Outline of mini lecture course, Mainz February 2016

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Warning: reference list is very much incomplete!

Lecture 1

In the first lecture I will introduce the tropical vertex group $\mathbb{V}$ of Kontsevich-Soibelman-Gross-Siebert in an elementary way and I will try to explain its relevance in enumerative problems, both for plane rational curves and for representations of certain algebras.

The relevance for this school "Irregular Hodge structures and stability conditions" comes from the fact that in the following lectures we will motivate and study (meromorphic, irregular) connections with monodromy in the group $\mathbb{V}$, parametrised by the central charge of a stability condition.

Plan + exercises

1. Elementary definition of tropical vertex group (in 2 variables) as subgroup $\mathbb{V} \subset \mathrm{Aut}_{\mathbb{C}[[t]]}[x^{\pm 1}, y^{\pm 1}][[t]]$.
   Exercise. Prove a formula for the inverse of generators, $(\theta_f)^{-1}$. (Hint: it's $\theta_{f^{-1}}$).
   Exercise. Give elementary proof that elements of $\mathbb{V}$ preserve $\frac{dx}{x} \wedge \frac{dy}{y}$. (We will give a more conceptual proof later).

2. Existence and uniqueness of factorisation

$$\theta_{f_2}^{-1}\theta_{f_1}\theta_{f_2}\theta_{f_1}^{-1} = \prod_{(a,b)}^{\rightarrow} \theta_{f_{a,b}}. \tag{1}$$

   Exercise. Give an elementary proof of existence and uniqueness for (1). (We will give a more conceptual proof later).

3. Brief discussion of weighted projective planes $\mathbb{P}(a, b, 1)$ and their blowups ([GPS] section 0.4). Formal definition of the invariants $N_{a,b}[P]$ enumerating rational curves on open blowups $X^o_{a,b}(P)$ of $\mathbb{P}(a, b, 1)$ in class β_P with tangency condition.
   Exercise. Count parameters and show that the expected dimension of the space of rational curves in class β_P with tangency condition is zero.
   Exercise. Say that a smooth surface S is log CY with respect to a

class β if $D \cdot \beta = c_1(S) \cdot \beta$ for all D. Prove that $X^o_{a,b}(P)$ is log CY with respect to class β_P. (See [GPS] section 6.1).

4. Enumerative meaning of the functions $f_{(a,b)}$ in Gromov-Witten theory (equation (2.3) in [FS]).
   Exercise. Prove by elementary algebraic geometry that

$$N_{(1,1)}[(1+1,1+1)] = 2, \quad N_{(1,3)}[1,1+1+1] = 1.$$

   Recover the result by a computation in the tropical vertex group. (This is [RW] Example 3.1).

5. Definition of (rational) tropical curves in $\mathbb{R}^2$ ([GPS] section 2.1). Definition of tropical invariants $N^{\mathrm{trop}}(\mathbf{w})$ ([GPS] Definition 2.6 and Proposition 2.7).
   Exercise. Compute $N^{\mathrm{trop}}((1,1),(1,2)) = 8$ by drawing funny pictures. See that the result doesn't change if you move around your boundary conditions.

6. Tropical meaning of the functions $f_{(a,b)}$ ([FS] equation (2.4)),

$$c_k^{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{R_{\mathbf{P}_i|\mathbf{w}_i}}{|\operatorname{Aut}(\mathbf{w}_i)|} N^{\mathrm{trop}}_{(a,b)}(\mathbf{w}), \tag{2}$$

7. Scattering diagrams $(\mathfrak{d}, \theta_{\mathfrak{d}})$. The factorisation/deformation method. Sketch of proof of (2) (bits of [GPS] sections 1 and 2).

8. From tropical counts to Gromov-Witten invariants ([FS] equation (2.5)),

$$N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)] = \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{R_{\mathbf{P}_i|\mathbf{w}_i}}{|\operatorname{Aut}(\mathbf{w}_i)|} N^{\mathrm{trop}}_{(a,b)}(\mathbf{w}). \tag{3}$$

   Exercise. Recover the result

$$N_{(1,1)}[(1+1,1+1)] = 2, \quad N_{(1,3)}[(1,1+1+1)] = 1.$$

   as well as the new result $N_{(2,3)}[(2,1+1+1)] = 1$ by using formula (3) and drawing funny pictures.

9. For those interested in quivers: the complete bipartite quiver $K(\ell_1, \ell_2)$ and its natural stability condition and moduli space $\mathcal{M}(P_1, P_2)$. The "GW/Kronecker" correspondence, in the coprime case

$$N[P_1, P_2] = \chi(\mathcal{M}(P_1, P_2)).$$

Exercise. If you know about quivers show that $\mathcal{M}(1+1,1+2)$ is just a point ([RW] section 5). On the other hand compute independently that $N[1+1,1+2]=1$ (e.g. using (3)).
Exercise. If you know about quivers try to compute

$$\chi(\mathcal{M}(1+1,1+1+1))=6$$

and independently (e.g. using (3)) $N[1+1,1+1+1]=6$.

Lecture 2

In the first part of the second lecture I will describe a more general and useful point of view on the group $\mathbb{V}$. Incidentally this leads naturally to a so-called q-deformation of $\mathbb{V}$. This comes with its own enumerative flavour, related on one side to refined curve-counting and on the other to Poincaré polynomials of moduli spaces of representations of certain algebras.

In the second part I will try to motivate the introduction of certain connections whose monodromy takes values in the group $\mathbb{V}$. This idea is due to Bridgeland-Toledano Laredo [BT1] and Gaiotto-Moore-Neitzke [GMN]. A key point is that identities such as (1) will say precisely that the generalised monodromy of these connections is constant (i.e. we have naturally isomonodromic families of connections).

Plan + exercises

1. The KS Poisson algebra $\mathfrak{g}_\Gamma$ of a lattice Γ with skew-symmetric bilinear form. The tropical vertex group $\mathbb{V}_{\Gamma,R}$ (over ring R). The "wall-crossing group" generated by $\theta^{\Omega}_{\alpha,1+\sigma e_\alpha}$ ([FS] section 3).
Exercise. Prove the formula $\theta_{\alpha,1+\sigma e_{m\alpha}} = \exp(1/m\,\mathrm{ad}(\mathrm{Li}_2(-\sigma e_{m\alpha})))$ ([FS] Lemma 3.4).

2. The q-deformed KS Poisson algebra $\mathfrak{g}_q$. The quantum dilog $\mathbf{E}(\sigma\hat{e}_\alpha)$. The q-deformed operators $\hat{\theta}^{\Omega}[\sigma\hat{e}_\alpha]=\mathrm{Ad}\,\mathbf{E}^{\Omega}(\sigma\hat{e}_\alpha)$.
Exercise. Find an explicit formula for the action of $\hat{\theta}^{\Omega}[\sigma\hat{e}_\alpha]$ (i.e. for the adjoint action of the quantum dilog) ([FS] Lemma 3.5).

3. 2d case. The q-analogue of the factorisation (1) ([FS] Lemma 3.7), and q-scattering diagrams $(\mathfrak{d},\hat{\theta}_{\mathfrak{d}})$.

4. The enumerative meaning of the functions $\hat{\theta}_{\mathfrak{d}}$ via Block-Göttsche tropical enumerative invariants ([FS] Corollary 4.9). Definition of a class of putative refined curve counts $\widehat{N}[P]$ ([FS] Definition 5.2). Relation to

the refined Severi degrees of Göttsche-Shende ([FS] section 5.2).

Exercise. (For those interested in Severi degrees and their refinement). Compute $N[1+1+1, 1+1+1] = 18$ using (3). Realise that this is in fact a Severi degree for $\mathbb{P}^2$ and compute it alternatively using the Caporaso-Harris recursion for Severi degrees of $\mathbb{P}^2$. Now q-deform both computations. On the one hand you will find $\widehat{N}[1+1+1, 1+1+1] = [3]_q([2]_q[2]_q + 2)$ using the q-deformation of (3). On the other you can look up the q-deformed Caporaso-Harris recursion proposed by Göttsche-Schende. You should find that the answers agree.

5. For those interested in quivers: the "GW/Kronecker" correspondence for quiver Poincaré polynomials, in the coprime case

$$\widehat{N}[P_1, P_2] = q^{-1/2 \dim \mathcal{M}(P_1,P_2)} P(\mathcal{M}(P_1, P_2))(q).$$

Exercise. Refine your previous computations for quiver Euler characteristics and ordinary curve counts to quiver Poincaré polynomials and refined curve counts.

6. Back to general case (not necessarily 2d). We can still define slope-ordered products if we fix a central charge $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$. Stability data: collection of numbers $\Omega(\alpha, Z)$ for $Z \in U \subset \mathrm{Hom}(\Gamma, \mathbb{C})$ such that $\prod^{\to, Z} \theta^{\Omega(\alpha,Z)}_{\alpha, 1+\sigma e_\alpha}$ is locally constant.

7. $2+1$ and $2+2$ meromorphic connections on $\mathbb{P}^1$ with monodromy in $\mathbb{V}$. Motivation from BTL (interpreting Joyce's holomorphic generating functions) and GMN (constructing certain HK metrics; canonical flat sections of the connections should give interesting Darboux coordinates).

8. Stability data supported on a convex cone define naturally families of $2+1$ and $2+2$ connections, with values in derivations of a completion $\widehat{\mathfrak{g}}$, parametrised by $Z \in U \subset \mathrm{Hom}(\Gamma, \mathbb{C})$. The generalised monodromy is given by the stability data and thus it is constant in the families.

Lecture 3

In the third lecture I will start by sketching one of the ways of constructing explicitly the isomonodromic family of $2+2$ connections $\nabla(Z)$ on $\mathbb{P}^1$ attached to stability data; one finds that their canonical flat sections $X(z, Z)$ are given by an infinite sum of certain graph integrals.

We will then study two scaling limits of the connections $\nabla(Z)$ under the scaling $Z \mapsto RZ$. The first one is called the conformal $R \to 0$ limit and maps $\nabla(Z)$ to the $2+1$ connections considered by BTL. The second is a large radius limit $R \to \infty$ and makes contact with the tropical aspects of the story discussed above.

Plan + exercises

1. Formulation of the Riemann-Hilbert factorisation problem for maps $X(z, Z) : \mathbb{C}^* \to \operatorname{Aut}(\widehat{\mathfrak{g}})$ with factors $\theta^{\Omega(\alpha,Z)}_{\alpha,1+\sigma e_\alpha}$. Equivalence with construction of $\nabla(Z)$ with monodromy $\theta^{\Omega(\alpha,Z)}_{\alpha,1+\sigma e_\alpha}$ ([FGS] section 3.2): $X(z, Z)$ are (essentially) canonical flat sections.

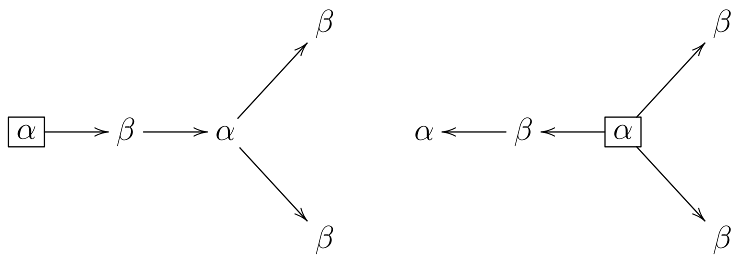
2. Explicit formula for $X(z, Z)$ in terms of iterated integrals $G_T(z, Z)$ of the basic automorphism $X^0(z, Z)$ ([FGS] Lemma 3.10),
$$X(z, Z)e_\alpha = e_\alpha \exp_* \left(z^{-1}Z(\alpha) + z\bar{Z}(\alpha) - \langle \alpha, \sum_T W_T(Z)G_T(z, Z)\rangle \right). \tag{4}$$
Exercise. Study the integral $\int X^0(z, Z)e_\alpha$ along a ray $-\mathbb{R}_{>0}Z(\alpha)$. In particular establish its precise exponential decay as $|Z| \to \infty$.
Exercise. Complete the details of the proof of (4).
Exercise. Convince yourself that the right hand side of (4) defines an algebra automorphism i.e. it's invertible. Realise that in principle it is even possible to invert it explicitly (you will need to recall the Lagrange inversion formula).

3. Conformal limit: there is a well-defined limit of $\nabla(z, Z)$ under the scaling $Z \mapsto RZ$, $z \mapsto Rz$ (up to explicit gauge transformations). The limit is a family of connections $\nabla^{BTL}(z, Z)$ with "conformal invariance" property ([FGS] Theorem 4.2). They coincide with the BTL connections; their residue is the Joyce holomorphic generating functions of the numbers $\mathrm{DT}(\alpha, Z)$ given by the Möbius transform of the $\Omega(\alpha, Z)$. The conformal invariance for these functions was observed by Joyce.

4. Wall-crossing: the flat sections $X(z, Z)$ are (locally) real analytic in Z, but the iterated integrals $G_T(z, Z)$ have intricate branch-cut behaviour along critical loci in Z.

5. Large radius limit $Z \mapsto RZ$, $R \to \infty$. The leading term as $R \to \infty$ of the branch-cut behaviour of $G_T(z, Z)$ can be described completely in terms of certain rational tropical curves in $\mathbb{R}^2$ ([FGS] Theorem 1.1).
Exercise. Suppose α, β are primitive classes with $Z(\alpha)$ crossing $Z(\beta)$

across a wall. Study the branch-cut behaviour, to leading order as $R \to \infty$, of the graph integrals for the graphs



Realise that each leading order contribution is attached to a rational tropical curve in $\mathbb{R}^2$.

6. Enumerative meaning of the branch-cut behaviour of graph integrals $G_T(z, Z)$ and so of the canonical flat sections $X(z, Z)$ in terms of tropical enumerative invariants ([FGS] Theorem 1.2).

References

[BT1] T. Bridgeland and V. Toledano-Laredo, *Stability conditions and Stokes factors*, Invent. Math. 187 (2012), no. 1, 61-98.

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[GMN] D. Gaiotto, G. Moore and A. Neitzke, *Four dimensional wall-crossing via three-dimensional field theory*, Comm. Math. Phys. 299 (2010), no. 1, 163-224.

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