) for $\mathcal{F}_1$ (letting $(\gamma_1, \gamma_2) = (\chi_{12}(t), \chi_{22}(t))$) and $\mathcal{F}_2$ (letting $(\gamma_1, \gamma_2) = (\chi_{11}(t), \chi_{21}(t))$), respectively. Taking the sum of the two inequalities, and integrating in time, we easily obtain

$$\sum_{j=1}^{2}\left(\frac{k}{2} \| (\chi_{j1} - \chi_{j2})(t) \|_H^2 + \eta \int_0^t \| \nabla(\chi_{j1} - \chi_{j2})(s) \|_H^2 \, ds\right)$$
$$- \sum_{j=1}^{2}\frac{k}{2} \| \chi_{j1}^0 - \chi_{j2}^0 \|_H^2 \leq \sum_{i=1}^{3} J^{(i)}(t), \tag{4.1}$$

for $t \in (0,T)$, where

$$J^{(1)}(t) = -\frac{\ell}{\vartheta^*}\int_0^t \big(\vartheta_1-\vartheta_2, \chi_{11}-\chi_{12}\big)_\Omega(s)ds,$$
$$J^{(2)}(t) = -\int_0^t \big(\alpha(\vartheta_2)\mathrm{div}(\boldsymbol{u}_1-\mathbf{u}_2), \chi_{21}-\chi_{22}\big)_\Omega(s)ds,$$
$$J^{(3)}(t) = -\int_0^t \big(\alpha(\vartheta_1)-\alpha(\vartheta_2)\mathrm{div}\boldsymbol{u}_1, \chi_{21}-\chi_{22}\big)_\Omega(s)ds.$$

Thanks to Hölder's inequality, the term $J^{(1)}(t)$ will be controlled as follows

$$J^{(1)}(t) \ \leq \frac{\ell}{2\vartheta^*}\int_0^t \parallel (\vartheta_1-\vartheta_2)(s) \parallel_H^2 ds + \frac{\ell}{2\vartheta^*}\int_0^t \parallel (\chi_{11}-\chi_{12})(s) \parallel_H^2 ds. \tag{4.2}$$

Concerning $J^{(2)}(t)$, an integration by parts with respect to space variables, the elementary inequality (3.13) and the Sobolev immersion $H^1(\Omega) \hookrightarrow L^6(\Omega)$, lead to ($\varepsilon > 0$ will be chosen later)

$$\begin{aligned}
J^{(2)}(t) &\leq \int_0^t \|(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)\|_{\boldsymbol{H}} \parallel \alpha' \parallel_{L^\infty(\mathbb{R})} \parallel \nabla\vartheta_2(s) \parallel_{(L^3(\Omega))^3} \|(\chi_{21}-\chi_{22})(s)\|_{L^6(\Omega)} ds \\
&+ \int_0^t \|(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)\|_{\boldsymbol{H}} \parallel \alpha \parallel_{L^\infty(\mathbb{R})} \|\nabla(\chi_{21}-\chi_{22})(s)\|_{\boldsymbol{H}} ds \\
&\leq \frac{1}{2\varepsilon} \parallel \alpha' \parallel_{L^\infty(\mathbb{R})}^2 \int_0^t \parallel \nabla\vartheta_2(s) \parallel_{(L^3(\Omega))^3}^2 \parallel (\boldsymbol{u}_1-\boldsymbol{u}_2)(s) \parallel_{\boldsymbol{H}}^2 ds \\
&+ \frac{\varepsilon}{2}c_\Omega \int_0^t \big(\parallel (\chi_{21}-\chi_{22})(s) \parallel_H^2 + \parallel \nabla(\chi_{21}-\chi_{22})(s) \parallel_{\boldsymbol{H}}^2\big) ds \\
&+ \frac{1}{2\varepsilon} \parallel \alpha \parallel_{L^\infty(\mathbb{R})}^2 \int_0^t \parallel (\boldsymbol{u}_1-\boldsymbol{u}_2)(s) \parallel_{\boldsymbol{H}}^2 ds + \frac{\varepsilon}{2}\int_0^t \parallel \nabla(\chi_{21}-\chi_{22})(s) \parallel_{\boldsymbol{H}}^2 ds, \quad \forall\varepsilon > 0.
\end{aligned} \tag{4.3}$$

Finally, we deal with $J^{(3)}(t)$. It is straightforward to obtain

$$\begin{aligned}
J^{(3)}(t) \leq c\frac{\varepsilon}{2}\int_0^t \big(\parallel (\chi_{21}-\chi_{22})(s) \parallel_H^2 + \parallel \nabla(\chi_{21}-\chi_{22})(s) \parallel_{\boldsymbol{H}}^2\big) ds \\
+\frac{1}{2\varepsilon} \parallel \mathrm{div}\boldsymbol{u}_1 \parallel_{L^\infty(0,T;H)}^2 \int_0^t \parallel \alpha(\vartheta_1)-\alpha(\vartheta_2) \parallel_{L^3(\Omega)}^2 ds.
\end{aligned} \tag{4.4}$$

Fixing ε small enough, taking into account (4.2 - 4.4) and recalling that (3.22) along with the semicontinuity of norms with respect to the weak topology, gives that $\parallel \mathrm{div}\boldsymbol{u} \parallel_{L^\infty(0,T;H)} \leq c$, from

(4.1), we get the following estimate

$$\sum_{j=1}^{2}\Big(\parallel (\chi_{j1}-\chi_{j2})(t)\parallel_H^2+\int_0^t\parallel \nabla(\chi_{j1}-\chi_{j2})(s)\parallel_{\boldsymbol{H}}^2 ds\Big)$$
$$\leq c\Big(\sum_{j=1}^{2}\parallel \chi_{j1}^0-\chi_{j2}^0\parallel_H^2+\int_0^t\parallel (\vartheta_1-\vartheta_2)(s)\parallel_H^2 ds$$
$$+\int_0^t\parallel \alpha(\vartheta_1)-\alpha(\vartheta_2)\parallel_{L^3(\Omega)}^2 ds+\sum_{j=1}^{2}\int_0^t\parallel (\chi_{j1}-\chi_{j2})(s)\parallel_H^2 ds$$
$$\int_0^t\Big(1+\parallel \nabla\vartheta_2(s)\parallel_{(L^3(\Omega))^3}^2\Big)\parallel (\boldsymbol{u}_1-\boldsymbol{u}_2)(s)\parallel_{\boldsymbol{H}}^2 ds\Big). \tag{4.5}$$

In order to get a control of the function $(\boldsymbol{u}_1-\boldsymbol{u}_2)$ with respect to the norm of $C^0([0,T];\boldsymbol{H})$, we first consider the difference between (2.22) written for for $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively and integrate over $(0,t)$ for $t\in(0,T)$. Then, we test by $\boldsymbol{v}=(\boldsymbol{u}_1-\boldsymbol{u}_2)(t)$, and integrate one more time over $(0,t)$. Thus, owing to (2.10) and (2.11) we obtain

$$\frac{1}{2}\parallel (\boldsymbol{u}_1-\boldsymbol{u}_2)(t)\parallel_{\boldsymbol{H}}^2-\frac{1}{2}\parallel \boldsymbol{u}_1^0-\boldsymbol{u}_2^0\parallel_{\boldsymbol{H}}^2+\frac{1}{2}a\left(\int_0^t(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds,\int_0^t(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds\right)$$
$$\leq<\boldsymbol{w}_1^0-\boldsymbol{w}_2^0,\int_0^t(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds>+\int_0^t<\int_0^s(\boldsymbol{G}_1-\boldsymbol{G}_2)(r)dr,(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)>ds$$
$$-\int_0^t\left(\int_0^s(\alpha(\vartheta_1)\chi_{21}-\alpha(\vartheta_2)\chi_{22})(r)dr,\mathrm{div}(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)\right)_\Omega ds. \tag{4.6}$$

Our next aim is to get a bound on the right side of (4.6). To this end, due to (3.13), we handle the first two terms as follows

$$<\boldsymbol{w}_1^0-\boldsymbol{w}_2^0,\int_0^t(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds>$$
$$\leq\frac{\varepsilon}{2}\left\|\int_0^t(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds\right\|_{\boldsymbol{V}}^2+\frac{1}{2\varepsilon}\left\|\boldsymbol{w}_1^0-\boldsymbol{w}_2^0\right\|_{\boldsymbol{V}'}^2, \tag{4.7}$$

and

$$\int_0^t<\int_0^s(\boldsymbol{G}_1-\boldsymbol{G}_2)(r)dr,(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)>ds$$
$$\leq\frac{1}{2}\int_0^t\left\|\int_0^s(\boldsymbol{G}_1-\boldsymbol{G}_2)(r)dr\right\|_{\boldsymbol{H}}^2+\frac{1}{2}\int_0^t\|(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)\|_{\boldsymbol{H}}^2 ds. \tag{4.8}$$

Integrating by parts with respect to time, we rewrite the last term as follows

$$-\int_0^t\left(\int_0^s(\alpha(\vartheta_1)\chi_{21}-\alpha(\vartheta_2)\chi_{22})(r)dr,\mathrm{div}(\boldsymbol{u}_1-\boldsymbol{u}_2)(s)\right)_\Omega ds\;=\;\sum_{i=4}^{7}J^{(i)}(t), \tag{4.9}$$

where

$$
\begin{aligned}
J^{(4)}(t) &= -\left(\int_0^t (\alpha(\vartheta_1)-\alpha(\vartheta_2))(s)\chi_{21}(s)ds, \operatorname{div}\int_0^t (\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds\right)_\Omega,\\
J^{(5)}(t) &= -\left(\int_0^t \alpha(\vartheta_2(s))(\chi_{21}-\chi_{22})(s)ds, \operatorname{div}\int_0^t (\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds\right)_\Omega,\\
J^{(6)}(t) &= \int_0^t \left((\alpha(\vartheta_1)-\alpha(\vartheta_2))(s)\chi_{21}(s), \operatorname{div}\int_0^s (\boldsymbol{u}_1-\boldsymbol{u}_2)(r)dr\right)_\Omega ds,\\
J^{(7)}(t) &= \int_0^t \left(\alpha(\vartheta_2(s))(\chi_{21}-\chi_{22})(s), \operatorname{div}\int_0^s (\boldsymbol{u}_1-\boldsymbol{u}_2)(r)dr\right)_\Omega ds.
\end{aligned}
$$

Thanks to Hölder's inequality and owing to (2.4), (2.6), (2.13), and (3.13), we infer that, for any $\varepsilon > 0$ to be chosen later

$$
\begin{aligned}
J^{(4)}(t)+J^{(6)}(t) \;&\le\; \frac{\sqrt{3}}{2}\Bigg(\varepsilon\left\|\int_0^t (\boldsymbol{u}_1-\mathbf{u}_2)(s)ds\right\|_{\boldsymbol{V}}^2 + \int_0^t \left\|\int_0^s (\boldsymbol{u}_1-\boldsymbol{u}_2)(r)dr\right\|_{\boldsymbol{V}}^2 ds\\
&+\; c_\kappa^2 \parallel \alpha' \parallel_{L^\infty(\mathbb{R})}^2 \left(\frac{T}{\varepsilon}+1\right)\int_0^t \parallel (\vartheta_1-\vartheta_2)(s) \parallel_H^2 ds\Bigg),
\end{aligned}
\tag{4.10}
$$

analogously,

$$
\begin{aligned}
J^{(5)}(t)+J^{(7)}(t) \;&\le\; \frac{\sqrt{3}}{2}\Bigg(\varepsilon\left\|\int_0^t (\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds\right\|_{\boldsymbol{V}}^2 + \int_0^t \left\|\int_0^s (\boldsymbol{u}_1-\boldsymbol{u}_2)(r)dr\right\|_{\boldsymbol{V}}^2 ds\\
&+\; \parallel \alpha \parallel_{L^\infty(\mathbb{R})}^2 \left(\frac{T}{\varepsilon}+1\right)\int_0^t \|(\chi_{21}-\chi_{22})(s)\|_H^2 ds\Bigg).
\end{aligned}
\tag{4.11}
$$

Now, fixing a proper ε and taking into account (4.10-4.11) and (2.3), (4.6) becomes

$$
\begin{aligned}
&\parallel (\boldsymbol{u}_1-\boldsymbol{u}_2)(t) \parallel_{\boldsymbol{H}}^2 + \left\|\int_0^t (\boldsymbol{u}_1-\boldsymbol{u}_2)(s)ds\right\|_{\boldsymbol{V}}^2\\
&\le c\Bigg(\parallel \boldsymbol{u}_1^0-\boldsymbol{u}_2^0 \parallel_{\boldsymbol{H}}^2 + \parallel \boldsymbol{w}_1^0-\boldsymbol{w}_2^0 \parallel_{\boldsymbol{V}'}^2\\
&+\int_0^t \left\|\int_0^s (\boldsymbol{G}_1-\boldsymbol{G}_2)(r)dr\right\|_{\boldsymbol{H}}^2 ds + \int_0^t \parallel (\vartheta_1-\vartheta_2)(s) \parallel_H^2 ds\\
&+\int_0^t \parallel (\boldsymbol{u}_1-\boldsymbol{u}_2)(s) \parallel_{\boldsymbol{H}}^2 ds + \int_0^t \left\|\int_0^s (\boldsymbol{u}_1-\mathbf{u}_2)(r)dr\right\|_{\boldsymbol{V}}^2 ds\\
&\int_0^t \parallel (\chi_{21}-\chi_{22})(s) \parallel_H^2 ds\Bigg).
\end{aligned}
\tag{4.12}
$$

Thus, adding (4.12) to (4.5) and applying the Gronwall Lemma, one can find a positive costant c

such that

$$\| \boldsymbol{u}_1 - \boldsymbol{u}_2 \|^2_{C^0([0,T];\boldsymbol{H})} + \sup_{t\in[0,T]} \left\| \int_0^t (\boldsymbol{u}_1 - \boldsymbol{u}_2)(s)ds \right\|^2_{\boldsymbol{V}} + \sum_{j=1}^{2} \| \chi_{j1} - \chi_{j2} \|^2_{L^2(0,T;V)\cap C^0([0,T];H)}$$
$$\leq c\Bigg(\sum_{j=1}^{2} \| \chi^0_{j1} - \chi^0_{j2} \|^2_H + \| \mathbf{u}^0_1 - \mathbf{u}^0_2 \|^2_{\boldsymbol{H}} + \| \mathbf{w}^0_1 - \mathbf{w}^0_2 \|^2_{\boldsymbol{V}'}$$
$$+ \| \boldsymbol{G}_1 - \boldsymbol{G}_2 \|^2_{L^2(0,T;\boldsymbol{H})} + \| \vartheta_1 - \vartheta_2 \|^2_{L^2(0,T;H)} + \| \alpha(\vartheta_1) - \alpha(\vartheta_2) \|^2_{L^2(0,T;L^3(\Omega))} \Bigg). \tag{4.13}$$

In order to get an estimate for $\| (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t \|_{C^0(0,T;\boldsymbol{V}')}$, let us integrate with respect to time the difference between (2.22) written for $\mathcal{F}_1$ and the same equation written for $\mathcal{F}_2$; hence, (4.13), and a comparison in the resulting relation, allow us to conclude the proof of (2.29). Now, the uniqueness result easily follows from the estimate (2.29). Indeed, if $\mathcal{F}_1 \equiv \mathcal{F}_2$, the right hand vanishes; hence we conclude immediately that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\chi_{j1} = \chi_{j2}$ $j = 1, 2$, $\forall t \in [0, T]$ and a.e. in Ω.

To conclude the proof of Theorem **2.4**, it remains to control the function $(\boldsymbol{u}_1 - \boldsymbol{u}_2)$ in the norm of $C^1([0,T];\boldsymbol{H}) \cap C^0([0,T];\boldsymbol{V})$. To this end, we take the difference between (2.22) written for $\mathcal{F}_1$ and $\mathcal{F}_2$, then we choose $\boldsymbol{v} = (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t$ as test function. All this is formal because $(\boldsymbol{u}_1 - \boldsymbol{u}_2)_t \in C^0([0,T];\boldsymbol{H})$. Nevertheless, it is possible to find a rigorous justification in [11, Appendix], to which we refer for the detailed computations. Thus, for $t \in (0, T)$, we formally obtain

$$\frac{1}{2}\frac{d}{dt} \|(\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t)\|^2_{\boldsymbol{H}} + \frac{1}{2}\frac{d}{dt} a\big(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2\big)(t) =< (\boldsymbol{G}_1 - \boldsymbol{G}_2)(t), (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t) >$$
$$- \big((\alpha(\vartheta_1)\chi_{21} - \alpha(\vartheta_2)\chi_{22})(t), \mathrm{div}(\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t)\big)_\Omega. \tag{4.14}$$

Arguing as in (4.8), one can easily bound the first term in the right side of(4.14). Our next aim is to control the other term; to this end, we integrate by parts with respect to space variables. It is straightforward to obtain

$$-\big((\alpha(\vartheta_1)\chi_{21} - \alpha(\vartheta_2)\chi_{22})(t), \mathrm{div}(\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t)\big)_\Omega = \sum_{i=8}^{11} J^{(i)}(t), \tag{4.15}$$

for $t \in (0, T)$ where

$$J^{(8)}(t) = \big(\nabla\chi_{21}(t)(\alpha(\vartheta_1) - \alpha(\vartheta_2))(t), (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t)\big)_\Omega,$$

$$J^{(9)}(t) = \big(\chi_{21}(t)\nabla(\alpha(\vartheta_1) - \alpha(\vartheta_2))(t), (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t)\big)_\Omega,$$

$$J^{(10)}(t) = \big(\alpha'(\vartheta_2(t))\nabla\vartheta_2(t)(\chi_{21} - \chi_{22})(t), (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t)\big)_\Omega,$$

$$J^{(11)}(t) = \big(\alpha(\vartheta_2(t))\nabla(\chi_{21} - \chi_{22})(t), (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t)\big)_\Omega.$$

To handle $\big\{J^{(i)}(t)\big\}_{i=8}^{11}$, we exploit the same techniques used in (4.10 - 4.11). Thus, in view of (2.3),

we integrate (4.14) in time over $(0,t)$ for $t \in (0,T)$ and we add it to (4.5), obtaining

$$
\begin{aligned}
&\| (\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(t) \|_{\boldsymbol{H}}^2 + \| (\boldsymbol{u}_1 - \boldsymbol{u}_2)(t) \|_{\boldsymbol{V}}^2 \\
&+ \sum_{j=1}^{2} \left(\|(\chi_{j1} - \chi_{j2})(t)\|_H^2 + \int_0^t \|\nabla (\chi_{j1} - \chi_{j2})(s)\|_{\boldsymbol{H}}^2 \, ds \right) \\
&\leq c \Bigg(\left\|\boldsymbol{w}_1^0 - \boldsymbol{w}_2^0\right\|_{\boldsymbol{H}}^2 + \left\|\boldsymbol{u}_1^0 - \boldsymbol{u}_2^0\right\|_{\boldsymbol{V}}^2 + \sum_{j=1}^{2} \left\|\chi_{j1}^0 - \chi_{j2}^0\right\|_H^2 + \|\boldsymbol{G}_1 - \boldsymbol{G}_2\|_{L^2(0,t;\boldsymbol{H})}^2 \\
&+ \|\vartheta_1 - \vartheta_2\|_{L^2(0,t;H)}^2 + \|\alpha(\vartheta_1) - \alpha(\vartheta_2)\|_{L^2(0,t;V)}^2 \\
&+ \int_0^t \left(1 + c_\kappa^2 + c_\Omega^2 + \|\chi_{21}(s)\|_{H^2(\Omega)}^2 + \|\nabla\vartheta(s)\|_{(L^3(\Omega))^3}^2\right) \|(\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(s)\|_{\boldsymbol{H}}^2 \, ds \\
&+ \int_0^t \left(1 + \|\nabla\vartheta_2(s)\|_{(L^3(\Omega))^3}^2\right) \|(\boldsymbol{u}_1 - \boldsymbol{u}_2)_t(s)\|_{\boldsymbol{H}}^2 + \int_0^t \| (\chi_{21} - \chi_{22})(s) \|_H^2 \, ds \Bigg)
\end{aligned}
\tag{4.16}
$$

Hence, applying the Gronwall Lemma to (4.16) and recalling that $\sum_{j=1}^{2} \| \chi_j \|_{L^2(0,T;H^2(\Omega))} \leq c$ thanks to (3.23) and to the semicontinuity of norms with respect to the weak topology, we easily conclude the proof of Theorem **2.4**.

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