Global Attractor for a Class of Doubly Nonlinear Abstract Evolution Equations *

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Abstract

In this paper we consider the Cauchy problem for the abstract nonlinear evolution equation in a Hilbert space $\mathscr{H}$

$$\begin{cases} \mathscr{A}(u'(t)) + \mathscr{B}(u(t)) - \lambda u(t) \ni f & \text{in } \mathscr{H} \text{ for a.e. } t \in (0,T) \\ u(0) = u_0, \end{cases}$$

where $\mathscr{A}$ is a maximal (possibly multivalued) monotone operator from the Hilbert space $\mathscr{H}$ to itself, while $\mathscr{B}$ is the subdifferential of a proper, convex and lower semicontinuous function $\varphi : \mathscr{H} \to (-\infty, +\infty]$ with compact sublevels in $\mathscr{H}$ satisfying a suitable compatibility condition. Finally, λ is a positive constant. The existence of solutions is proved by using an approximation-a priori estimates-passage to the limit procedure. The main result of this paper is that the set of all the solutions generates a *Generalized Semiflow* in the sense of John M. Ball [Bal97] in the phase space given by the domain of the potential φ. This process is shown to be point dissipative and asymptotically compact; moreover the global attractor, which attracts all the trajectories of the system with respect to a metric strictly linked to the constraint imposed on the unknown, is constructed. Applications to some problems involving PDEs are given.

Key words: Global attractor, doubly nonlinear evolution equation, abstract Cauchy problem, generalized semiflow, existence, nonuniqueness.

AMS (MOS) Subject Classification: 37L30, 35B41, 35K55.

1 Introduction

Let $\mathscr{H}$ be a Hilbert space endowed with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We are given the following Cauchy problem for the abstract evolution equation

$$\begin{cases} \mathscr{A}(u'(t)) + \mathscr{B}(u(t)) - \lambda u(t) \ni f & \text{in } \mathscr{H} \text{ for a.e. } t \in (0,+\infty) \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where $u' := du/dt$, and the nonlinear and possibly multivalued operators $\mathscr{A}$ and $\mathscr{B}$ act from $\mathscr{H}$ to $2^{\mathscr{H}}$, the space of all subsets of $\mathscr{H}$. Moreover, λ is a positive constant and u_0 and f are given

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data. In this paper we aim to analyze the asymptotic stability of (1.1) from the point of view of the global attractor under suitable assumptions on the structure of the two nonlinear maximal monotone operators $\mathscr{A}$ and $\mathscr{B}$. More precisely, in our analysis we suppose that $\mathscr{A}$ is bounded, so that $D(\mathscr{A}) \equiv \mathscr{H}$, and with at most linear growth. We recall that a maximal monotone operator $\mathscr{A}$ has linear growth whenever there exists a positive constant $\mathcal{C}_{\mathscr{A}}$ such that

$$\| w \|^2 \leq \mathcal{C}_{\mathscr{A}} \big(1 + \| v \|^2 \big) \quad \forall \ [v, w] \in \mathscr{A}. \tag{1.2}$$

As regards the other operator, we ask $\mathscr{B}$ to be the subdifferential of a convex, proper and lower semicontinuous function $\varphi : \mathscr{H} \to (-\infty, +\infty]$, with (proper) domain $D(\varphi) := \big\{ v \in \mathscr{H} : \varphi(v) < +\infty \big\}$ and compact sublevels in $\mathscr{H}$. Moreover, we ask φ to fulfill a suitable compatibility condition. We do not require any growth condition on $\mathscr{B}$. Finally, for the initial datum u_0 and the forcing function f, we suppose that

$$u_0 \in D(\varphi), \quad f \in \mathscr{H}. \tag{1.3}$$

For f, we are thus requiring that it is independent of time. Our system is thus autonomous.
In Section 4 we will prove that this problem admits at least one solution. Our technique relies on an approximation by regularization. More precisely, we introduce a regularized version of (1.1), which will be solved by means of standard ODE techniques. Subsequently, we will derive some uniform (in the approximating parameter) a priori estimates on the solution of the approximating problem. Thus, the passage to the limit procedure will be finally achieved by exploiting some compactness argument and the monotonic structure of the two nonlinearities. We have to stress that, while the existence result for the $\lambda = 0$ case is not new (as the papers [CV90] and [Col92] show in the Hilbert and in the Banach space case, respectively), no existence result is available, up to our knowledge, for the perturbed equation in (1.1). The key argument in proving the existence result is a compatibility condition between φ and the non convex quadratic perturbation $-\lambda \| \cdot \|^2$. With this position, the proof of our existence theorem substantially reduces to the proof of the unperturbed case (i.e. $\lambda = 0$), which is similar to the one given in [CV90]. However, our abstract framework is slightly different: more precisely, while in [CV90] the problem is settled down in the usual Hilbert triplet $V \subset H \equiv H' \subset V'$ (see, e.g., [LM72]) and the potential φ is taken coercive with respect to the V-norm, here we only ask the potential φ to have compact sublevels in $\mathscr{H}$.

As regards the uniqueness of solutions to systems of the form (1.1), it is well known that genuine non uniqueness may occur. This means, in particular, that equation (1.1) does not generates a semigroup and thus the standard theory for the construction of a global attractor (see, e.g., [Tem97] and [BV92]) is not applicable. Anyway, we are able to overcome the lack of uniqueness and prove the existence of the global attractor by exploiting the theory recently proposed by J.M. Ball for the study of the long time behavior of the Navier-Stokes equation and the semilinear damped wave equation (see [Bal97], [Bal98] and [Bal04]). The basic concept in the study of the asymptotic behavior for systems for which non uniqueness of solutions may occur, is the concept of *generalized semiflow*. This is defined as a family of maps $\mathfrak{F} : [0, +\infty) \to \mathcal{X}$ (where $\mathcal{X}$ is the proper phase space) satisfying some axioms relating to existence, time translations, concatenation and upper-semicontinuity with respect to initial data (see the next Section 2). It is possible to extend to *generalized semiflows* standard definitions for semiflows, such as the concepts of *positive orbit, ω-limit sets, attractor, point dissipativity and asymptotic compactness.* Furthermore, it has been proved in [Bal97] (see also Theorem 2.5 in this paper) that if the *generalized semiflow* $\mathfrak{F}$ is point dissipative and asymptotically compact, then it has a global attractor in the proper phase space $\mathcal{X}$. Regarding the regularity imposed on the initial datum u_0 (see (1.3)), it seems to us that the most natural phase space $\mathcal{X}$ for our problem (1.1) is the domain of the potential φ. This space becomes a metric space with the metric "induced" by φ, that is

$$d_{\mathcal{X}}(u,v) := \parallel u - v \parallel + |\varphi(u) - \varphi(v)|, \quad \forall u, v \in \mathcal{X}.$$

We stress that our phase space is reminiscent of the phase space introduced in [RS04] for the study of the global attractor for the Penrose-Fife model for phase transitions. In the next Section 4, we will show that the set of all solutions to (1.1) (see definition 3.6) is a *generalized semi-flow* on the phase space $\mathcal{X} = D(\varphi)$. In this abstract setting, the tricky and far from obvious part consists in showing the upper semicontinuity of the solutions to (1.1) with respect to initial data (see the axiom $(H4)$ in the Definition 2.3 below) in the phase space $\mathcal{X} = D(\varphi)$ endowed with the metric $d_{\mathcal{X}}$. Anyway, the regularization effect of equation (1.1), that gives that the selection $w \in \partial\varphi(u)$ has $\parallel w(t) \parallel^2$ finite, for almost any $t > 0$, and a careful application of the Helly Theorem will allow us to overcome this difficulty. Finally, in Section 5 we will show that the *generalized semiflow* associated with (1.1) is point dissipative with respect to the metric $d_{\mathcal{X}}$ and asymptotically compact and thus, following Ball [Bal97], it admits a global attractor in $\mathcal{X}$.
Actually, there are other possible strategies to overcome the difficulty of non uniqueness of solutions. One alternative method (see [Sel73]) is to recover uniqueness of solutions by working in a space of semi trajectories $u : [0, +\infty) \to \mathcal{X}$ and thus defining a corresponding semiflow $T(\cdot)$ by $T(t)u = u^\tau$, for $\tau \geq 0$, where $u^\tau(t) = u(t + \tau)$. This approach has been used by Sell in [Sel96] to prove the existence of a global attractor for the 3D incompressible Navier-Stokes equations. However, this method has the disadvantage of proving the existence of the global attractor in a space of semi trajectories and not in the physical phase space.
Another method, which is more closely related to Ball's approach, is to consider a set-valued trajectory $t \to T(t)z$ in which $T(t)z$ consists of all possible points reached at time t by solutions with initial data z. For works based on this second approach, we refer, among the others, to [KMV03], [MV00].

It does not seem, up to our knowledge, that the characterization of the global attractor for equation of the form (1.1) has yet been tackled (both in the $\lambda = 0$ and in the $\lambda \neq 0$ case). More concern has been devoted to the existence of a global attractor for equations of the type (see [Shi00])

$$\big(\mathscr{A}u(t)\big)' + \mathscr{B}u(t) + g(t, u(t)) \ni f(t) \quad \text{in } \mathscr{H}, \quad t > 0 \tag{1.4}$$

with $\mathscr{A}$ and $\mathscr{B}$ still nonlinear and satisfying proper assumptions. In particular, in [Shi00] $\mathscr{A}$ is a continuous and bi-Lipschitz subdifferential of a continuous and convex function on $\mathscr{H}$, while $\mathscr{B}$ is the subdifferential of a time dependent proper and lower semicontinuous function with compact sublevels on $\mathscr{H}$. Moreover, g is a single-valued operator in $\mathscr{H}$ and f is a given function. However, it is worthwhile noting, that in this case the author is able to prove uniqueness of the solutions and thus he shows the existence of the global attractor by using the usual theory developed for semigroups. Moreover, the large time behavior of the dynamical system associated to (1.4) is characterized by means of that of a proper limiting autonomous dynamical system. Finally, regarding equation (1.4), the particular case in which $\mathscr{A}$ is an increasing locally lipschitz continuous function from $\mathbb{R}$ to $\mathbb{R}$ and $\mathscr{B}$ is the p-Laplacian operator has been analyzed in [EHEO02].

Doubly non linear equations like (1.1) rather than being purely mathematical objects, have a number of interesting physical applications (as we will show in section 6). For instance, they may represent a gradient flow in presence of a pseudo potential of dissipation $\psi(\partial_t u)$, with ψ proper, convex and lower semicontinuous such that $\partial\psi = \mathscr{A}$ (see, e.g, [BDG89], [CV93], [Ger73] [Vis96] and Remark 3.4 in this paper) or a generalization of some kind of hysteresis process (see [Vis94, Sects. VI.3, VI.4]).

This paper is organized as follows. In the next Section 2, we present some preliminary tools about maximal monotone operators and their approximation. Moreover, we recall from [Bal97] some notions on *generalized semiflows*, especially in connection with their long-time behavior. The subsequent Section 3 will be dedicated to the presentation of the main results of the paper. In Section 4 the *generalized semiflow* is constructed and in Section 5 we will characterize its global attractor. Finally, in Section 6, we present some possible application of our theory to certain doubly nonlinear physical models.

2 Preliminaries

In this section we introduce some notation and recall some preliminary machinery which is needed to state our problem in a rigorous way.
Since we deal with time dependent functions defined on all the positive line $(0,+\infty)$, for a Banach space X, we let $H^m_{loc}(0,+\infty,X)$ stand for the set of all measurable functions v from $(0,+\infty)$ to X such that $v \in H^m(0,T;X)$ for all $T>0$ (for the definition of this last space we refer to Lions and Magenes [LM72, pg. 7]). Now, we recall some basics facts about maximal monotone operators which will be intensively used throughout the paper. The reader is referred to [Att84], [Bré71] and [Bré73] for the details of the proofs. Given a Hilbert space $\mathscr{H}$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, we consider the multivalued map $\mathscr{B}$ from $\mathscr{H}$ to $2^{\mathscr{H}}$, and we let the expression $w\in\mathscr{B}v$ to denote that $[v,w]\in\mathscr{B}$. In fact, we have implicitly identified the operator $\mathscr{B}$ with its graph in $\mathscr{H}\times\mathscr{H}$. An operator $\mathscr{B}$ is called monotone if, for every $[v_1,w_1],[v_2,w_2]\in\mathscr{B}$, there holds $\big\langle w_1-w_2, v_1-v_2\big\rangle \geq 0$. Moreover, we say that $\mathscr{B}$ is maximal monotone if it is maximal in the sense of inclusion of graphs within the class of monotone operators. The Minty Theorem gives an equivalent way to characterize maximal monotone operators, that is to require the existence of some $\varepsilon>0$ such that $\mathcal{R}(I+\varepsilon\mathscr{B})=\mathscr{H}$, where $\mathcal{R}$ indicates the range of the operator. For any maximal monotone operator and for any $\varepsilon>0$, we introduce the resolvent $J_\varepsilon := \big(Id+\varepsilon\mathscr{B}\big)^{-1}$ which turns out to be a one to one contraction mapping defined on all $\mathscr{H}$. Then, we define the Yosida approximation $\mathscr{B}_\varepsilon$ of $\mathscr{B}$ by letting $\mathscr{B}_\varepsilon := \dfrac{1}{\varepsilon}\big(I-J_\varepsilon\big)$. This approximation, which will be the main tool in proving the existence result for (1.1), is an everywhere Lipschitz continuous mapping with Lipschitz constant equal to ε^{-1}. Now, we introduce the notion of subdifferential operators, which will be extremely relevant for the forthcoming analysis. Let $\varphi:\mathscr{H}\to(-\infty,+\infty]$ be a proper, convex and lower semicontinuous function, then we define its subdifferential $\partial\varphi:\mathscr{H}\to 2^{\mathscr{H}}$ (the power set) as follows

$$\partial\varphi := \big\{[u,v]\in\mathscr{H}\times\mathscr{H} : \varphi(u)-\varphi(w)\leq\langle v,u-w\rangle, \forall w\in D(\varphi)\big\}, \tag{2.1}$$

where $D(\varphi)$ is the effective domain of φ, i.e., the set $D(\varphi)=\big\{v\in\mathscr{H}:\varphi(v)<+\infty\big\}$. It is well known that, under the above assumptions on φ, the subdifferential $\partial\varphi$ turns out to be maximal monotone in the sense specified above (see, e.g., [Bré73, pg. 25]). Concerning the approximation of subdifferential mapping using the Yosida approximation, we have the following

Proposition 2.1. *Let $\mathscr{B}=\partial\varphi$, with φ convex, proper and lower semicontinuous from $\mathscr{H}$ to $\mathbb{R}^+$, define*

$$\varphi_\varepsilon(u) := \min_{z\in\mathscr{H}}\left\{\frac{1}{2\varepsilon}\parallel u-z\parallel^2 + \varphi(z)\right\}. \tag{2.2}$$

Then, φ_ε is convex, Fréchet-differentiable in $\mathscr{H}$ and its subdifferential coincides with $\mathscr{B}_\varepsilon$. Moreover,

$$\varphi_\varepsilon(u) = \frac{\varepsilon}{2}\parallel \mathscr{B}_\varepsilon u\parallel^2 + \varphi(J_\varepsilon u), \quad \forall u\in\mathscr{H}, \ \ \forall\varepsilon>0, \tag{2.3}$$

$$\varphi_\varepsilon(u)\nearrow\varphi(u), \quad \forall u\in\mathscr{H} \quad \text{as } \ \varepsilon\searrow 0. \tag{2.4}$$

Next, we give the notion of convergence *in the sense of Mosco* for a sequence φ_n of convex, proper and lower semicontinuous functions. More precisely, we say that φ_n converges to φ *in the sense of Mosco* in $\mathcal{H}$ if

$$\forall u_n \to u \quad \text{weakly in} \quad \mathcal{H}, \quad \varphi(u) \leq \liminf_{n \nearrow +\infty} \varphi_n(u_n) \quad \text{and}$$

$$\forall u \in \mathcal{H} \ \exists \{u_n\} \ \text{such that} \quad u_n \to u \ \text{strongly in} \quad \mathcal{H} \ \text{and} \ \varphi(u) = \lim_{n \nearrow +\infty} \varphi_n(u_n).$$

We conclude this part by reporting the fundamental Chain rule Lemma

Lemma 2.2 (Chain rule). *Let $\varphi : \mathscr{H} \to]-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. If $u \in H^1(0,T;\mathscr{H})$, $v \in L^2(0,T;\mathscr{H})$ and $v(t) \in \partial\varphi(u(t))$ for a.e. $t \in]0,T[$, then the function $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0,T]$, and for a.e. $t \in]0,T[$*

$$\frac{d}{dt}\varphi(u(t)) = \langle w, u'(t) \rangle, \quad \forall w \in \partial\varphi(u(t)) \tag{2.5}$$

Now, we define the object of our study, the *generalized semiflow*. Namely, we summarize some definitions and results from [Bal97] concerning *generalized semiflows* and their long-time behavior.

Suppose we are given a metric space (not necessarily complete) $\mathcal{X}$ with metric $d_{\mathcal{X}}$. If C is a subset of $\mathcal{X}$ and b is point in $\mathcal{X}$, we set $\rho(b,C) := \inf_{c \in C} d_{\mathcal{X}}(b,c)$, consequently, if $C \subset \mathcal{X}$ and $B \subset \mathcal{X}$, we set $\operatorname{dist}(B,C) := \sup_{b \in B} \rho(b,C)$.

Definition 2.3. A *generalized semiflow* $\mathfrak{F}$ on $\mathcal{X}$ is a family of maps $u : [0,+\infty) \to \mathcal{X}$, called solutions, satisfying the following hypotheses:
(H1) (*Existence*) For each $v \in \mathcal{X}$ there exists at least one $u \in \mathfrak{F}$ with $u(0) = v$.
(H2) (*Translates of solutions are still solutions*) If $u \in \mathfrak{F}$ and $\tau \geq 0$, then $u^\tau \in \mathfrak{F}$ where $u^\tau(t) := u(t+\tau)$, $t \in (0,+\infty)$.
(H3) (*Concatenation*) If $u, v \in \mathfrak{F}$, $t \geq 0$ with $u(t) = v(0)$ then $w \in \mathfrak{F}$ where

$$w(\tau) := \begin{cases} u(\tau) & \text{for} \quad 0 \leq \tau \leq t, \\ v(\tau - t) & \text{for} \quad t < \tau. \end{cases}$$

(H4) (*Upper semi-continuity with respect to initial data*) If $u_n \in \mathfrak{F}$ with $u_n(0) \to v$, then there exist a subsequence u_{n_k} of u_n and $u \in \mathfrak{F}$ with $u(0) = v$ such that $u_{n_k}(t) \to u(t)$ for each $t \geq 0$.

Furthermore, a *generalized semiflow* can satisfy (or not) the following continuity properties.
(C1) Each $u \in \mathfrak{F}$ is continuous from $(0,+\infty)$ to $\mathcal{X}$.
(C2) If $u_n \in \mathfrak{F}$ with $u_n(0) \to v$, then there exists a subsequence u_{n_k} of u_n and $u \in \mathfrak{F}$ with $u(0) = v$ such that $u_{n_k}(t) \to u(t)$ uniformly for t in compact subsets of $(0,+\infty)$.
(C3) Each $u \in \mathfrak{F}$ is continuous from $[0,+\infty)$ to $\mathcal{X}$.
(C4) If $u_n \in \mathfrak{F}$ with $u_n(0) \to v$, then there exists a subsequence u_{n_k} of u_n and $u \in \mathfrak{F}$ with $u(0) = v$ such that $u_{n_k}(t) \to u(t)$ uniformly for t in compact subsets of $[0,+\infty)$.
For other interesting properties on *generalized semiflows*, especially relating measurability and continuity we refer to [Bal97]. These results are extension to *generalized semiflows* of the results of [Bal76] concerning semiflows originally given for nonlinear evolutionary processes on metric spaces .

Now, we extend to *generalized semiflow* the standard definition concerning absorbing sets and attractors given for semiflows and semigroups (cf. [SY02] and [Tem97]) . Let $\mathfrak{F}$ be a *generalized semiflow* and let $E \subset \mathcal{X}$. For any $t \geq 0$, we define

$$T(t)E = \{u(t) : u \in \mathfrak{F} \ \text{with} \ u(0) \in E\}, \tag{2.6}$$

so that $T(t) : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$, denoting by $2^{\mathcal{X}}$ the space of all subsets of $\mathcal{X}$. It is worthwhile to note that, thanks to $(H2)$ and $(H3)$, $\{T(t)\}_{t\geq 0}$ defines a semigroup on $2^{\mathcal{X}}$. On the other hand, $(H4)$ implies that $T(t)z$ is compact for any $z \in \mathcal{X}$.
The *the positive orbit* of $u \in \mathfrak{F}$ is the set $\gamma^+(u) = \{u(t) : t \geq 0\}$. If $E \subset \mathcal{X}$ then the *positive orbit* of E is the set $\gamma^+(E) = \bigcup_{t\geq 0} T(t)E = \bigcup \{\gamma^+(u) : u \in \mathfrak{F}, u(0) \in E\}$.
The $\omega - limit$ set of $u \in \mathfrak{F}$ is the set

$$\omega(u) = \big\{v \in X : u(t_n) \to v \quad \text{for some sequence } \ t_n \nearrow +\infty\big\},$$

while the $\omega - limit$ set of E, is the set

$$\omega(E) := \big\{u^\infty \in \mathcal{X} : \quad \text{there exist} \quad u_n \in \mathfrak{F} \quad \text{with} \quad u_n(0) \in E, \quad u_n(0) \quad \text{bounded,} \\ \text{and a sequence} \quad t_n \nearrow +\infty \ \text{ with } \ u_n(t_n) \to u^\infty\big\}$$

A *complete orbit* is a map $\Psi : \mathbb{R} \to \mathcal{X}$ such that, for any $s \in \mathbb{R}$, $\Psi^s \in \mathfrak{F}$. Then, if Ψ is a *complete orbit*, we can define the $\alpha - limit$ set of Ψ as

$$\alpha(\Psi) := \big\{z \in \mathcal{X} : \Psi(t_n) \to z \quad \text{for some sequence } \ t_n \to -\infty\big\}.$$

We say that the subset $\mathcal{U} \subset \mathcal{X}$ *attracts* a set E if $\operatorname{dist}(T(t)E, \mathcal{U}) \to 0$ as $t \to +\infty$.
We say that $\mathcal{U}$ is *positively invariant* if $T(t)\mathcal{U} \subset \mathcal{U}$ for all $t \geq 0$, while $\mathcal{U}$ is *invariant* if $T(t)\mathcal{U} = \mathcal{U}$ for all $t \geq 0$.
The subset $\mathcal{U} \subset \mathcal{X}$ is a global attractor if $\mathcal{U}$ is compact, invariant, and attracts all bounded sets.
$\mathfrak{F}$ is *eventually bounded* if, given any bounded $B \subset \mathcal{X}$, there exists $\tau \geq 0$ with $\gamma^\tau(B)$ bounded.
$\mathfrak{F}$ is *point dissipative* if there exists a bounded set B_0 such that, for any $u \in \mathfrak{F}$, $u(t) \in B_0$ for all sufficiently large $t \geq 0$.
$\mathfrak{F}$ is *asymptotically compact* if for any sequence $u_n \in \mathfrak{F}$ with $u_n(0)$ bounded, and for any sequence $t_n \nearrow +\infty$, the sequence $u_n(t_n)$ has a convergent subsequence.
$\mathfrak{F}$ is *compact*, if for any sequence $u_n \in \mathfrak{F}$ with $u_n(0)$ bounded, there exists a subsequence u_{n_k} such that $u_{n_k}(t)$ is convergent for any $t > 0$.
The next Proposition, whose (simple) proof is to be found in [Bal97, Prop.3.2], will be relevant in proving the existence of the global attractor for our system

Proposition 2.4. *Let $\mathfrak{F}$ eventually bounded and compact. Then $\mathfrak{F}$ is asymptotically compact.*

We now quote the general abstract criterion providing a sufficient and necessary condition for the existence of the attractor.

Theorem 2.5 (Ball 1997). *A generalize semiflow $\mathfrak{F}$ has a global attractor if and only if $\mathfrak{F}$ is point dissipative and asymptotically compact. The attractor $\mathcal{U}$ is unique. Moreover, $\mathcal{U}$ is the maximal compact invariant set of $\mathcal{X}$ and it is given by*

$$\mathcal{U} = \bigcup \{\omega(B) : \quad B \quad \textit{a bounded set of } \ \mathcal{X}\} = \omega(\mathcal{X}) \tag{2.7}$$

For the proof of this result the reader is referred to [Bal97, Theorem **3.3**].

We conclude this section by quoting a classical result, due to Helly, on compactness of monotone functions with respect to the pointwise convergence. For the proof of this result, the reader is referred to, e.g., [AGSar].

Proposition 2.6 (Helly). *Suppose that ϕ_n is a sequence of non increasing functions defined in $[0,T]$ with values in $[-\infty,+\infty]$. Then, there exist a subsequence $n(k)$ and a non increasing map $\phi:[0,T]\to[-\infty,+\infty]$ such that $\phi(t)=\lim_{k\nearrow+\infty}\phi_{n(k)}(t)$ for any $t\in[0,T]$.*

3 Main results

We begin specifying the assumptions on the operators $\mathscr{A}$ and $\mathscr{B}$, on the potential φ and on data. We ask

$$\mathscr{A} \text{ is a maximal monotone graph in } \mathscr{H}\times\mathscr{H}, \text{ with } 0\in\mathscr{A}0, \tag{3.1}$$

$$\exists\, C_1, C_2>0: \quad \langle \xi, v\rangle \geq C_1 \parallel v\parallel^2 - C_2, \quad \forall[v,\xi]\in\mathscr{A}, \tag{3.2}$$

$$\exists\, C_{\mathscr{A}}>0: \quad \parallel \xi\parallel^2 \leq C_{\mathscr{A}}\big(\parallel v\parallel^2+1\big), \quad \forall[v,\xi]\in\mathscr{A}, \tag{3.3}$$

$$\mathscr{B} \text{ is a maximal monotone graph in } \mathscr{H}\times\mathscr{H} \text{ given by } \mathscr{B}=\partial\varphi, \tag{3.4}$$

$$\varphi:\mathscr{H}\to\mathbb{R}^+\cup\{+\infty\}, \text{ proper, convex and lower semicontinuous,} \tag{3.5}$$

$$\forall c\in\mathbb{R}, \text{ the set } \Big\{u\in\mathscr{H}:\varphi(u)\leq c\Big\} \text{ is locally compact in } \mathscr{H}, \tag{3.6}$$

$$\exists C_{\varphi_1}, C_{\varphi_2}\in\mathbb{R} \text{ with } 0<C_{\varphi_1}<1 \text{ and } C_{\varphi_2}\geq 0 \text{ such that}$$

$$\varphi(v)-\lambda\parallel v\parallel^2\geq C_{\varphi_1}\varphi(v)-C_{\varphi_2}, \quad \forall v\in D(\varphi), \tag{3.7}$$

$$0\in D(\varphi), \tag{3.8}$$

$$f\in\mathscr{H}, \quad u(0)=u_0\in\mathcal{X}. \tag{3.9}$$

Remark 3.1. Note that the assumptions (3.2-3.3) restrict the behavior of $\mathscr{A}$ at infinity but allow the presence of horizontal and vertical segments in its graph. In particular, $\mathscr{A}$ could be multivalued.

Remark 3.2. The assumption (3.8) is not restrictive since with a proper translation we can deal with the general case in which $0\notin D(\varphi)$.

Remark 3.3. Note that the compatibility condition (3.7) could be read as a further coercivity condition on the potential φ. In fact, since the constant C_{φ_1} in (3.7) is strictly smaller than 1, then (3.7) becomes

$$\varphi(v)\geq\frac{\lambda}{1-C_{\varphi_1}}\parallel v\parallel^2-\frac{C_{\varphi_2}}{1-C_{\varphi_1}}, \quad \forall\, v\in D(\varphi). \tag{3.10}$$

This reformulation of (3.7) will be extremely useful in proving the dissipativity of our *generalized semiflow*.

Remark 3.4 (A gradient flow in presence of a pseudo potential of dissipation). By introducing the notion of Fréchet subdifferential for a proper and lower semicontinuous function ψ (not necessarily convex!) (see, e.g., [DGMT80]), that is the set

$$\partial_F\psi(v)=\big\{w\in\mathscr{H}:\psi(z)-\psi(v)-\langle w,z-v\rangle\geq o(\parallel z-v\parallel) \quad \text{as } z\to v \text{ strongly in } \mathscr{H}\big\}, \tag{3.11}$$

where the Landau notation should be understood as

$$\liminf_{z\to v}\frac{\psi(z)-\psi(v)-\langle w,z-v\rangle}{\parallel z-v\parallel}\geq 0, \tag{3.12}$$

we can equivalently rewrite Problem (1.1) as

$$\mathscr{A}(u'(t)) + \partial_F \tilde{\varphi}(u(t)) \ni f \text{ in } \mathscr{H} \text{ for a.e. } t \in (0,T), \tag{3.13}$$

$$u(0) = u_0, \tag{3.14}$$

where $\tilde{\varphi}$ is the quadratic perturbation of φ given by

$$\tilde{\varphi}(v) := \varphi(v) - \frac{\lambda}{2} \parallel u \parallel^2, \forall v \in D(\varphi), \tag{3.15}$$

and the infinitesimal term in (3.11) is of the form $o(r) := -\frac{\lambda}{2} r^2$. Functional of the type of $\tilde{\varphi}$ are usually named λ-*convex*. Thus, (1.1) could interpreted as a gradient flow for the λ-*convex* function $\tilde{\varphi}$ (see (3.15)) in presence of the pseudo potential of dissipation $\psi(u')$, with ψ proper positive, convex, lower semicontinuous, such that $\psi(0) = 0$ and $\partial\psi = \mathscr{A}$. However, our assumptions do not force $\mathscr{A}$ to be a subdifferential, thus allowing us to consider also systems (see Example 2 in this paper for a discussion in this direction). We conclude this remark by noting that assumptions (3.5) and (3.7) entail that

$$\tilde{\varphi}(v) := \varphi(v) - \frac{\lambda}{2} \parallel v \parallel^2 \geq -C_{\varphi_2} \quad \forall v \in D(\varphi). \tag{3.16}$$

Now, we have to fix the phase space $\mathcal{X}$ in order to study the long-time dynamic of the system (1.1). Regarding the regularity imposed on the initial datum u_0, we think that the natural phase space $\mathcal{X}$ for our problem is the effective domain of φ, that is $D(\varphi) := \{u \in \mathscr{H} \text{ such that } \varphi(u) < +\infty\}$. This space becomes a metric space with the following distance

$$d_{\mathcal{X}}(u,v) := \parallel u - v \parallel + |\varphi(u) - \varphi(v)|, \quad \forall u, v \in \mathcal{X}. \tag{3.17}$$

Remark 3.5. It is worthwhile to note that the metric space $\mathcal{X} = D(\varphi)$ with the metric $d_{\mathcal{X}}$ defined above in this general abstract setting is not complete, but anyway the completeness of the phase space is not essential for the theory of *generalized semiflow*.

Definition 3.6 (Definition of solution). A function $u : [0, +\infty) \to \mathscr{H}$ is called a solution of (1.1) if
(s1): $u \in H^1(0,T;\mathscr{H}) \cap C^0([0,T];\mathcal{X})$ for all $T > 0$,
(s2): there exist $\xi, w : (0, +\infty) \to \mathscr{H}$ with $\xi \in L^2(0,T;\mathscr{H})$ $\forall T > 0$ and $w \in L^2(0,T;\mathscr{H})$ $\forall T > 0$ such that

$$\xi(t) \in \mathscr{A}(u'(t)) \quad \text{for a.a. } t > 0, \tag{3.18}$$

$$w(t) \in \partial\varphi(u(t)) \quad \text{for a.a. } t > 0, \tag{3.19}$$

$$\xi(t) + w(t) - \lambda u(t) = f \quad \text{for a.a } t > 0. \tag{3.20}$$

Let (DNE) denote the set of all solutions to (1.1). Theorem **3.7** below shows that given any $u_0 \in \mathcal{X}$ there exists at least one solution to (1.1) with $u(0) = u_0$. The set (DNE) will be constructed by using an approximation by regularization, as we shall see in the next section 4.

Theorem 3.7 (Existence). *Under assumptions (3.1-3.9), problem (1.1) admits at least one solution.*

In the Theorem **3.8** below, we show that (DNE) is a *generalized semiflow* on $\mathcal{X}$.

Theorem 3.8. *Under the assumptions of Theorem* **3.7**, *(DNE) is a generalized semiflow on* $\mathcal{X}$ *satisfying* $C1$.

Finally, we can prove that the *generalized semiflow* (DNE) has a global attractor in the phase space $\mathcal{X} = D(\varphi)$ which attracts all the trajectories of the system with respect to the metric (3.17).

Theorem 3.9. *Under assumptions (3.1-3.9), there exists a unique global attractor $\mathcal{U}$ for (DNE) that is given by $\mathcal{U} = \omega(D(\varphi))$.*

Remark 3.10 (The $\lambda = 0$ case). In the case $\lambda = 0$ the compatibility condition (3.7) is unnecessary for the proof of Theorems **3.7** and **3.8** since it reduces to ask that φ should be bounded from below (recall that $\varphi(v) \geq 0 \ \ \forall v \in D(\varphi)$). However, in order to prove Theorem **3.9** for the $\lambda = 0$ case, a coercivity condition on the potential φ of the type of (3.10) is mandatory.

The proofs of Theorems **3.8** and **3.9** will be outlined in the next section 4 and 5.

4 The generalized semiflow generated by doubly non linear equations

This section is devoted to the proof of Theorems **3.7** and **3.8**. In this direction, first we regularize problem (1.1) by replacing the multivalued operator $\partial\varphi$ with its Yosida regularization, then we solve (P_ε), the regularized version of (1.1), by means of ODE techniques. Subsequently, we will derive some uniform a-priori estimates on the approximated solution and finally the passage to the limit procedure will be achieved by means of monotonicity and compactness arguments.

First we regularize (1.1). To this aim, we replace the multivalued operator $\partial\varphi$ with its Yosida approximation $\partial\varphi_\varepsilon$ and we consider the following approximating problem.

PROBLEM (P_ε): Let $0 < \varepsilon \leq 1$ be given. Find $u_\varepsilon \in C^1([0,T];\mathscr{H})$, $\forall T > 0$ and $\xi_\varepsilon \in C^0([0,T];\mathscr{H})$, $\forall T > 0$ such that

$$\varepsilon u_\varepsilon'(t) + \xi_\varepsilon(t) + \partial\varphi_\varepsilon(u_\varepsilon(t)) - \lambda u_\varepsilon(t) = f, \ \ \forall t \in [0,T] \tag{4.1}$$

$$\xi_\varepsilon(t) \in \mathscr{A}(u_\varepsilon'(t)), \quad \forall t \in [0,T] \tag{4.2}$$

$$u_\varepsilon(0) = u_0 \tag{4.3}$$

For problem (P_ε) there holds the following

Proposition 4.1. *Under assumptions (3.1-3.6), Problem P_ε admits a unique solution.*

Proof. Since $\big(\varepsilon Id + \mathscr{A}\big)^{-1}$ and $\partial\varphi_\varepsilon - \lambda Id$ are Lipschitz continuous from $\mathscr{H}$ to $\mathscr{H}$, we can use a Cauchy-Lipschitz-Picard type argument (see, e.g., [Bre83, pg. 104]) to deduce there exist a unique $u_\varepsilon : [0,+\infty) \to \mathscr{H}$ with $u_\varepsilon \in C^1([0,T];\mathscr{H})$ $\forall T > 0$, satisfying the Cauchy condition and

$$u_\varepsilon'(t) - \big(\varepsilon Id + \mathscr{A}\big)^{-1}\big(f - \partial\varphi_\varepsilon(u_\varepsilon(t) + \lambda u_\varepsilon(t)\big) = 0, \ \ \forall t > 0 \tag{4.4}$$

which is equivalent to (4.1), while $\xi_\varepsilon \in C^0([0,T];\mathscr{H})$, $\forall T > 0$ is given by

$$\xi_\varepsilon(t) = f - \varepsilon u_\varepsilon'(t) - \partial\varphi_\varepsilon(u_\varepsilon(t)) + \lambda u_\varepsilon(t), \ \ \forall T > 0. \tag{4.5}$$

$\square$

Now, we turn our attention to the construction of the *generalized semiflow* for (1.1). In particular, recalling the list of axioms defining a *generalized semiflow*, we start by showing the Existence result of Theorem **3.7**. The strategy of the proof relies on some *a priori* estimates on the solution of P_ε. This estimates are uniform with respect to the approximation parameter ε, allowing us to pass to the limit as $\varepsilon \searrow 0$ in Problem P_ε in a proper sense. Before deriving the a priori estimates on the approximated solution, we give advice to the reader that in the sequel we widely use the convention to denote with C different constants which depend only on the constants and on the norms of the functions involved in (3.1-3.9) and on the final time T. Thus, let us test (4.1) by u'_ε and integrate in time in $(0,t)$, with $t \leq T$. Since an analogous of the coercivity condition (3.10) holds also for the $\varepsilon-$approximation φ_ε of φ, easy manipulations give

$$\int_0^t \| u'_\varepsilon(s) \|^2 \, ds + \varphi_\varepsilon(u_\varepsilon(t))$$
$$\leq C\left(1 + \| f \|^2 + \varphi_\varepsilon(u_0) + \int_0^t \varphi_\varepsilon(u_\varepsilon(s)) ds\right), \;\; \forall t \leq T. \tag{4.6}$$

Where the constant C depends on $T, C_1, C_2, \lambda, C_{\varphi_1}$ and C_{φ_2} Now, the Gronwall Lemma entails

$$\int_0^t \| u'_\varepsilon(s) \|^2 \, ds + \varphi_\varepsilon(u_\varepsilon(t)) \leq C, \;\; \forall t \leq T, \tag{4.7}$$

where the positive constant C depends on $T, C_1, C_2, \lambda, C_{\varphi_1}, C_{\varphi_2}, \varphi(u_0)$ and $\| f \|$ but is independent of ε thanks to the convergence in (2.4). Thus, we get

$$\| u'_\varepsilon \|_{L^2(0,T;\mathscr{H})} \leq C. \tag{4.8}$$

Moreover, combining (4.8) with the contraction property of the resolvent operator J_ε, there holds

$$\| u'_\varepsilon \|_{L^2(0,T;\mathscr{H})} + \| (J_\varepsilon u_\varepsilon)' \|_{L^2(0,T;\mathscr{H})} \leq C. \tag{4.9}$$

Now, owing to (2.3), (4.9) and (4.7) we deduce that

$$\| \varphi(J_\varepsilon u_\varepsilon(t)) \| \leq C. \tag{4.10}$$

Finally, (4.9) and the growth condition on $\mathscr{A}$ give

$$\| \xi_\varepsilon \|_{L^2(0,T;\mathscr{H})} \leq C, \tag{4.11}$$

while a comparison in (4.1) shows that

$$\| \partial\varphi_\varepsilon(u_\varepsilon) \|_{L^2(0,T;\mathscr{H})} \leq C. \tag{4.12}$$

Now we are ready to pass to the limit as $\varepsilon \searrow 0$ in Problem P_ε. Estimates (4.9) and (4.11)-(4.12) guarantees that we can use the usual weak and weak star compactness results for a proper diagonal subsequence, which we do not relabel, to obtain the existence of two functions ξ, w, which belong to $L^2_{loc}(0,+\infty;\mathscr{H})$ such that,

$$\xi_\varepsilon \to \xi \quad \text{weakly in} \quad L^2(0,T;\mathscr{H}) \;\text{ for all } \; T > 0, \tag{4.13}$$
$$\partial\varphi_\varepsilon(u_\varepsilon) \to w \quad \text{weakly in} \quad L^2(0,T;\mathscr{H}) \;\text{ for all } \; T > 0. \tag{4.14}$$

We note that

$$u_\varepsilon - J_\varepsilon u_\varepsilon = \varepsilon\partial\varphi_\varepsilon(u_\varepsilon) \to 0 \quad \text{strongly in} \quad L^2(0,T;\mathscr{H}), \;\text{ for all } \; T > 0 \tag{4.15}$$

Moreover, by using the Ascoli-Arzelá Theorem (see [Sim87, Lemma 1, pg. 71]) for the sequence $J_\varepsilon u_\varepsilon$ (recall (3.6), (4.9), (4.10) and (4.15)), we infer that there exists a function $u \in H^1(0,T;\mathscr{H})$ $\forall T$ such that

$$J_\varepsilon u_\varepsilon \to u \quad \text{strongly in } \; C^0([0,T];\mathscr{H}), \quad \text{for all} \quad T>0 \tag{4.16}$$

$$u_\varepsilon \to u \quad \text{strongly in} \quad L^2(0,T;\mathscr{H}) \; \text{ and weakly in } \quad H^1(0,T;\mathscr{H}) \; \text{ for all } \quad T>0. \tag{4.17}$$

Convergences (4.16), (4.17) and the lower semicontinuity of φ give that $u \in L^\infty_{loc}(0,+\infty;\mathcal{X})$, while convergences (4.14) and (4.17) give immediately the identification of w in $\partial\varphi(u)$, that is $w(t) \in \partial\varphi(u(t))$, for almost any $t \geq 0$. Thus, it remains to prove that $\xi(t) \in \mathscr{A}(u'(t))$ for almost any $t \geq 0$, and thanks to [Bré73, Prop 2.5. pg. 27], we have to show that

$$\limsup_{\varepsilon \searrow 0} \int_0^T \big\langle \xi_\varepsilon(t), u_\varepsilon'(t) \big\rangle dt \leq \int_0^T \big\langle \xi(t), u'(t) \big\rangle dt. \tag{4.18}$$

To this end, we test (4.1) by u_ε'; easy manipulations, (1.3), (2.4), (4.3) and (4.17) show that proving (4.18) turn out to be equivalent to prove

$$\liminf_{\varepsilon \searrow 0} \varphi_\varepsilon(u_\varepsilon(T)) \geq \varphi(u(T)). \tag{4.19}$$

This last inequality follows since φ_ε converges *in the sense of Mosco* (see [Att84, Prop. 3.56, pg. 354]) to φ and we have that $u_\varepsilon(T) \to u(T)$ weakly in $\mathscr{H}$. Collecting all this information, we have proved that u solves almost everywhere in $(0,+\infty)$ the problem (1.1). Moreover, there hold $u \in H^1_{loc}(0,+\infty;\mathscr{H}) \cap L^\infty_{loc}(0,+\infty;\mathcal{X})$ and $\xi, w \in L^2_{loc}(0,+\infty;\mathscr{H})$. It remains to show that any solution of (1.1) is continuous with values in $\mathcal{X}$ on every bounded set of $[0,+\infty)$. But, recalling the definition of the metric $d_{\mathcal{X}}$ on $\mathcal{X}$, this continuity property follows since the function $t \mapsto \varphi(u(t))$ is absolutely continuous thanks to the Chain rule Lemma 2.2. Theorem **3.7** is thus completely proved.

We now prove that the set (DNE) generates a *generalized semiflow* on $\mathcal{X}$. Hypothesis $(H1)$ follows from Theorem **3.7**, while $(H2)$ and $(H3)$ easily follow from the definition of solution. On the contrary, the proof of $(H4)$ requires some additional work. Let u_n be a sequence of solutions of (1.1) with the initial datum $u_n(0) \to u_0$ in $\mathcal{X}$. We have to prove that there exist a subsequence u_{n_k} of u_n and a function u with $u(0) = z$, $u_{n_k}(t) \to u(t)$ in $\mathcal{X}$ for any $t \geq 0$ and such that u solves (1.1). First of all, by simply testing equation (1.1) written for u_n by $u_n'(t)$ and using (3.2), the chain rule (2.5) in Lemma 2.2 and (3.7), we have the following

$$\int_0^t \| u_n'(s) \|^2 \, ds + \varphi(u_n(t)) \leq C\big(1 + \varphi(u_n(0)) + \frac{\lambda}{2} \| u_n(0) \|^2 + T\big), \tag{4.20}$$

where the positive constant C depends only on $\| f \|, C_1, C_2, \lambda, C_{\varphi_1}$ and C_{φ_2}. In particular, the right hand side of (4.20) is bounded independently of n thanks to the convergence $u_n(0) \to u_0$ in $\mathcal{X}$. This means that u_n is bounded in $H^1(0,T;\mathscr{H})$ for any $T>0$ and that $\big\{u_n(t), \; n \in \mathbb{N}, \; \forall t \in (0,T]\big\}$ lies in a compact set of $\mathscr{H}$ thanks to (3.6). Moreover, condition (3.3) and a comparison in (1.1) written for u_n, gives that the two selections ξ_n and w_n are bounded in $L^2(0,T;\mathscr{H})$ for any $T>0$. Thus, using the usual weak compactness results combined with the Ascoli-Arzelá Theorem for a diagonal subsequence, which we do not relabel, we can find three functions $u, \xi, w : [0,+\infty) \to \mathscr{H}$ such that

$$u_n \to u \text{ weakly in } \quad H^1(0,T;\mathscr{H}) \quad \forall T>0, \tag{4.21}$$

$$u_n \to u \text{ strongly in } C^0([0,T];\mathscr{H}) \quad \forall T>0, \tag{4.22}$$

$$\xi_n \to \xi \text{ weakly in } L^2(0,T;\mathscr{H}) \quad \forall T>0, \tag{4.23}$$

$$w_n \to w \text{ weakly in } L^2(0,T;\mathscr{H}) \quad \forall T>0, \tag{4.24}$$

Convergences (4.21-4.24) are enough to conclude that u, ξ and w solve (1.1) with $u(0) = u_0$. In fact, (4.22) and (4.23) gives immediately that $w(t) \in \partial\varphi(u(t))$ for almost any $t \in (0,T)$, while (4.21), (4.22) and the lower semicontinuity technique we outlined in the existence proof [1] gives the second identification, that is $\xi(t) \in \mathscr{A}(u'(t))$ for almost any $t \in (0,T)$. To conclude, it remains to verify the point wise convergence of u_n with respect to the metric of $\mathcal{X}$. Since (4.22) implies that $u_n(t) \to u(t)$ in $\mathscr{H}$, for all $t \geq 0$, we only have to prove that $\varphi(u_n(t)) \to \varphi(u(t))$ for all $t \geq 0$. This property, as we will see in a moment, follows from Proposition 2.6 and from the fact that $w_n \in \partial\phi(u_n)$ remains bounded in $L^2(0,T;\mathscr{H})$ for all $T > 0$. First of all, we introduce the sequence of auxiliary functions ζ_n defined in $[0,T]$ with values in $(-\infty,+\infty]$ given by (analogous definition for $\zeta(t)$)

$$\zeta_n(t) := \varphi(u_n(t)) - \frac{\lambda}{2} \parallel u_n(t) \parallel^2 - \langle f, u_n(t)\rangle - C_2 t \tag{4.25}$$

Then, testing equation (1.1) written for u_n by $u_n'(t)$ and recalling (3.2) one readily obtain that $\dfrac{d}{dt}\zeta_n(t) \leq 0$, thus ζ_n is non increasing. Thanks to Proposition 2.6, there exists a non-increasing function $\phi : [0,+\infty) \to \mathbb{R}$ such that

$$\phi(t) := \lim_{k\nearrow+\infty} \zeta_n(t), \quad \forall t \geq 0, \tag{4.26}$$

for a proper subsequence n_k of n. Now, (4.23) combined with the Fatou Lemma, gives that

$$\liminf_{n\nearrow+\infty} \parallel w_n(t) \parallel^2 < +\infty, \quad \text{almost everywhere in} \quad (0,T). \tag{4.27}$$

Thus, for almost any t, we can select a proper subsequence n_{k_λ} of n_k such that $\parallel w_{n_{k_\lambda}}(t) \parallel^2$ is convergent as $\lambda \nearrow +\infty$. Now, the definition of subdifferential (2.1) written for $\varphi(u_{n_{k_\lambda}}(t))$, gives

$$\varphi(u_{n_{k_\lambda}}(t)) \leq \langle w_{n_{k_\lambda}}(t), u_{n_{k_\lambda}}(t) - u(t)\rangle + \varphi(u(t)) \tag{4.28}$$

from which it follows that, passing to the $\liminf$ as $\lambda \nearrow +\infty$ in (4.28) and recalling (4.22), there holds

$$\liminf_{\lambda\nearrow+\infty} \varphi(u_{n_{k_\lambda}}(t)) \leq \varphi(u(t)). \tag{4.29}$$

Actually, the extraction of the subsequence in (4.29) is uniform with respect to t since the following inequalities hold

$$\liminf_{k\nearrow+\infty} \varphi(u_{n_k}(t)) \leq \liminf_{\lambda\nearrow+\infty} \varphi(u_{n_{k_\lambda}}(t)) \leq \varphi(u(t)) \leq \liminf_{k\nearrow+\infty} \varphi(u_{n_k}(t)),$$
$$\text{for almost any } t \text{ in } (0,T). \tag{4.30}$$

[1] Actually in this case things are much more easy, since proving the limsup inequality in (4.18) is equivalent to prove that $\liminf_{n\nearrow+\infty} \varphi(u_n(T)) \geq \varphi(u(T))$. But this follows from the convergence (4.22) and the lower semicontinuity of φ. Thus we do not have to invoke the *Mosco Convergence*.

Thus, recalling (4.22),

$$\liminf_{k\nearrow+\infty} \zeta_{n_k}(t) = \zeta(t) \quad \text{for almost any } \ t \ \text{ in } \ (0,T). \tag{4.31}$$

Now, the limit (4.26), the monotonicity of ϕ and ζ combined with the continuity of ζ (actually much more is true, thanks to Lemma 2.2), gives that

$$\phi(t) = \zeta(t) = \lim_{k\nearrow+\infty} \zeta_{n_k}(t), \quad \forall t > 0, \tag{4.32}$$

and thus, recalling (4.25), (4.22), the fact that $\varphi(u_n(0)) \to \varphi(u_0) = \phi(0)$ and that $u(0) = u_0$, we have $\lim_{k\nearrow+\infty} \varphi(u_{n_k}(t)) = \varphi(u(t)) \quad \forall t \geq 0$. Hence $(H4)$ holds and (DNE) is a *generalized semiflow*.

5 Existence of the global attractor

In this section we prove Theorem **3.9**. Following Ball's approach (see Theorem **2.5**), we have to show that the *generalized semiflow* generated by (1.1) is point dissipative and asymptotically compact. Concerning this last property, we will actually show that our *generalized semiflow* is compact and eventually bounded and thus asymptotically compact thanks to Proposition 2.4. We begin by proving the point dissipativity of our system. There holds the following crucial lemma

Lemma 5.1. *Let* $u : [0, +\infty) \to \mathscr{H}$ *a solution of* (1.1) *in the sense of Definition* (3.6) *and let* φ *satisfy (3.5), (3.7) (or its analog (3.10)) and (3.8). Then there holds*

$$\sigma\big[\varphi(u(t)) - \frac{\lambda}{2} \parallel u(t) \parallel^2 \big] + \frac{d}{dt}\big[\varphi(u(t)) - \frac{\lambda}{2} \parallel u(t) \parallel^2 \big] \leq \mathcal{C}(1+ \parallel f \parallel^2), \ \ \forall t > 0, \tag{5.1}$$

where the positive constant $\mathcal{C}$ *depends only on* $C_1, C_2, C_{\mathscr{A}}, C_{\varphi_1}, C_{\varphi_2}, \lambda, \varphi(0)$ *and* σ *is a proper (and computable) scaling constant greater than* 0.

Proof. The proof of this result is reached via a number of a priori estimates. Along the proof, we agree to denote by C a generic positive constant depending on data. Moreover, we denote by c_ε a constant allowed to depend in addition on a positive (small) parameter (here ε). In particular, we make use of the Young inequality in the following form

$$ab \leq \varepsilon a^2 + c_\varepsilon b^2, \quad \forall a, b \in \mathbb{R}, \forall \varepsilon > 0. \tag{5.2}$$

We stress that all the subsequent calculations are completely justified in our regularity framework.
First estimate:
test (1.1) by $u(t)$. Recalling the definition of subdifferential, (3.8), (3.7) and (3.10), and using the inequality (5.2) one obtains

$$\varphi(u(t)) \leq C\bigg(1 + c_\varepsilon\big(\parallel f \parallel^2 + \parallel u'(t) \parallel^2 \big) + \varepsilon\varphi(u(t))\bigg), \tag{5.3}$$

where the constant C depends only on $\varphi(0), C_{\mathscr{A}}, \lambda, C_{\varphi_1}$ and C_{φ_2}.
Second estimate:
test (1.1) by $u'(t)$. Recalling (5.2), we obtain

$$\parallel u'(t) \parallel^2 + \frac{d}{dt}\big[\varphi(u(t)) - \frac{\lambda}{2} \parallel u(t) \parallel^2 \big] \leq C\bigg(1 + c_{\varepsilon_1} \parallel f \parallel^2 + \varepsilon_1 \parallel u'(t) \parallel^2 \bigg), \tag{5.4}$$

where $C > 0$ is a constant depending on C_1 and C_2. By choosing ε and ε_1 small enough and summing (5.3) multiplied by a proper scaling constant $\delta > 0$ to (5.4), we readily obtain (5.1). $\square$

Now, we show that this estimate, combined with (3.7), actually entails the existence of a bounded set B_0 such that, for any solution u to (1.1), there holds $u(t) \in B_0$ for all sufficiently large $t \geq 0$. As a first step, we note that a set B of $\mathcal{X}$ is bounded with respect to the metric $d_{\mathcal{X}}$ whenever

$$\exists R_B > 0 : \quad d_{\mathcal{X}}(z, 0) \leq R_B, \quad \forall\, z \in B. \tag{5.5}$$

Thus, by applying the Gronwall Lemma in the differential form to (5.1) (recall (3.16)) and finally using the compatibility condition (3.7), we can find a proper finite time t^* and a radius R_{B_0}, both computable in terms of the data, such that

$$\forall u \in (DNE), \quad u(t) \in B_0, \;\; \forall t \geq t^*, \tag{5.6}$$

that is the *generalized semiflow* (DNE) is point dissipative. The eventually boundedness of the *generalized semiflow* follows from a similar argument. Now, we have to prove that the *generalized semiflow* generated by (DNE) is compact. Then, suppose we are given a sequence $u_n \in (DNE)$ such that $u_n(0)$ is bounded in $\mathcal{X}$, we have to show that there exists a subsequence n_k of n such that $u_{n_k}(t)$ is convergent in $\mathcal{X}$ *for all* $t > 0$. Actually, as we will see in a moment, we do not need to know that $u_n(0) \to u_0$ in $\mathcal{X}$ to conclude that $u_n(t) \to u(t)$ in $\mathcal{X}$ for all $t > 0$, with u a solution of (1.1) with $u(0) = u_0$. In fact, since $u_n(0)$ is bounded in $\mathcal{X}$ by assumption, we can argue as in (4.20) and obtain, for a subsequence n_k of n, the convergences (4.21-4.24). Moreover, since φ is lower semicontinuous and has compact sublevels in $\mathscr{H}$ (see (3.6)), there exist $u_0 \in X$ such that $u_n(0) \to u_0$ in $\mathscr{H}$. This gives that the limit u in (4.21-4.24) solves (1.1) with $u(0) = u_0$ and thus belongs to (DNE). Also the pointwise convergence in $\mathcal{X}$ can be deduced by exploiting the same arguments used in proving $(H4)$. Obviously, we can not conclude that $u_n(t) \to u(t)$ in $\mathcal{X}$ for all $t \geq 0$, but only for t strictly greater than 0. We have thus proved that (DNE) is compact. Proposition 2.4 and Theorem **2.5** apply and so we conclude that there exist a unique global attractor for (DNE). This ends the proof of Theorem **3.9**.

6 Applications

In this section we shall give some applications of the previous results to initial and boundary value problems for partial differential equations and systems. These examples are just intended to suggest a class of problems that can be solved by our Theorems, and not to cover all the possible range of applications.
Henceforth, we shall denote by Ω a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. The notations for Sobolev spaces are the same as in [LM72].
Thus, in Example 1 we deal with a generalized form, devised by M. E. Gurtin (see [Gur96]), of the well known Allen-Cahn equation. In Examples 2 and 3 on the contrary, we analyze some models that fits the $\lambda = 0$ situation. These two last Examples are of independent interest since, although the existence of solution is known (one can argue as in [CV90]), the existence of the global attractor is completely new. More precisely, in Example 2 we will deal with an interior obstacle problem for a quasi linear elliptic operator with a nonlinear time relaxation dynamics. Next, in Example 3, we will see that the equation describing the martensitic dynamics in the Frémond model for shape memory alloys (in which a non smooth pseudo potential of dissipation is taken into account) perfectly complies with our assumptions.

6.1 Example 1

In this Example we aim to show that the generalized version of the Allen-Cahn equation derived by Gurtin in ([Gur96]) can be rewritten as a doubly nonlinear abstract evolution equation of the form (1.1), and thus one can apply the abstract machinery of Theorems **3.7-3.9** to obtain that the set of all the solutions is a *generalized semiflow* and that it has a unique global attractor.
The Allen-Cahn equation [2] plays a central and major role in material sciences. In fact, it describes very important and interesting for applications qualitative features of two phase systems, that is the ordering of atoms within unit cells. Thus, the scalar (actually we can deal also with vector value functions) function $u : \Omega \times (0,T) \to \mathbb{R}$ will represent the order parameter. Moreover, although other choices are possible, we impose Neumann boundary condition for the unknown function u. Finally, regarding the domain Ω, we restrict the analysis to the physically significant case of the dimension $N = 2, 3$. Thus Ω will be a regular and bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$. Moreover, we let $\mathscr{H}$ to be $L^2(\Omega)$. The Allen-Cahn equation is based on a free energy of the form

$$\Psi(u) := \int_\Omega \big(\varepsilon|\nabla u(x)|^2 + \frac{1}{\varepsilon}\mathcal{W}(u(x))\big)dx, \quad \forall\, u \in H^1(\Omega), \tag{6.1}$$

where ε is a positive parameter. The term with $\mathcal{W}$ is the so called *double well potential* whose wells characterize the phases of the material. A thermodynamically consistent choice is provided by a non smooth potential of the form

$$\mathcal{W}(v) := 1 - v^2 + I_{[-1,1]}(v) = \begin{cases} 1 - v^2 & \text{if} \quad |v| \le 1, \\ +\infty & \text{otherwise} \end{cases} \qquad \forall\, v \in \mathbb{R}. \tag{6.2}$$

It is worthwhile to note that the two terms $\varepsilon|\nabla u|^2$ and $\dfrac{1}{\varepsilon}\mathcal{W}(u)$ in (6.1) are in competition. In fact, when u is not uniform the second one penalizes the deviation from the pure states $|u(x)| = 1$ for almost any $x \in \Omega$, whereas the first one penalizes the hight gradients that are induced by sharp variations of u. Moreover, for small values of the parameter ε any absolute minimum of the functional Ψ attains values close to the pure state $u = \pm 1$ in the whole domain Ω but for thin transition layers. In the real world systems, the parameter ε is taken so small that the layer thickness is of the order of 10^{-7} cm. This length scale is known as microscopic length scale, since it is close to that of molecular phenomena. Finally, the term with $I_{[-1,1]}$ is the indicator function of the interval $[-1, 1]$ and forces the order parameter to attain values only in $[-1, 1]$, that is $-1 \le u(x) \le 1$ for almost any $x \in \Omega$. By denoting the convex and lower semicontinuous (hence subdifferentiable) part of the free energy by φ, that is,

$$\varphi(u) := \int_\Omega \big(\varepsilon|\nabla u(x)|^2 + I_{[-1,1]}(u) + \frac{1}{\varepsilon}\big)dx, \quad \forall u \in H^1(\Omega), \tag{6.3}$$

it is easy to see that the domain of φ is $D(\varphi) = H^1(\Omega) \cap K := \big\{v \in L^2(\Omega) : -1 \le v(x) \le 1 \text{ for a.e. } x \in \Omega\big\}$, and that φ has compact sublevels in $\mathscr{H} = L^2(\Omega)$. Moreover, the subdifferential of φ with respect to the Hilbert structure of $L^2(\Omega)$ has, thanks to [Bré73, Prop 2.17], the simple expression

$$\begin{gathered} w^* \in \partial\varphi(w) \quad \Leftrightarrow \quad w^* \in -\varepsilon\Delta w + \partial I_K(w), \\ D(\partial\varphi) = H^2(\Omega) \cap K. \end{gathered} \tag{6.4}$$

[2] Actually, this equation is sometimes attributed to S.K. Chan (see [Cha77] and [AC79]), to L.D. Landau and I.M. Khalatnikov (see [LL65] where this equation is named Ginzburg-Landau equation).

Finally, it is not difficult to show that φ satisfies the compatibility condition (3.7) with respect to the non convex quadratic perturbation

$$q(u) := -\int_\Omega |u(x)|^2 dx, \tag{6.5}$$

for proper constants C_{φ_1} and C_{φ_2}. By considering a balance of the microforces, that should be taken into account since it is plausible that their work accompanies changes in the order parameter u, and the particular form of our free energy (6.1), one can follow Gurtin's approach and obtain the following generalized Allen-Cahn equation with double obstacles ± 1. (we refer to [Gur96] for the details of the derivation as well as for the presentation of the theory of microforce balance):

$$\beta \frac{\partial u(t)}{\partial t} - \varepsilon \Delta u(t) + \partial I_K(u(t)) - \frac{2}{\varepsilon} u(t) \ni 0, \quad \text{in } \mathscr{H}, \text{ for almost any } t \in (0,T), \tag{6.6}$$

where the constitutive modulus β depends on $\frac{\partial u}{\partial t}$ in a suitable way. Actually, in the most general situation, β could depend also on $u, \nabla u$ and $\nabla \frac{\partial u}{\partial t}$ but anyway it turns out that our choice is consistent with the laws of thermodynamic and with the derivation of the model. Finally, by focusing on the β's such that the resulting operator $\mathscr{A}$, given by

$$\mathscr{A}(v) := \beta(v)v, \;\; \forall\, v \in \mathscr{H}, \tag{6.7}$$

is maximal monotone and satisfies the assumptions (3.1-3.3), we see that the generalized Allen-Cahn equation in (6.6) can be rewritten as an abstract doubly non linear evolution equation of the type of (1.1) with $\lambda = \frac{\varepsilon}{2}$. Thus, Theorems **3.7-3.9** apply in the phase space $\mathcal{X} = H^1(\Omega) \cap K$. We refer, for example, to [CM99] and to the references therein for other contributions to the mathematical analysis of such models.

6.2 Example 2

Let $\mathscr{H} := (L^2(\Omega))^M$. Given an obstacle $g \in (W^{1,p}(\Omega))^M$ satisfying $g \le 0$ on $\partial\Omega$, let

$$\begin{aligned} \mathcal{K} := \big\{ \boldsymbol{v} = (v_1, \ldots, v_M) \in (W_0^{1,p}(\Omega))^M : v_i(x) \ge g_i(x) \text{ for all } i = 1, \ldots, M \\ \text{and for a.e. } x \in \Omega \big\} \end{aligned} \tag{6.8}$$

be the convex set of admissible configurations and let $I_\mathcal{K}$ be its indicator function

$$I_\mathcal{K} := \begin{cases} 0 & \text{if } \;\; v \in \mathcal{K}, \\ +\infty & \text{otherwise;} \end{cases} \tag{6.9}$$

note that $I_\mathcal{K}$ is a non smooth lower semicontinuous function. Then, we consider for[3] $p \ge 2$ the potential

$$\varphi(\boldsymbol{v}) := \int_\Omega G(x, \nabla v(x)) dx + I_\mathcal{K}(v(x)), \;\; D(\varphi) = \mathcal{K}. \tag{6.10}$$

[3] Actually, for the construction of the *generalized semiflow* for (1.1) it suffices to take $p > 1$. On the other hand, the assumption $p \ge 2$ is crucial in order to deduce from (6.11) the analog of the condition (3.10) (see Remark 3.3) and prove the existence of the global attractor.

Here $G(x,\boldsymbol{z}) : \Omega \times \mathbb{R}^{N\times M} \to \mathbb{R}$ is a convex Carathéodory function and continuously differentiable [4] with respect to $\boldsymbol{z} \in \mathbb{R}^{N\times M}$, for almost any $x \in \Omega$. Moreover, we ask G to satisfy together with its differential $\boldsymbol{a}(x,\boldsymbol{z})$ with respect to $\boldsymbol{z}$, the p-growth condition (see, e.g., [DiB93, Chap II])

$$G(x,\boldsymbol{z}) \geq \alpha_1|\boldsymbol{z}|^p - \alpha_2, \quad |\boldsymbol{a}(x,\boldsymbol{z})| \leq \alpha_3(|\boldsymbol{z}|^{p-1}+1) \quad \forall \boldsymbol{z} \in \mathbb{R}^N \times \mathbb{R}^M, \tag{6.11}$$

where α_1, α_2 are given positive constant. With this position, φ is proper, convex and lower semicontinuous in $\mathscr{H}$, hence subdifferentiable in $\mathscr{H}$. Moreover, the assumption (6.11) guarantees that φ satisfies the analogous of the growth condition (3.10) and the coercivity (3.6), for a proper choice of the dimension N and of the exponent p according to the Rellich-Kondrachov Theorem (see, e.g., [Ada75, Chaps. V, VI]).
Next, let α be a maximal monotone graph (not necessarily a subdifferential) in $\mathbb{R}^M \times \mathbb{R}^M$, and let $\mathscr{A}$ be the following multivalued operator from $(L^2(\Omega))^M$ to $(L^2(\Omega))^M$:

$$\xi \in \mathscr{A}(v) \text{ if and only if } \xi(x) \in \alpha(v(x)) \text{ for a.e. } x \in \Omega.$$

The operator $\mathscr{A}$ is thus maximal monotone (see [Bré73, Exemple 2.3.3]). Moreover, if we assume that there are three positive constants C_1, C_2 and $C_{\mathscr{A}}$ such that

$$\sum_{i=1}^M \xi_i u_i \geq C_1 \sum_{i=1}^M |u_i|^2 - C_2, \qquad \sum_{i=1}^M |\xi|^2 \leq C_{\mathscr{A}}\Big(\sum_{i=1}^M |u_i|^2 + 1\Big), \tag{6.12}$$

for any vector $u = (u_1, \dots, u_M) \in \mathbb{R}^M$ and any $\xi = (\xi_1, \dots, \xi_M) \in \alpha(u)$, then it is easy to see that the above defined operator $\mathscr{A}$ satisfies (3.2) and (3.3).

The following result is thus a direct consequence of the Theorems **3.7**, **3.8** and **3.9** (see also Remarks 3.3 and 3.10)

Corollary 6.1. *Let α be a maximal monotone graph in $\mathbb{R}^M \times \mathbb{R}^M$ fulfilling (6.12). Given $u_0 \in \mathcal{K} = \mathcal{X}$ and $f \in (L^2(\Omega))^M$, then there exist $u = (u_1, \dots, u_M) \in H^1(0,T;(L^2(\Omega))^M) \cap C^0([0,T];\mathcal{X})$ $\forall T > 0$, and $\xi = (\xi_1, \dots, \xi_M), w = (w_1, \dots, w_M) \in (L^2((0,T)\times\Omega))^M$, $\forall T > 0$ satisfying, a.e. in $]0,T[$,*

$$\xi + w = f, \quad \xi \in \mathscr{A}(u'), \quad w \in \partial\varphi(u), \tag{6.13}$$

$$u(0) = u_0. \tag{6.14}$$

Moreover, the set of the solution to (6.13) is a generalized semiflow on the phase space $\mathcal{X} = \mathcal{K}$. Finally, this generalized semiflow has a global attractor which attracts all the trajectories of the system with respect to the following metric

$$d_{\mathcal{X}}(u,v) = \| u - v \|_{(L^2(\Omega))^M} + \left| \int_\Omega \Big(G(x, \nabla u(x)) - G(x, \nabla v(x)) \Big) dx \right|. \tag{6.15}$$

Remark 6.2. In the special case $M = 1$, if we choose $G(x,\boldsymbol{z}) = \frac{1}{p}|\boldsymbol{z}|^p$ we obtain the usual obstacle problem for the p-Laplacian operator with a nonlinear relaxation dynamics, in fact the subdifferential of φ can be easily computed as

$$w^* \in \partial\varphi(u) \Leftrightarrow w^* \in -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + \partial I_{\mathcal{K}}(u) \text{ in } L^2(\Omega). \tag{6.16}$$

[4] We can indeed consider weakly differentiable properties of G. For instance, we can deal with a fairly general function G which is subdifferentiable with respect to its second variable $\boldsymbol{z}$ and satisfies a proper growth condition similar to (6.11).

6.3 Example 3

With the notations of the previous example, we set $M = 2$, $p = 2$ and $N = 3$, thus Ω is a bounded regular domain of $\mathbb{R}^3$. Moreover, we replace the convex in (6.8) with the following bounded triangular convex set in $(L^2(\Omega))^2$

$$K := \Big\{ v = (v_1, v_2) \in (L^2(\Omega))^2 : 0 \le v_i(x) \le 1;\ \ v_1(x) + v_2(x) \le 1 \text{ for a.e. } x \in \Omega \Big\}. \tag{6.17}$$

The potential φ in (6.10) becomes

$$\varphi(v_1, v_2) = \frac{1}{2} \sum_{j=1}^{2} \int_\Omega |\nabla v_j(x)|^2 dx + I_K(v_1, v_2), \quad \forall (v_1, v_2) \in (H^1(\Omega))^2 \cap K, \tag{6.18}$$

where the term with I_K is the indicator function of the convex K and is equal to 0 if $(v_1, v_2) \in K$ and equal to $+\infty$ otherwise. It is clear that the domain of φ is $(H^1(\Omega))^2 \cap K$. Thanks to the boundedness of the convex K it is a standard matter to verify that φ has compact sublevels in $(L^2(\Omega))^2$ and that the coercivity condition (see Remark 3.10)

$$\varphi(v) \ge \alpha \parallel v \parallel^2 - \beta, \ \ \forall v \in D(\varphi) \text{ with } \alpha, \beta > 0$$

is satisfied. Moreover, the subdifferential of φ has, thanks to [Bré73, prop 2.17], the simple expression

$$(w_1^*, w_2^*) \in \partial\varphi(w_1, w_2) \quad \Leftrightarrow \quad \begin{pmatrix} w_1^* \\ w_2^* \end{pmatrix} \in - \begin{pmatrix} \Delta w_1 \\ \Delta w_2 \end{pmatrix} + \partial I_K(w_1, w_2),$$
$$D(\partial\varphi) = (H^2(\Omega))^2 \cap K. \tag{6.19}$$

Then, we introduce the operator $\mathscr{A}$ as the realization in $(L^2(\Omega))^2$ of the following operator $\alpha = Id + S$, where Id is the identity operator in $\mathbb{R}^2$ and S is the following maximal monotone graph

$$S(w_1, w_2) := \begin{cases} \dfrac{[w_1, w_2]}{\sqrt{\sum_{j=1}^2 |w_j|^2}} & \text{if } [w_1, w_2] \neq [0, 0], \\ \big\{ \mathbf{v} \in \mathbb{R}^2 : \parallel \mathbf{v} \parallel \le 1 \big\} & \text{if } [w_1, w_2] = [0, 0]. \end{cases} \tag{6.20}$$

It is easy to see that $\mathscr{A}$ satisfies assumptions (3.1-3.3), thus, if $u_0 = (u_{01}, u_{02}) \in (H^1(\Omega))^2 \cap K$ and given $f \in (L^2(\Omega))^2$, Theorems **3.7**, **3.8** and **3.9** still apply in the phase space $\mathcal{X} = (H^1(\Omega))^2 \cap K$.

Equation (1.1) with the choice of $\mathscr{A}$ and φ outlined above has a physical motivation. As we will see in fact, equation (1.1) represents an abstract version of the equation, derived by M. Frémond, ruling the evolution of the martensites in shape memory alloys. The latter are metallic alloys that exhibit some surprising thermo-mechanical behaviors, namely a super elastic effect and a shape memory effect. The latter one in particular consists in the property of recovering, once deformed, the original shape just by thermal means. Although the phenomenon has been interpreted (see, e.g., [AEK87, Mül79]), at a microscopic scale, as the effect of a structural phase transition between two different configurations of the metallic lattice, the austenite and the martensite, the modeling approach of M. Frémond is macroscopic. Thus, we let u_1, u_2, u_3 denote the volumetric ratios of the two martensitic (u_1, u_2) and of the austenite u_3 variants. In particular, we ask these quantities to fulfill the constraint

$$u_1, u_2, u_3 = 1, \quad 0 \le u_i \le 1 \text{ for } i = 1, 2, 3. \tag{6.21}$$

From a physical point of view, (6.21) means that we are requiring no void nor overlapping between the phases. Because of the relationship (6.21), one of the u's can be selected, for instance $u_3 = 1 - u_1 - u_2$, and eliminated. Thus, the constraint (6.21) reduces to (6.17). Moreover, we assume that the temperature ϑ and the spheric component of the strain tensor, i.e., div$\boldsymbol{u}$, are known and constant in time.[5] We refer to [Fré02] for the detailed derivation of the model by means of the theory of microscopic movements. Here we only present the free energy, which is given as follows

$$\Phi(u_1, u_2) := \varphi(u_1, u_2) + \sum_{i=1}^{3} \big(F_i(\vartheta, \varepsilon(\boldsymbol{u})), u_i\big), \tag{6.22}$$

where $F = (F_1, F_2, F_3)$, depending on ϑ and $\varepsilon(\boldsymbol{u})$, is the volume free energy of the single phases and comes from the classical Landau-Ginzburg theory. The term with φ (see (6.18)) is an interaction energy term, more precisely the term involving the gradients of the phase parameters corresponds to assume that the micro-structure of the material at one point is influenced by its neighborhood, while the indicator function of the convex K forces the phases to attain only physically admissible values, that is $(u_1, u_2) \in K$ during all the evolution. Now, we include the dissipation in the model, and thus the evolution, by following the approach proposed by Moreau (see [Mor70]). Thus, we introduce the pseudo-potential of dissipation as a real positive convex function of the dissipative variables, that in our model are $\beta_t = (\beta_{1t}, \beta_{2t})$. We choose as pseudo potential of dissipation the following convex, lower semicontinuous (non smooth) function

$$\Psi(u_{1t}, u_{2t}) = \frac{1}{2} \sum_{j=1}^{2} |u_{jt}|^2 + \sqrt{\sum_{j=1}^{2} |u_{jt}|^2} \tag{6.23}$$

This particular choice is induced by experimental results: the first term is related to viscous aspect, while the second one is related to the permanent deformations that can influence the direction in the triangle K of the evolution of the phases. Finally, the pseudo potential Ψ is subdifferentiable and it is easy to see that its subdifferential (actually its realization in $(L^2(\Omega))^2$) coincides with the operator $\mathscr{A}$ introduced above, while the vector in $(L^2(\Omega))^2$ given by $f = \begin{pmatrix} F_1 - F_3 \\ F_2 - F_3 \end{pmatrix}$ plays the role of the right hand side in the resulting equation (1.1).

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