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Global Attractors for the Quasistationary Phase Field Model: a Gradient Flow Approach

Antonio Segatti

Abstract. In this note we summarize some results of a forthcoming paper (see [15]), where we examine, in particular, the long time behavior of the so-called quasistationary phase field model by using a gradient flow approach. Our strategy in fact, is inspired by recent existence results which show that gradient flows of suitable non-convex functionals yield solutions to the related quasistationary phase field systems. Thus, we firstly present the long-time behavior of solutions to an abstract non-convex gradient flow equation, by carefully exploiting the notion of *generalized semiflows* by J.M. Ball and we provide some sufficient conditions for the existence of the global attractor for the solution semiflow. Then, the existence of the global attractor for a proper subset of all the solutions to the quasistationary phase field model is obtained as a byproduct of our abstract results.

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1. Introduction

In this paper we are interested in the study of the asymptotic stability from the point of view of the global attractor of the so-called quasistationary phase field system

$$\partial_t(\vartheta + \chi) - \Delta\vartheta = g, \tag{1.1}$$

$$-\Delta\chi + W'(\chi) = \vartheta, \tag{1.2}$$

in $\Omega \times (0,T)$, where Ω is a bounded domain, occupied by a medium liable to phase transition in the time interval $(0,T)$, for $T > 0$. Here, ϑ is the *relative* temperature of the system, and χ is the phase variable. Further, the function W' is the derivative of the double well potential (e.g., $W(\chi) := (\chi^2 - 1)^2/4$, but in our analysis we can cover also the case in which W presents some singular parts) and g

is a source term that will be taken independent of time. The system (1.1)–(1.2) can be formally obtained from the standard parabolic phase field system (firstly discussed by Caginalp in [6]) by suppressing the time derivative of χ in the equation for the order parameter. However, proving the convergence of solutions of the phase field system to the solutions of its quasistationary version is still an open and apparently difficult problem. Indeed, the problem of existence of solutions to (1.1)–(1.2) is intrinsically difficult, because of its mixed elliptic-parabolic nature, i.e., the lack of the term $\partial_t \chi$ in (1.2), which prevents from directly controlling the variation in time of the order parameter. Thus, standard approximation arguments are not straightforwardly available. However, the existence of a suitable solution to (1.1)–(1.2) has been proved by Plotnikov & Starovoitov in [11] and by Schätzle [16] in the technically different cases of Dirichlet and of Neumann boundary condition for ϑ. In their approach, the proof of the convergence of the discrete approximation to the solution to (1.1)–(1.2) relies on Holmgren uniqueness continuation theorem or on refined spectral analysis tools and in both cases essentially depends on the particular shape and regularity of the double well potential $W = (\chi^2 - 1)^2/4$. More recently, Rossi and Savaré in [13, 14] obtained an existence results by a procedure which somehow exploits the underlying physics of the system. In particular, their analysis relies on the crucial observation that (1.1)–(1.2) stems as a gradient flow for a non convex function strictly related to the entropy of the system. Thus the existence of solutions to (1.1)–(1.2) is obtained as a by product of the general existence theory for gradient flows for non convex functional developed in [13, 14]. We recall that also the quasistationary version of the Penrose-Fife model for phase transitions has recently received a good deal of interest, as the paper [8] shows.

Our approach to the analysis of the long-time behavior of (1.1)–(1.2) actually follows from the existence analysis of [13, 14], which we briefly recall. For later convenience, we recast (1.1)–(1.2) by introducing the *internal energy* (*or enthalpy*) variable $u := \vartheta + \chi$, thus obtaining

$$\partial_t u - \Delta(u - \chi) = g, \text{ in } \Omega \times (0,T) \tag{1.3}$$

$$-\Delta\chi + W'(\chi) = u - \chi, \text{ in } \Omega \times (0,T) \tag{1.4}$$

$$u - \chi = 0, \quad \partial_n \chi = 0 \text{ on } \partial\Omega \times (0,T). \tag{1.5}$$

In [13, 14] relation (1.4) is interpreted as the Euler-Lagrange equation for the minimization, with respect to χ and for fixed u, of the functional

$$\mathscr{F}(u,\chi) := \int_\Omega \left(\frac{1}{2}|u-\chi|^2 + \frac{1}{2}|\nabla\chi|^2 + W(\chi) \right) dx,$$

whose gradient flow with respect to the variable u also yields (1.3). Namely, we turn to the system ($f := (-\Delta^{-1})g$)

$$\begin{cases} (-\Delta^{-1})\partial_t u - \frac{\delta\mathscr{F}}{\delta u} = f \quad \text{in } \Omega \times (0,T), \\ \frac{\delta\mathscr{F}}{\delta\chi} = 0 \quad \text{in } \Omega \times (0,T), \end{cases} \tag{1.6}$$

which has a clear variational structure. In fact, in [14] it has been rigorously proved that (1.6) may be interpreted as the *gradient flow* equation (for a suitable notion of subdifferential which we introduce below), in the Hilbert space $H^{-1}(\Omega)$ (recall the conditions (1.5) on $u-\chi$), of the functional defined by $\phi : H^{-1}(\Omega) \to (-\infty, +\infty]$

$$\phi(u) := \inf_{\chi \in H^1(\Omega)} \int_\Omega \left(\frac{1}{2}|u-\chi|^2 + \frac{1}{2}|\nabla\chi|^2 + W(\chi) \right) dx, \quad \text{with } D(\phi) = L^2(\Omega). \tag{1.7}$$

Let us point out that ϕ is a concave perturbation of a quadratic functional, hence it is *non-convex*. In [14] (see also Theorem 3.1 in this paper) existence and approximation results have been obtained for the abstract Cauchy problem

$$u'(t) + \partial_s\phi(u(t)) \ni f \quad \text{a.e. in } (0,T), \quad u(0) = u_0, \tag{GF}$$

for a given initial datum $u_0 \in D(\phi)$ and source term f. The term with $\partial_s\phi$ is a suitable limiting version (cf. the forthcoming Section 3 for the rigorous definition) of the Fréchet subdifferential of the (general) proper and lower semicontinuous functional $\phi : \mathscr{H} \to (-\infty, +\infty]$, not necessarily convex, defined in a (separable) Hilbert space $\mathscr{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|_{\mathscr{H}}$ (which we will often simply denote by $|\cdot|$). Such techniques have been then applied in order to deduce existence and approximation results for the quasistationary phase field evolution problem (1.3)–(1.4), supplemented with Dirichlet boundary conditions. Thus, in order to study the long time behavior of (1.3)–(1.4), we are naturally led to study the long time behavior of (GF) with the potential ϕ given by (1.7) in the Hilbert space $H^{-1}(\Omega)$. For the investigation of the long time behavior of a gradient flow equation for a fairly general non-convex function, the reader is referred to the forthcoming paper [15].

The long time dynamics of gradient flow equations of the type (GF) when the potential ϕ is a convex and lower semicontinuous function (thus $\partial_s\phi$ reduces to the subdifferential of the convex analysis) is rather well known. In particular, the existence of the global attractor has been proved and the long-time convergence to single stationary states investigated even in the non-autonomous situation (see, among the others, [4] and [19]). When ϕ is non convex nor a smooth perturbation of a convex function (as (1.7)), things are remarkably more difficult. In fact, due to the non convexity of the potential ϕ, the uniqueness of the solutions is no longer to be expected (as in the concrete case of the quasistationary phase field system (1.3)–(1.4)). Hence, (GF) does not generate a semigroup, and we cannot rely on the theory of [20] for the study of the long-term dynamics of the solutions.

In recent years, several approaches have been developed in order to address the asymptotic behavior of solutions of differential problems without uniqueness. Without any claim of completeness, we may refer the reader to, e.g., the results by Sell [17] (but see also Chepyzhov & Vishik [7]), Melnik & Valero [10], and, especially, to the work of J.M. Ball, [1, 2]. In particular, we will especially focus here on the theory of *generalized semiflows* proposed in [1]. By definition, a generalized semiflow is a family of functions on $[0, +\infty)$ taking values in a given phase space (we have to think for instance to the solutions to a given differential problem), and complying with suitable existence, stability for time translation, concatenation, and upper semicontinuity axioms. Within this setting, it is possible to introduce a suitable notion of *global attractor*, and to characterize the existence of such an attractor in terms of suitable boundedness and compactness properties on the generalized semiflow. We refer the reader to Section 2 for an overview of these general notions and results, which we have exploited in connection with (GF) in the framework of the metric space (see (3.4))

$$\mathcal{X} = D(\phi), \quad d_{\mathcal{X}} = |u - v| + |\phi(u) - \phi(v)| \ \ \forall u, v \in \mathcal{X}.$$

Indeed, we define the phase space in terms of the energy functional ϕ (see [12, 18] for some analogous choices), which turns out to be a *Lyapunov function* for the system.

We have shown that the set of the solutions to (GF) on the half-line $[0, +\infty)$ is a generalized semiflow, cf. Theorem 3.3 later on, and that it possesses a global attractor, cf. Theorem 3.4. Referring to the forthcoming paper [15] for all the details, we just stress that the energy identity and a proper chain rule for the potential ϕ combined with the compactness of the sublevels of ϕ (see Theorem 3.1) will play a key role both in the proof of the upper-semicontinuity axiom, and of the boundedness and compactness properties of the trajectories. Eventually, we apply our abstract results Theorems 3.3 and 3.4 to the investigation of the long-time behavior of the *energy solutions* to (1.3)–(1.4), i.e., the solutions deriving from the related gradient flow equation (GF), for the functional ϕ in (1.7). Thus, we show that, under suitable conditions, the energy solutions to (1.3)–(1.4) form a generalized semiflow, which admits the global attractor. Let us stress that this gradient flow approach does not provide the description of the long-term behavior of the *whole* set of solutions to (1.3)–(1.4), but it is rather concerned with a proper subclass of trajectories (i.e., the solutions to the gradient flow).

2. Generalized semiflows

Suppose we are given a metric space (not necessarily complete) $\mathcal{X}$ with metric $d_{\mathcal{X}}$. If C is a subset of $\mathcal{X}$ and b is a point in $\mathcal{X}$, we set $\rho(b, C) := \inf_{c \in C} d_{\mathcal{X}}(b, c)$; consequently, if $C \subset \mathcal{X}$ and $B \subset \mathcal{X}$, we set $\operatorname{dist}(B, C) := \sup_{b \in B} \rho(b, C)$.

Definition 2.1. A *generalized semiflow* $\mathfrak{F}$ on $\mathcal{X}$ is a family of maps $u : [0, +\infty) \to \mathcal{X}$, called solutions, satisfying the following hypotheses:

(H1) (*Existence*) For each $v \in \mathcal{X}$ there exists at least one $u \in \mathfrak{F}$ with $u(0) = v$.

(H2) (*Translates of solutions are still solutions*) If $u \in \mathfrak{F}$ and $\tau \geq 0$, then $u^\tau \in \mathfrak{F}$ where $u^\tau(t) := u(t + \tau)$, $t \in (0, +\infty)$.

(H3) (*Concatenation*) If $u, v \in \mathfrak{F}$, and $t \geq 0$ with $u(t) = v(0)$ then the function w defined by

$$w(\tau) := \begin{cases} u(\tau) & \text{for} \quad 0 \leq \tau \leq t, \\ v(\tau - t) & \text{for} \quad t < \tau, \end{cases}$$

belongs to $\mathfrak{F}$.

(H4) (*Upper semi-continuity with respect to initial data*) If $u_n \in \mathfrak{F}$ with $u_n(0) \to v$, then there exist a subsequence u_{n_k} of u_n and $u \in \mathfrak{F}$ with $u(0) = v$ such that $u_{n_k}(t) \to u(t)$ for each $t \geq 0$.

It is possible to extend to *generalized semiflows* the standard concepts concerning absorbing sets, ω-limit sets and attractors given for semiflows and semigroups (see [1]). In particular, for a given *generalized semiflows* $\mathfrak{F}$ any $t \geq 0$, we define

$$T(t)E = \{u(t) : u \in \mathfrak{F} \text{ with } u(0) \in E\}, \tag{2.1}$$

where $E \subset \mathcal{X}$. It is clear that $T(t) : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$, denoting by $2^{\mathcal{X}}$ the space of all subsets of $\mathcal{X}$. Moreover, thanks to $(H2)$ and $(H3)$, $\{T(t)\}_{t \geq 0}$ defines a semigroup on $2^{\mathcal{X}}$. On the other hand, $(H4)$ implies that $T(t)z$ is compact for any $z \in \mathcal{X}$. We say that the subset $\mathcal{U} \subset \mathcal{X}$ *attracts* a set E if $\operatorname{dist}(T(t)E, \mathcal{U}) \to 0$ as $t \to +\infty$.

We say that $\mathcal{U}$ is *invariant* if $T(t)\mathcal{U} = \mathcal{U}$ for all $t \geq 0$.

The subset $\mathcal{U} \subset \mathcal{X}$ is a global attractor if $\mathcal{U}$ is compact, invariant, and attracts all bounded sets.

$\mathfrak{F}$ is *eventually bounded* if, given any bounded $B \subset \mathcal{X}$, there exists $\tau \geq 0$ with $\gamma^\tau(B)$ bounded.

$\mathfrak{F}$ is *point dissipative* if there exists a bounded set B_0 such that, for any $u \in \mathfrak{F}$, $u(t) \in B_0$ for all sufficiently large $t \geq 0$.

$\mathfrak{F}$ is *asymptotically compact* if for any sequence $u_n \in \mathfrak{F}$ with $u_n(0)$ bounded, and for any sequence $t_n \nearrow +\infty$, the sequence $u_n(t_n)$ has a convergent subsequence. We will also make use of the notion of *Lyapunov function*, which can be introduced starting from the following definitions: we say that a complete orbit $g \in \mathfrak{F}$ is *stationary* if there exists $x \in \mathfrak{F}$ such that $g(t) = x$ for all $t \in \mathbb{R}$ – such x is then called a *rest point*. We denote the set of rest points of $\mathfrak{F}$ by $Z(\mathcal{X})$. A function $V : \mathcal{X} \to \mathbb{R}$ is said to be a *Lyapunov function* for $\mathfrak{F}$ if: V is continuous, $V(g(t)) \leq V(g(s))$ for all $g \in \mathfrak{F}$ and $0 \leq s \leq t$ (i.e., V decreases along solutions), and, whenever the map $t \mapsto V(g(t))$ is constant for some complete orbit g, then g is a stationary orbit. Finally, we say that a global attractor A for $\mathfrak{F}$ is *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \subset \mathfrak{F}$ with $\operatorname{dist}(E, A) \leq \delta$, then $\operatorname{dist}(T(t)E, A) \leq \varepsilon$ for all $t \geq 0$. The following Theorem (see [1, Theorem 5.1]) gives sufficient conditions for a generalized semiflow to have a global attractor.

Theorem 2.2 (Ball 1997). *Assume that each element $u \in \mathfrak{F}$ is continuous from $(0, +\infty) \to \mathcal{X}$ and that $\mathfrak{F}$ is asymptotically compact. Suppose further that there exists a Lyapunov function V for $\mathfrak{F}$ and that the sets of its rest points $Z(\mathcal{X})$ is bounded. Then, $\mathfrak{F}$ is also point dissipative, and thus admits a global attractor A.*

Moreover for all trajectories $u \in \mathfrak{F}$, the limit sets $\alpha(u), \omega(u)$ are connected subsets of $Z(\mathcal{X})$ on which V is constant. When $Z(\mathcal{X})$ is totally disconnected the limits

$$z_- = \lim_{t \searrow -\infty} u(t), \quad z_+ = \lim_{t \nearrow +\infty} u(t) \tag{2.2}$$

exist and z_-, z_+ are rest points; moreover, $v(t)$ tends to a rest point as $t \nearrow +\infty$ for every $v \in \mathfrak{F}$.

3. Abstract gradient flows in Hilbert spaces and their long time behavior

In this section we briefly recall the existence theorem of Rossi and Savaré in [14] and we state, without proofs, our main results. The interested reader is referred to the forthcoming paper [15] for the proofs and for some related remarks.

First of all, we have to introduce the notion of subdifferential we aim to use in our analysis. Since the function ϕ is *non convex*, a preliminary choice could be the so called Fréchet subdifferential defined by

$$\xi \in \partial\phi(v) \quad \Leftrightarrow \quad v \in D(\phi), \ \liminf_{w \to v} \frac{\phi(w) - \phi(v) - \langle \xi, w - v \rangle}{|w - v|} \geq 0. \tag{3.1}$$

It is easy to see that the Fréchet subdifferential reduces to the usual one as soon as ϕ is convex. Unfortunately, (3.1) has some drawbacks. In particular, easy finite dimensional examples show that the graph of the Fréchet subdifferential may not be strongly-weakly closed, which is one of the major features of the convex case. We thus define the *strong limiting subdifferential* $\partial_s\phi$ at a point $v \in D(\phi)$ as the set of the vectors ξ such that there exists sequences

$$v_n, \xi_n \in \mathscr{H} \quad \text{with} \quad \xi_n \in \partial\phi(v_n), \ v_n \to v, \ \xi_n \to \xi, \ \phi(v_n) \to \phi(v), \tag{3.2}$$

as $n \uparrow +\infty$. Of course, $\partial_s\phi$ reduces to the usual subdifferential $\partial\phi$ whenever ϕ is convex. Let us now recall one of the existence results proved in [14] for the Cauchy problem (GF).

Theorem 3.1 (Rossi-Savaré 04). *Let $\phi : \mathscr{H} \to (-\infty, +\infty]$ be a proper and lower semicontinuous function which complies with the* coercivity *assumption*

$$\exists\, \kappa \geq 0: \quad v \mapsto \phi(v) + \kappa|v|^2 \quad \text{has compact sublevels,} \tag{COMP}$$

and with the Chain Rule *condition for any bounded interval (a, b)*

$$\begin{aligned} &\text{if } v \in H^1(a, b; \mathscr{H}),\ \xi \in L^2(a, b; \mathscr{H}),\ \xi \in \partial_s\phi(v) \ \textit{a.e. in } (a, b), \\ &\qquad \textit{and } \phi \circ v \textit{ is bounded}, \text{ then } \phi \circ v \in AC(a, b) \textit{ and} \\ &\qquad \frac{d}{dt}\phi(v(t)) = \langle \xi(t), v'(t) \rangle \quad \textit{for a.e. } \ t \in (a, b). \end{aligned} \tag{CHAIN}$$

Then, for any $u_0 \in D(\phi)$, $T > 0$ and $f \in \mathscr{H}$, the Cauchy problem

$$u'(t) + \partial_s\phi(u(t)) \ni f \quad \textit{a.e. in } (0, T), \quad u(0) = u_0,$$

admits a solution $u \in H^1(0,T;\mathscr{H})$*. Moreover, there holds the* energy identity

$$\int_s^t |u'(\sigma)|^2\, d\sigma + \phi(u(t)) = \phi(u(s)) + \int_s^t \langle f, u'(\sigma)\rangle\, d\sigma, \quad \forall s,t \in [0,T]. \tag{3.3}$$

3.1. Long time behavior of (GF)

The assumption $u_0 \in D(\phi)$ in the existence Theorem 3.1, suggests to investigate the long time behavior of (GF) in the phase space (not complete in general)

$$\mathcal{X} := D(\phi), \quad \text{with} \quad d_{\mathcal{X}}(u,v) := |u-v| + |\phi(u) - \phi(v)| \quad \forall u,v \in \mathcal{X}. \tag{3.4}$$

Definition 3.2. *We denote by* $\mathcal{S}$ *the set of all solutions* $u \in H^1_{loc}(0,+\infty;\mathscr{H})$ *to the gradient flow equation*

$$u'(t) + \partial_s\phi(u(t)) \ni f \quad \textit{for a.e. } t \in (0,+\infty). \tag{3.5}$$

We can now state our main results (see [15] for the proofs).

Theorem 3.3 (The generalized semiflow). *Let* ϕ *comply with the assumptions of Theorem* 3.1*. In addition, assume that*

$$\exists K_1, K_2 \geq 0: \quad \phi(u) \geq -K_1|u| - K_2 \quad \forall u \in \mathscr{H}. \tag{3.6}$$

Then, $\mathcal{S}$ *is a generalized semiflow on* $(\mathcal{X}, d_{\mathcal{X}})$*.*

In order to study the long time behavior for our gradient flow equation, we assume some additional continuity property for the potential ϕ, that is

$$v_n \to v, \quad \sup_n \big(|(\partial_s\phi(v_n))^\circ|, \phi(v_n)\big) < +\infty \ \Rightarrow \phi(v_n) \to \phi(v), \tag{CONT}$$

where $|(\partial_s\phi(v))^\circ| := \inf_{\xi\in\partial_s\phi(v)} |\xi|$. The latter is indeed a natural request. In fact, (CONT) is readily fulfilled by lower semicontinuous convex functionals. We thus have

Theorem 3.4 (The global attractor). *Let* ϕ *fulfill the above assumptions of Theorem* 3.1*,* (CONT) *and*

$$\exists J_1, J_2 > 0: \quad \phi(u) \geq J_1|u| - J_2 \quad \forall u \in \mathscr{H}. \tag{3.7}$$

Further, let $\mathcal{D}$ *be a non-empty subset of* $\mathcal{X}$ *satisfying*

$$\begin{gathered} \mathcal{T}(t)\mathcal{D} \subset \mathcal{D} \quad \forall t \geq 0, \\ Z(\mathcal{S}) \cap \mathcal{D} := \{u \in D(\partial_s\phi) \,:\, 0 \in \partial_s\phi(u) - f\} \cap \mathcal{D} \textit{ is bounded in } (\mathcal{X}, d_{\mathcal{X}}). \end{gathered} \tag{3.8}$$

Then, there exists a unique, Lyapunov stable attractor A *for* $\mathcal{S}$ in $\mathcal{D}$*, given by* $A := \cup\{\omega(D) \,:\, D \subset \mathcal{D} \textit{ bounded}\}$*.*

4. Long time behavior of the quasistationary phase field system

First of all, we have to specify the class of solutions of the quasistationary phase field model (1.3)–(1.4) for which we construct the global attractor. We introduce the following

Definition 4.1 (Energy solutions). We say that a function

$$u \in H^1_{loc}(0,+\infty; H^{-1}(\Omega)) \cap L^\infty_{\mathrm{loc}}(0,+\infty; L^2(\Omega))$$

is an *energy solution* to Problem (1.3)–(1.4) with the boundary conditions (1.5) if u solves the gradient flow equation

$$\begin{gathered} u'(t) + \partial_s \phi(u(t)) \ni f \quad \text{for a.e. } t \in (0,+\infty), \\ \text{in the Hilbert space } \mathscr{H} := H^{-1}(\Omega), \text{ for the functional} \\ \phi(u) := \inf_{\chi \in H^1(\Omega)} \int_\Omega \left(\frac{1}{2}|u-\chi|^2 + \frac{1}{2}|\nabla\chi|^2 + W(\chi) \right) dx, \quad u \in L^2(\Omega). \end{gathered} \tag{4.1}$$

We denote by $\mathcal{E}$ the set of all energy solutions.

The set $\mathcal{E}$ is not empty thanks to Theorem 3.1. In fact, in [14] it has been proved that the potential ϕ in (1.7) is proper and lower semicontinuous and satisfies the chain rule (CHAIN) and the coercivity condition (COMP) in the Hilbert space $\mathscr{H} = H^{-1}(\Omega)$. As a by product of our main results Theorem 3.3 and Theorem 3.4 we thus have the following result ($D(W)$ is the realization in $L^2(\Omega)$ of the domain of W) in the framework of the phase space (see (4.1))

$$\mathcal{X} = L^2(\Omega) \quad d_{\mathcal{X}}(u,v) = \|u-v\|_{H^{-1}(\Omega)} + |\phi(u)-\phi(v)| \;\; \forall u,v \in L^2(\Omega) \tag{4.2}$$

Theorem 4.2. *Let the double well potential W in (1.4) be such that: there exist constants $\kappa_1,\ \kappa_2 > 0$ such that for all $v \in H^1(\Omega) \cap D(W)$*

$$\int_\Omega W(v)dx \geq \kappa_1 \|v\|^2_{L^2(\Omega)} - \kappa_2, \tag{4.3}$$

and either one of the following

1. *the set $H^1(\Omega) \cap D(W')$ is bounded in $(L^2(\Omega), d_{\mathcal{X}})$,* (4.4)
2. *there exist two positive constants κ_3, κ_4 such that for all $v \in H^1(\Omega) \cap D(W')$*

$$\int_\Omega W'(v)v \geq \kappa_3 \|v\|_{L^2(\Omega)} - \kappa_4. \tag{4.5}$$

Then, the set $\mathcal{E}$ of all the energy solutions to Problem (1.3)–(1.4) is a generalized semiflow in the phase space $(L^2(\Omega), d_{\mathcal{X}})$ (see (4.2)). Moreover, $\mathcal{E}$ possesses a unique global attractor $A_{\mathcal{E}}$, which is Lyapunov stable. Finally, for any trajectory $u \in \mathcal{E}$ and for any $u_\infty \in \omega(u)$, we have

$$\begin{aligned} -\Delta u_\infty + W'(u_\infty) &= f, \\ \partial_n u_\infty &= 0 \end{aligned} \tag{4.6}$$

Meaningful examples of potentials W satisfying the coercivity assumption (4.3) and (4.2) (or(4.5)) are the standard double-well potential

$$W(\chi) := \frac{(\chi^2 - 1)^2}{4}, \tag{4.7}$$

but also

$$W(\chi) := I_{[-1,1]}(\chi) + (1 - \chi)^2; \tag{4.8}$$

$$W(\chi) := c_1 \left((1 + \chi) \ln(1 + \chi) + (1 - \chi) \ln(1 - \chi)\right) - c_2\chi^2 + c_3\chi + c_4, \tag{4.9}$$

with $c_1, c_2 > 0$ and $c_3, c_4 \in \mathbb{R}$ (see, e.g., [5, 4.4, p. 170] for (4.9), [3, 21] for (4.8)). In particular, the term with $I_{[-1,1]}$ is the indicator function of $[-1, 1]$, thus forcing χ to lie between -1 and 1.

Remark 4.3. We stress that the question of the convergence of *all* the trajectory $u(t)$ to a single solution of equation (4.6) is a nontrivial one and is not answered by the preceding Theorem. This problem would have an affirmative answer if the set of all the solution would be totally disconnected (see Theorem 2.2). Unfortunately, it is well known (see [9]) that problem (4.6) may well admit a continuum of solutions.

Remark 4.4 (The Neumann-Neumann boundary condition case). If one replace the first in (1.5) with $\partial_n(u - \chi) = 0$ (i.e., homogeneous Neumann boundary conditions for the temperature ϑ), we get the so-called quasistationary phase field model with Neumann-Neumann boundary condition. This situation is very delicate since with this type of (non coercive) boundary conditions problem (1.3)–(1.4) does not have a gradient flow structure (see [14]). In [14] however, the existence of solutions has been deduced by means of a suitable approximation with more regular problems of gradient flow type. This kind of approximation has been reconsidered in [15] from the point of view of the long time dynamics. More precisely, in [15] we show that the set of all the solutions to (1.3)–(1.4) obtained with the above mentioned approximation still retain a (kind of) generalized semiflow structure. In particular this set, named $\mathcal{E}_N$, does not satisfy the concatenation property, but complies with some substantial properties, which allow us to prove the existence of a suitable weak notion of global attractor $A_{\mathcal{E}_N}$. Here weak means that this subset of the phase space is no longer *invariant* but only *quasi invariant* in the sense that for any $v \in A_{\mathcal{E}_N}$ there exists a complete orbit w with $w(0) = v$ and $w(t) \in A_{\mathcal{E}_N}$ for all $t \in \mathbb{R}$.

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Antonio Segatti
Department of Mathematics "F. Casorati", Università di Pavia
Via Ferrata 1, I-27100 Pavia, Italy
e-mail: `antonio.segatti@unipv.it`