Stability and wall-crossing in algebraic and differential geometry

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STABAGDG

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1. Introduction
General principle (after Narasimhan-Seshadri, Atiyah-Bott, Hitchin-Kobayashi, Donaldson, Yau, Tian...)

Solutions to natural $PDE$s in complex geometry

$\uparrow \downarrow$

Natural objects in complex $algebraic$ geometry

In terms of families of solutions and objects or $moduli$ $spaces$:

$\mathcal{M}_{PDE} \cong \mathcal{M}_{alg}$
**General principle**

This point of view has been spectacularly successful.

The aim of our project is to **attack a number of open problems which fit in this context**.

At the same time we will **expand this principle** bringing in **new insights and problems coming from recent work in the geometry of quantum field theories**.

Total duration: **4 years**.
Examples: gauge theory

- **Hermitian Yang-Mills connections:** \((X, \omega)\) a Kähler manifold (i.e. complex manifold with Riemannian metric which is perfectly adapted to complex structure).

\(E \rightarrow X\) holomorphic vector bundle; \(A\) on \(E\) compatible connection \(\Rightarrow\) a complex analogue of usual fields in physical gauge theory.

Field-strength = \(F_A = \text{curvature} = dA + [A, A]\), 2-form with values in \(\text{ad}(E)\).

Equation of motion: \(\text{Tr} \ F_A = \lambda I\).

**Thm (Hitchin-Kobayashi, Donaldson, Uhlenbeck-Yau)**

A solution exists iff \(E\) satisfies a purely algebro-geometric condition, *slope polystability*. This is a constraint on *all the holomorphic subsheaves*. 
Examples: gauge theory

- **Higgs bundles:** $\Sigma$ a Riemann surface, $E \to \Sigma$ rank 2 complex vector bundle with structure group $SU(2)$.
  
  $A$ a compatible connection.
  
  $\Phi$ a Higgs field: 1-form with values in $\text{ad}(E) \otimes \mathbb{C}$.
  
  Equations of motion are Hitchin’s equations:
  
  $$F_A + [\Phi, \Phi^*] = 0$$
  $$\overline{\partial}_A \Phi = 0.$$

  **Theorem** (Hitchin): solution iff $(E, \overline{\partial}_A)$ holomorphic, $(E, \Phi)$ stable.

  **(Open) problem** (Hitchin, Simpson...):
  
  Understand (hyperkähler) space of solutions $(\mathcal{M}, g)$ (general gauge groups, singular fields...).
Examples: Kähler geometry

- **Kähler-Einstein metrics:** \((X, g)\) Kähler; \(\omega_g\) Kähler differential 2-form.

  Ricci form: \(\text{Ric}(\omega_g) = -\partial\bar{\partial} \log \det g_{k\bar{l}}\).

  Einstein’s equation in the Kähler world:

  \[
  \text{Ric}(\omega_g) = \lambda \omega_g.
  \]

  Topological constraint: \(c_1(X) = \lambda [\omega_g]\).

  **Aubin-Calabi-Yau Theorem:** If \(c_1(X) \leq 0\) this is the only constraint.

  **Kähler-Einstein (open) problem (Calabi, Yau, Tian, Donaldson...)**

  Solve the equation for positive Ricci curvature. **Main conjecture (Yau-Tian-Donaldson):** this is a purely algebro-geometric problem.
Examples: Kähler geometry

- **CscK and extremal metrics:** prescribe *scalar curvature* to be constant, or as constant as possible:

  Scalar curvature: \( s(g) = -g^{ij} \partial_i \bar{\partial}_j \log \det g_{k\bar{l}}. \)

  Csck equation and extremal equations:

  \[
  s(g) = \tilde{s} = \text{a topological constant.}
  \]

  \[
  \nabla^{1,0} s(g) = \chi = \text{ess. unique holomorphic vector field.}
  \]

**CscK (open) problem (Calabi, Yau, Tian, Donaldson...)**

Which manifolds have a cscK (or extremal) metric? **Main conjecture (Yau-Tian-Donaldson):** when \([\omega] = c_1(L)\) for \(L \to X\) ample, this is a purely algebro-geometric problem.
2. Stability and canonical metrics
**K-stability**

- **K-(semi, poly)stability:** $[\omega_g] = c_1(L)$. Embed $X \hookrightarrow \mathbb{P}^{N_k}$ using powers $L^k$. K-(semi,poly)stability is a constraint on all degenerations $X'$ of $(X, L)$ induced by flowing under a $\mathbb{C}^* \curvearrowleft \mathbb{P}^{N_k}$, $F(X') > 0 \geq 0$.

**Theorem (Donaldson)**

$\omega_g \in c_1(L) \text{ cscK} \Rightarrow (X, L) \text{ K-semistable.}$

**Theorem (S.): “blow-up method”**

If moreover $\text{Aut}(X, L)$ discrete $\Rightarrow$ K-*stable*. In the extremal case get *relative* K-*polystability* (with Székelyhidi).

**Conjecture (Yau-Tian-Donaldson)**

In general, cscK $\Leftrightarrow$ K-*polystable.*
K-stability: problems and limitations

- **No uniform control:** for analytic purposes need uniform bound on $F(\mathcal{X})/\|\mathcal{X}\|$, but it's lacking.

- **Might not be sufficient:** conjectural counterexample by Apostolov, Calderbank, Gauduchon and Tonnesen-Friedman.

- **Not natural:** natural statements like stability $\Rightarrow$ reductivity, Zariski openness, Atiyah-Bott type theorems on worst degenerations are all *very hard* conjectures.

- **Not compatible with Gromov-Hausdorff limits:** in the approach of Donaldson (also with Chen, ...) Gromov-Hausdorff limits lead to a different notion: b-stability.
Our main objectives

General objective

Conjecture A and similar problems
Prove naturality properties for our new notion, and a posteriori for K-stability: equivariance and compatibility to maximal tori, Zariski openness, reductivity of Aut(X)...

b-stability
Obtain a detailed understanding of Donaldson’s b-stability (the algebraic counterpart of Gromov-Hausdorff limits). Prove that a cscK manifold is b-stable.
Sketch of methodology

Candidate: stability by filtrations
Encode degenerations by filtrations of $\bigoplus_k H^0(X, L^k)$, not necessarily finitely generated (Witt-Nystrom, Székelyhidi). Better behaved, but all the main problems still open. Bring in methods from birational geometry and the analysis of ample linear series.

Variations of the blow-up method
Donaldson made progress on cscK $\Rightarrow$ b-stable using blow-up method, in the equivariant case. Develop a non-equivariant version for the blow-up method.

Approximation
Develop an approximation theory for Donaldson’s general families in b-stability by one dimensional objects.
3. Hyperkähler metrics and wall-crossing
Seiberg-Witten and Higgs bundles

- **Seiberg-Witten:** Study $\mathcal{N} = 2$ Yang-Mills theories on $\mathbb{R}^3 \times S^1_R \Rightarrow$ a $\sigma$-model $\mathbb{R}^3 \rightarrow$ hyperkähler $\mathcal{M}$.

- **Higgs bundles:** Claim $(\mathcal{M}, g)$ is a moduli space of solutions to $F_A + R[\Phi, \Phi^*] = \overline{\partial}_A \Phi = 0$, with prescribed singularities!

Study the **geometry of moduli of Higgs bundles** $(\mathcal{M}, g)$ (especially Hitchin fibration $\mathcal{M} \rightarrow \mathcal{B}$) **via** the structure of $\mathcal{N} = 2$ Yang-Mills (i.e. its operators, spectrum...).
BPS states and Gaiotto-Moore-Neitzke

- **BPS spectrum:** states in $\mathcal{N} = 2$ Yang-Mills killed by half the supersymmetry operators.

Find mathematical framework (Donaldson-Thomas theory?): Bridgeland, Smith...

- **Wall-crossing:** the number of BPS states is a function $\Omega(\gamma, u)$ of charge and coupling constant $u \in \mathcal{B}$.

$\Omega(\gamma, u)$ jump when $u$ crosses critical locus: **wall-crossing** formulae as in Joyce-Song, Kontsevich-Soibelman.

**Gaiotto-Moore-Neitzke conjecture(s)**

$\Omega(\gamma, u)$ **completely determines** $(\mathcal{M}, g)$ by (nonperturbative) instanton corrections in $R$ through **integral equation**. Wall-crossing reflects **continuity** of $g$. 
Our main objectives

General objective
Study the conjecture(s) of Gaiotto-Moore-Neitzke for a large class of $(\mathcal{M}, g)$. Prove *existence and uniqueness of solutions* for their integral equation.

Asymptotic expansion
Study the *convergence of the natural asymptotic expansion* for solution emerging from GMN.

Additional structures
Study natural additional structures on $(\mathcal{M}, g)$: *hyperholomorphic connections*. What is the *mirror* for these constructions?
Sketch of methodology

Finite BPS spectra
Concentrate initially on examples where $\Omega(\gamma, u)$ is everywhere finite.

Comparisons
Establish precise comparisons with algebro-geometric work of Joyce and Bridgeland, Toledano-Laredo. Conjecture (S.): asymptotic expansion recovers Joyce’s theory (checked in many cases).

Nahm-type equations
Recast as infinite dimensional Nahm-type equations. Bring in recent ideas of Donaldson on infinite dimensional Nahm and geodesics in the space of Kähler potentials.
4. The team

- **Jacopo Stoppa** (Università di Pavia and Trinity College, Cambridge) - Principal Investigator. *CscK, K-stability, DT theory, quivers and tropical vertex, GMN theory.*

- **Gabor Székelyhidi** (University of Notre Dame) - Team Member. *CscK and extremal metrics, Monge-Ampère equations, Kahler-Ricci flow, Sasaki geometry.*

- **Two postdocs** (2 + 2 years). With suitable expertise in algebraic or differential geometry.

- **One doctoral student** (3 years). Initially working in low-dimensional examples for b-stability or finite spectrum examples in GMN theory.
5. Timeliness and flexibility

Very active research areas: recent or forthcoming works of Donaldson (also with Chen, Sun); Bridgeland, Smith...

Forthcoming Junior Research Trimester at HIM, Bonn, Sep-Dec 2012 (P.I. will be group leader for “BPS states” group).

Project is flexible and can adapt to the area of expertise of highly qualified post-doctoral members.

Good range of PhD problems to attract an excellent candidate.