Etale Cohomology

Tom Sutherland

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Note: All schemes are taken to be Noetherian and separated

1 Introduction

The development of etale cohomology was motivated by work on the Weil conjectures, which state that local-zeta functions $\zeta(X, s)$, i.e. the generating function for the number of rational points of a projective variety defined over a finite field $\mathbb{F}_q$ in all finite extensions, satisfy certain properties analogous to those of the Riemann zeta function.

Weil observed that the number of points defined over $\mathbb{F}_{q^k}$ is given by the number of fixed points of $X(\mathbb{F}_q)$ of the k-th power of the Frobenius automorphism. Thus given an appropriate cohomology theory for varieties over a finite field (with co-efficients in a field of characteristic zero), this could be computed using the Lefschetz trace formula.

In particular we require a so-called Weil cohomology theory, which shares the properties of singular cohomology for varieties defined over the complex numbers - namely a contravariant functor $H^i(-, K)$ from non-singular irreducible complete varieties over an algebraically closed field k to finite dimensional vector spaces over a characteristic zero field K, satisfying (amongst other things) the following:

(Dimension) If X has dimension d, $H^i(X, K) = 0$ outside $0 \leq i \leq 2d$
(Finiteness) $H^i(X, K)$ is a finite dimensional vector space
(Duality) There exists a perfect pairing $H^i \times H^{2d-i} \to K$

Grothendieck's ℓ-adic cohomology, defined as

$$H(X, \mathbb{Q}_\ell) = lim_{\leftarrow} H_{et}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

where ℓ is coprime to the characteristic of the base field k, provided the first example of such a theory for varieties defined over an arbitrary field.

Grothendieck's major contribution was to define the etale topology, whose cohomology groups H_{et} agree with the complex topology when dealing with coefficients in *torsion* sheaves. The ℓ-adic cohomology is approximated by the etale cohomology with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$ for higher and higher n.

2 The etale approach

It is instructive in considering how we may develop an algebraic approach to the topology of varieties to consider the fundamental group.

Consider the simple example $\mathbb{C}^*$, the affine line with a point removed. This has universal cover $\mathbb{C} \xrightarrow{exp} \mathbb{C}^*$, which of course cannot be constructed algebraically. The fundamental group is the group of $\mathbb{C}^*$-automorphisms of $\mathbb{C}$, which is just $\mathbb{Z}$.

The etale approach is to approximate this by *finite* covers of the form $\mathbb{C}^*_{(n)} \xrightarrow{z \mapsto z^n} \mathbb{C}^*$, ind-representing the universal cover. Then we can define the *etale fundamental group*

$$\pi_1^{et}(X) = \varprojlim Aut_{\mathbb{C}^*}(\mathbb{C}^*_{(n)}) = \varprojlim \mu_n(\mathbb{C}) = \hat{\mathbb{Z}}$$

to be the corresponding profinite group, which in our example is indeed the profinite completion of $\mathbb{Z} = \pi_1(X(\mathbb{C}))$

Indeed the Riemann existence theorem (11.1) says that there is an equivalence of categories between finite etale coverings of a smooth variety X defined over $\mathbb{C}$ and the finite covers of $X(\mathbb{C})$, so the relationship $\pi_1(\hat{X(\mathbb{C})}) = \pi_1^{et}(X)$ holds for all smooth varieties defined over $\mathbb{C}$. By the Lefschetz principle it holds for all ground fields of characteristic zero.

Remark 2.1. *We encounter some problems in characteristic p - the etale fundamental group is not stable under extensions of an algebraically closed base field. For example, the class of Artin-Schreier coverings $z \mapsto z^p - z + \alpha$, $\alpha \in k$ grows if we extend the base field k. We will see that we can't hope for the etale topology to give the "correct" answers for things of degree p in characteristic p.*

We cannot realize these coverings in the Zariski topology, as $z \mapsto z^n$ is not a local isomorphism as it is in the complex topology. This means that the Zariski topology does not contain enough acyclic open sets, which are provided by small open balls in the complex topology. Indeed in our example, a Zariski open subset of $\mathbb{C}^*$ is just the complex plane with a finite number of points removed, which is homotopy equivalent to a bouquet of circles and so is not acyclic.

The 1-cycles can however by killed by passing to an unramified covering. For example if a closed curve winds r times around the origin in our punctured complex plane, then it is no longer a cycle after having been lifted by the cover $z \mapsto z^n$ for $n > r$.

This led Serre to consider locally isotrivial bundles on a variety, i.e. those which become trivial after pullback by by a finite unramified map. These allow for a correct definition of H^1_{et}, the first etale cohomology group. We will now see how Grothendieck generalized this intuition to define etale cohomology in all dimensions.

3 Grothendieck topologies

In the previous section, we saw that we could "unwind" a Zariski open subset U of $\mathbb{C}^*$ by considering finite unramified covers $U' \to U$.

The notion of a Grothendieck topology, a generalization of the usual topology defined by open sets, allows us to consider such a covering map as an "open piece" of U.

Definition 3.1. *A* Grothendieck topology *on a category* $\mathbf{C}$ *is a system of coverings of all objects* $U \in Ob(\mathbf{C})$*, that is for each* U *a distinguished collection of morphisms* $(U_i \to U)_{i\in I}$ *satisfying the following axioms:*

1. *The identity map* $U \xrightarrow{id} U$ *is a covering of* U
2. *If* $(U_i \to U)_{i\in I}$ *is a covering of* U*, and* $(V_{ij} \to U_i)_{j\in J_i}$ *is a covering of each* U_i*, then the family of composites* $(V_{ij} \to U)_{i,j}$ *is a covering of* U
3. *If* $V \to U$ *is any morphism in* $\mathbf{C}$ *and* $(U_i \to U)_{i\in I}$ *is a covering of* U*, then the fibre products* $U_i \times_U V$ *exist, and* $(U_i \times_U V \to V)_{i\in I}$ *is a covering of* V

Remark 3.2. *We can recover the usual definition of a topology on a space* X *by taking* $\mathbf{C}$ *to be the collection of open subsets of* X*, and the coverings to be collections of open inclusions* $(U_i \hookrightarrow U)$ *covering* U*. Then the fibre product* $U \times_X V = U \cap V$*, so axiom 3 is familiar.*

Definition 3.3. *A* site $\mathbf{T}$ *is a category* $\mathbf{C}$ *together with a Grothendieck topology.*

Example 3.4. *The Zariski site* X_{zar} *on an algebraic variety* X *is the one obtained from viewing* X *as a topological space with the Zariski topology.*

The usefulness of these definitions is they abstract the information required to define a sheaf.

Definition 3.5. *A presheaf on a site* $\mathbf{T}$ *is a contravariant functor* $\mathcal{F} : \mathbf{C} \to Sets$ *from the underlying category to sets*

The sheaf condition translates into the language of coverings as follows

Definition 3.6. *A presheaf* $\mathcal{F}$ *on* T*is a sheaf if for all coverings* $(U_i \to U)_i$

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$$

is exact, i.e. the image of the first map is excatly where the second pair of maps agree.

Remark 3.7. *Note that there may be more than one covering map* $U \to X$*, and* $U \times_X U$ *may not be equal to* U *Indeed if we take* U *to be the universal cover* $\tilde{X}$ *of a topological space* X*, then the connected components of* $\tilde{X} \times_X \tilde{X}$ *are in bijection with the elements of fundamental group of* X*.*

4 The etale site

To define the etale topology on a variety X, we need to give the coverings, i.e. the trivial neighbourhoods for the topology.

We first define these notions for local rings in the opposite category of k-algebras.

Definition 4.1. *A morphism $f : A \to B$ of local k-algebras is unramified if $m_A B = m_B$ and the map on residue fields $A/m_A \to B/m_B$ is a finite separable extension*

It is flat if the functor A-mod $\to$ B-mod given by $M \mapsto M \otimes_A B$ is exact.

Definition 4.2. *A morphism of schemes $\phi : Y \to X$ is unramified if it is of finite type and the maps on the local rings $\mathcal{O}_{X,\phi(y)} \to \mathcal{O}_{Y,y}$ are unramified for all $y \in Y$*

It is flat if the maps on the local rings $\mathcal{O}_{X,\phi(y)} \to \mathcal{O}_{Y,y}$ are flat for all $y \in Y$

Definition 4.3. *A morphism $Y \to X$ is etale if it is flat and unramified.*

Definition 4.4. *The etale site X_{et} on a scheme X has underlying category Et/X, whose objects are etale morphisms $U \to X$, and the coverings of U are surjective families of etale morphisms $(U_i \to U)_i$*

We have already seen that finite etale maps are precisely the etale covers in the sense of Section 2. The trivial maps for finite-etale site are the isotrivial maps.

Definition 4.5. *The finite-etale site on X has underlying category Et/X and coverings surjective families of finite etale morphisms $(U_i \to U)_i$ onto a Zariski open subset U*

A lot of the cohomological theorems we will see only depend on flatness, so we can consider the "finer" flat topology. It may give information where the etale cohomology groups are anomalous, for example with co-efficients in $\mathbb{Z}/p\mathbb{Z}$ in characteristic p.

Definition 4.6. *The flat site on X has category all X-schemes Sch/X and coverings surjective families of flat morphisms of finite type.*

It can be shown that the category of sheaves on the etale site is Abelian and has enough injectives. We can now define etale cohomology as the right derived functors of the global section functor.

Definition 4.7. $H^i_{et}(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$

Example 4.8. *An open immersion is etale as it induces isomorphisms on the local rings*

Example 4.9. *A closed immersion, although unramified, is not etale as the map $A/a \hookrightarrow A$ is not flat*

Proposition 4.10. *Etale morphisms are open*

Proof. This follows from the more general fact that flat morphisms of finite type are open. ([Mil] I.2.12) □

Proposition 4.11. *Etale morphisms are quasi-finite, i.e. have finite fibres*

Proof. Unramified morphisms are quasi-finite as the extension on residue fields is finite □

Example 4.12. *The etale maps into a point* $X \to \mathrm{Spec}k$ *can be considered in the opposite category of etale* k*-algebras as etale maps* $k \to A$*. As* A *is unramified over a field* k*, no maximal ideal of* A *contains a prime ideal. Thus* A *has dimension 0 and so is Artinian ([AM] 8.5), so* A *is a product of local rings ([AM] 8.7). Hence* A *is a product of finite separable extensions of* k*.*

We now give the local structure of etale morphisms.

Definition 4.13. *A standard etale morphism is given by*
$\mathrm{Spec}A \to \mathrm{Spec}((A[t]/(f(t)))_b)$ *where* $f(t) \in A[t]$ *is monic, and* $f'(t)$ *in invertible in* $A[t]/(f(t))_b$*.*

Proposition 4.14. *Locally every etale morphism* $Y \to X$ *is a standard etale morphism, i.e there exists affine open neighbourhoods* V *of* y *and* U *of* $f(y)$ *such that* $f|_V$ *is standard etale.*

Proof. See [Mil] I.3.14 □

Etale morphisms are the algebraic analogue of local isomorphisms in the sense of the following proposition

Proposition 4.15. *A regular map of* varieties $f : X \to Y$ *defined over an algebraically closed field* k *is etale if and only if the corresponding map on tangent cones is a local isomorphism at all points*

The tangent cone at a point x is just $gr(\mathcal{O}_{X,x}) = \sum_0^\infty m^n/m^{n+1}$, where m is the unique maximal ideal of the local ring $\mathcal{O}_{X,x}$. By a standard result in commutative algebra, f induces an isomorphism on tangent cones iff it does so on the completions of the local rings.

As the statement of the proposition is local, we may work in the opposite category of local etale k-algebras. The equivalent formulation is

Proposition 4.16. *If* $f : A \to B$ *is a regular local homomorphism of local* k*-algebras then the induced homomorphism on the completions* $\hat{f} : \hat{A} \to \hat{B}$ *is an isomorphism iff* f *is flat and unramified.*

Proof. ($\Rightarrow$) If $\hat{f}$ is an isomorphism, then it is certainly flat and unramified. Thus it suffices to show that if $\hat{f}$ is flat or unramified, then so is f.

If $\hat{f}$ is flat, let (S): $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Then

$$\begin{aligned}
\text{(S) exact} &\Rightarrow \hat{B} \otimes_{\hat{A}} \text{(S) exact} \\
&\Rightarrow \hat{B} \otimes_A \text{(S) exact, as } A \to \hat{A} \text{ is flat} \\
&\Rightarrow B \otimes_A \text{(S) exact, as } B \to \hat{B} \text{ is faithfully flat}
\end{aligned}$$

so $f : A \to B$ is flat

If $\hat{f}$ is unramified then $m_A B$ and m_B generate the same ideal in $\hat{B}$. But as $B \to \hat{B}$ is faithfully flat, they also generate the same ideal in B, as $aS \cap R = a$ for any ideal a of R and any faithfully flat homomorphism $R \to S$

($\Leftarrow$) Zariski's Main Theorem allows us to assume that B is the localisation of a finite A-algebra C at a maximal ideal, and as in the proof of 4.14, we may take $C = A[T]/(g(T))$ for a monic polynomial $g(T)$. Thus $\hat{B}$ is the completion of $\hat{A}/g(t)$ at some maximal ideal lying over the maximal ideal of $\hat{A}$. But a complete local ring is Henselian, so the factorisation $\bar{g} = {g'_1}^{e_1} \dots {g'_n}^{e_n}$ in $k[T]$ lifts uniquely to a factorisation $g = g_1 \dots g_n$ in $\hat{A}[T]$. But then $\hat{B} = \hat{A}/(g_i(T))$ for some i. Thus $\hat{A} \to \hat{B}$ is finite.

We know $\hat{f}$ is injective. It is also unramified, and the map on the residue fields $k \cong \hat{A}/m_{\hat{A}} \to \hat{B}/m_{\hat{B}} \cong k$ is an isomorphism. Thus $\hat{B} = \hat{f}(\hat{A}) + m_{\hat{A}} B$. But as $\hat{f}$ is finite, we can apply Nakayama's Lemma, so $\hat{B} = \hat{f}(\hat{A})$ so $\hat{f}$ is indeed an isomorphism.

□

The following are immediate from the definitions

Proposition 4.17. *(Permanence properties of etale morphisms)*

1. *The composite of two etale morphisms is etale*
2. *Any base change of two etale morphisms is etale*

Etale morphisms also preserve a wide variety of properties of a scheme.

Proposition 4.18. *Let $Y \to X$ be etale. If X is reduced, normal, regular, Cohen-Macaulay or excellent then so is Y*

5 The local ring

We will see below that it is a geometric point, i.e. the spectrum of a *separably closed* field, which plays the role of a point in the etale topology. Thus the definitions of all our local notions are with respect to a geometric point, which we will generally denote $\bar{x}$

Definition 5.1. *An etale neigbourhood of a geometric point $\bar{x} \to X$ of a scheme X is an etale map $U \to X$ such that $\bar{x} \to X$ factors through $\bar{x} \to U \to X$.*

There is at most one map between two etale neighbourhoods, so they form a directed set. The definition of the stalk of a sheaf $\mathcal{F}$ on the etale site is formally the same as for the Zariski topology.

Definition 5.2. *The stalk of a sheaf $\mathcal{F}$ at a geometric point $\bar{x}$ is given by $\mathcal{F}_{\bar{x}} = \lim_{\rightarrow} \mathcal{F}(U)$, where the direct limit is over all etale neighbourhoods U of $\bar{x}$.*

As with the Zariski topology, to check a map of sheaves is injective or surjective it suffices to do so on stalks.

Definition 5.3. *The stalk of the sheaf $\mathcal{O}_X$ at a geometric point $\bar{x}$ is called the strictly local ring at the point $\bar{x}$ and is denoted $\mathcal{O}_{X,\bar{x}}$.*

This is the analogue of the local ring for the Zariski topology, and we have a map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\bar{x}}$. Although it is not an etale neighbourhood in the above sense (it is *pro*-etale), it is a homotopically trivial object, just like a sufficiently small open ball in the complex topology. It has another important property.

Proposition 5.4. *The strictly local ring $\mathcal{O}_{X,\bar{x}}$ is Henselian.*

Proof. To show a local ring is Henselian, it suffices to prove that for any n polynomials in n unknowns $f_1, \ldots, f_n \in A[t_1, \ldots, t_n]$, any common zero $\mathbf{a_0}$ of the f_i in k^n where the Jacobian is non-zero lifts to a common zero in A^n. (Lifting a factorisation of a monic polynomial in $k[t]$ to $A[t]$ amounts to solving n equations in n unknowns for the co-efficients)

Suppose $f_1, \ldots, f_n \in \mathcal{O}_{X,\bar{x}}[t_1, \ldots, t_n]$. We can find an sufficiently small etale neighbourhood U such that the $f_i \in k[U][t_1, \ldots, t_n]$. Let $V = \text{Spec}k[U][t_1, \ldots, t_n]/(f_1, \ldots, f_n)$. Then the common zero $\mathbf{a_0}$ defines a point $v \in V$ lying over $u = \bar{x} \in U$. Moreover as $Jac(f_1, \ldots, f_n)\mathbf{a_0} = 0$, the map $V \to U$ is etale at v.

Thus for some $g \in C$ not in the maximal ideal m_v, the map $D(g) \to U$ is etale, where $D(g)$ denotes the open set where g is non-zero, and hence a map $C_g \to \mathcal{O}_{X,\bar{x}}$ where the inverse image of $m_{\bar{x}}$ is m_v. This defines a common zero of the f_i in $\mathcal{O}_{X,\bar{x}}$ lifting $\mathbf{a_0}$. □

Definition 5.5. *We say a Henselian ring is strictly Henselian if its residue field s separably closed, i.e. every monic polynomial $f(T) \in A[T]$ with $\bar{f}(T) \in k[T]$ separable splits* completely *in $A[T]$*

Hence the strictly local ring is *strictly Henselian*

We note a few properties of Henselian rings

Proposition 5.6. *Let A be a local Henselian ring with residue field $k = A/m_A$ and B an etale A-algebra. Then the extension of the residue fields $B/m_B = k \times C$ lifts to a decomposition $B = A \times C$.*

Proof. We can assume that $A \to B$ is standard etale, so $B = (A[t]/(f(t))_b$ where $f'(t)$ is invertible in $B[t]$. Then the decomposition $B/m_B = k \times C$ is defined by the choice of a simple root $\bar{a}$ of $\bar{f}[t] \in k[t]$. But such a root lifts to $a \in A$ and thus defines a decomposition $B = A \times C$. □

Corollary 5.7. *The category of etale extensions of a local Henselian ring is equivalent to the category of finite separable extensions of its residue field.*

Corollary 5.8. $H^q(A, \mathcal{F}) = H^q(k, \mathcal{F})$ *for all sheaves* $\mathcal{F}$*. In particular if* A *is* strictly *Henselian, it is acyclic.*

Proposition 5.9. *Let* A *be a Henselian ring, and* $A \to B$ *a finite homomorphism. Then* B *is the product of finitely many local (Henselian) rings*

Proof. This is theorem 3.1 in [FK]. It is in fact an equivalent definition of the Henselian property. □

6 Sheaves on the etale site

The following proposition provides an easier criterion to check when a etale presheaf is a sheaf.

Proposition 6.1. *A presheaf* $\mathcal{F}$ *on* X_{et} *is a sheaf if it satisfies the sheaf condition for Zariski open coverings, and for etale coverings* $V \to U$ *consisting of a single surjective map where both* V *and* U *are affine.*

Proof. If $\mathcal{F}$ satisfies the sheaf condition for Zariski open coverings, then $F(\coprod U_i) = \prod \mathcal{F}(U_i)$, where $\coprod$ denotes the disjoint union. If the index set is finite, then checking that $\mathcal{F}$ satisfies the sheaf condition for $(U_i \to U)_{i \in I}$ is equivalent to checking it for the single morphism $(\coprod U_i \to U)$, as

$$(\coprod U_i) \times_U (\coprod U_i) = \coprod (U_i \times_U U_j)$$

Thus we have shown $/F$ satisfies the sheaf condition for coverings $(U_i \to U)_{i \in I}$ where I is finite and the U_i are affine. The remaining steps can be found in [Mil] II.1.5 □

This allows us to make use of Grothendieck's descent theory, as surjective etale maps are faithfully flat.

Theorem 6.2. *Let* $f : A \to B$ *be a faithfully flat ring homomorphism, and* M *be an* A*-module. Then the complex* $\mathbf{K}$

$$M \xrightarrow{d_0} M \otimes_A B \xrightarrow{d_1} M \otimes_A B \otimes_A B \to \ldots \xrightarrow{d_n} M \otimes_A B^{\otimes^n} \to \ldots$$

is exact, where

$$d_n(m \otimes b_1 \ldots \otimes b_n) = \sum_{i=0}^{n} m \otimes b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \ldots \otimes b_n$$

Proof. First note that as $f : A \to B$ is faithfully flat, it suffices to prove the complex $\mathbf{K} \otimes_A B$ is exact. Thus we may take $f : B \to B \otimes_A B, b \mapsto 1 \otimes b$. Then f has a section $g : B \otimes_A B \to B, b \otimes b' \mapsto bb'$.

Thus we can assume $f : A \to B$ has a section, and we define a contracting homotopy $s_n : M \otimes_A B^{\otimes^n} \to M \otimes_A B^{\otimes^{n-1}}$ as follows

$$s_n(m \otimes b_1 \otimes \ldots \otimes b_n) = g(b_1)m \otimes \ldots \otimes b_n$$

This is well-defined, and $ds + sd = id$.

□

The following result is the analogue of GAGA for the etale topology. Note that the result holds for arbitrary schemes (we do not need to assume X to be complete as in GAGA itself)

Theorem 6.3. *Any quasi-coherent sheaf $\mathcal{F}$ on X defines an etale sheaf $\mathcal{F}_{et}$ on X_{et} given by $U \mapsto \Gamma(U, \varphi^*\mathcal{F})$ for any etale map $\varphi : U \to X$. Moreover there is a canonical isomorphism*

$$H^q(X_{zar}, \mathcal{F}) \cong H^q(X_{et}, \mathcal{F}_{et})$$

Proof. By 6.1, it suffices to check this a surjective etale map $V \to U$ where $V = \mathrm{Spec}B$ and $U = \mathrm{Spec}A$ are affine. Then $\mathcal{F}$ is defined by an A-module M, so the sheaf condition is the exactness of

$$M \to M \otimes_A B \xrightarrow{b\otimes 1 - 1\otimes b} M \otimes_A B \otimes_A B$$

which is an immediate consequence of descent (6.2).

Moreover, the cohomology of the covering ([Dan] p24) $V \to U$ is given by the cohomology of the complex

$$M \otimes_A B \to M \otimes_A B \otimes_A B \to \ldots$$

which is acyclic, again by 6.2. □

Descent theory also allows us to show that the presheaf represented by a group scheme defines an abelian sheaf on the etale site.

Proposition 6.4. *Let Z be a group scheme defined over X and $\mathcal{F} : Et/X \to Ab$ be given by $\mathcal{F}(U) = \mathrm{Hom}_X(U, Z)$ for $U \to X$ etale. Then $\mathcal{F}$ defines a sheaf on X_{et}.*

Proof. By 6.1, we only need to check the sheaf condition for Zariski open coverings, and a single covering $V \to U$, with U and V affine. But the latter follows from the exactness of

$$\mathrm{Hom}_A(C, A) \to \mathrm{Hom}_A(C, B) \to \mathrm{Hom}_A(C, B \otimes_A B)$$

which is exact by our descent theorem (6.2)

□

Example 6.5. $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[t])$*, the affine line considered as an additive group defines a etale sheaf on X by $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$ for $U \to X$ etale*

Example 6.6. $\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[t, t^{-1}])$, *the affine line with the origin removed considered as a multiplicative group, defines* $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^*$.

Example 6.7. $\mu_n = \operatorname{Spec}(\mathbb{Z}[t]/(t^n - 1)$. *Then* $\mu_n(U)$ *is the* n*th roots of unity in* $\Gamma(U, \mathcal{O}_U)$

Earlier we saw the group of automorphisms of the cover $\mathbb{C}^*_{(n)} \xrightarrow{z \to z^n} \mathbb{C}^*$ was just $\mu_n(\mathbb{C}^*)$. The kernel of the map on sheaves $\mathbb{G}_m \xrightarrow{t \to t^n} \mathbb{G}_m$ on a scheme X is the sheaf of roots of unity, μ_n.

In the Zariski topology, the map $\mathbb{G}_m \xrightarrow{t \to t^n} \mathbb{G}_m$ is not surjective, as an nth root of a regular function on a variety need not be algebraic. However, as our intuition from Section 2 requires, it is exact for the etale topology, provided n is invertible on X, i.e. does not divide the characteristic of the residue field at any point of X.

Proposition 6.8. *The* Kummer sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{t \to t^n} \mathbb{G}_m \to 0$ *is exact, provided* n *is invertible on* X.

Proof. Suppose $U \to X$ is etale and $a \in \mathbb{G}_m(U)$. Let $U' = \mathcal{O}_U(T)/(T^n - a)$. Then $U' \to U$ is clearly surjective, and is etale as $T^n - a$ is separable. But by construction a has an nth root in $\mathbb{G}_m(U')$, and so must also in $\mathbb{G}_m(U)$ □

Remark 6.9. *The Kummer sequence is the analogue of the exponential sequence* $0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{exp} \mathbb{C}^* \to 0$ *in the complex topology. It is exact on the flat site for any* n

We can define the notion of direct and inverse image of sheaves.

Definition 6.10. *If* $f : Y \to X$ *is a morphism of schemes, the direct image of a sheaf* $\mathcal{F}$ *on* Y_{et} *the sheaf on* X_{et} *defined by* $f_*\mathcal{F}(U) = \mathcal{F}(U \times_X Y)$, *for* $U \to X$ *etale.*

The functor f_* *is left exact, and has a left adjoint, the inverse image functor* f^*.

The stalks of the right derived functors of f_* are given by $(R^q f_* \mathcal{F})_{\bar{x}} = \varinjlim_{U} H^q(U \times_X Y)$, where the direct limit is over all etale neighbourhoods U of $\bar{x}$.

Recall that the strict localisation $\tilde{X} = \operatorname{Spec}(\mathcal{O}_{X,\bar{x}})$ captures all the local information for the etale topology. We can extend the sheaf $\mathcal{F}$ to $\tilde{Y} = Y \times_X \tilde{X}$ as follows:

Let $\tilde{V} \to \tilde{Y}$ be etale. Then there exists an etale neighbourhood U of $\bar{x}$ and an etale map $V \to Y \times_X U$ such that $\tilde{V} = V \times_U \tilde{X}$. Define

$$\mathcal{F}(\tilde{V}) = \varinjlim_{U'} \mathcal{F}(V \times_U U')$$

where the limit is taken over all etale neighbourhoods U' of $\bar{x}$ dominating U.

Thus we have a description of the fibres of the higher direct images of a sheaf $\mathcal{F}$

Proposition 6.11. $(R^q f_* \mathcal{F})_{\bar{x}} = H^q(\tilde{Y}, \mathcal{F})$

From this we can deduce the acyclicity of finite morphisms

Proposition 6.12. *Let $f : Y \to X$ be a finite morphism. Then $R^q f_* \mathcal{F} = 0$ for all $q \geq 0$.*

Proof. Using the same notation as above, it suffices to show that $H^q(\tilde{Y}, \mathcal{F}) = 0$. But $\tilde{Y}$ is the product of local strictly Henselian rings by 5.9, which are acyclic (5.8). □

7 Cohomology of a point

In this section we show that the etale cohomology of Spec$k$ is given by the Galois cohomology of the field.

Theorem 7.1. *Let $X = \mathrm{Spec} k$, k_{sep} a separable closure of k and $G = Gal(k_{sep}/k)$ the Galois group of k_{sep} with its usual topology. Then for any sheaf $\mathcal{F}$ on X,*

$$H^q(X_{et}, \mathcal{F}) = H^q(G, \mathcal{F}_{k_{sep}})$$

where $\mathcal{F}_{k_{sep}}$ denotes the geometric fibre

This follows immediately from the following description of sheaves on Spec$k$ as we have an equality of the derived functors

Proposition 7.2. *The functor (Sheaves on* Spec$k$*) $\to$ (discrete G-modules) given by $\mathcal{F} \mapsto \mathcal{F}_{k_{sep}}$ is an equivalence of categories.*

Proof. $\mathcal{F}_{k_{sep}}$ is a discrete G-module as G acts on $\mathcal{F}_{k_{sep}} = \varinjlim \mathcal{F}(k')$, where k' runs over finite Galois extensions of k, in the obvious way through a finite subgroup.

To construct the inverse, suppose A is an etale k-algebra, i.e. is a finite product of finite separable field extensions of k (4.12). Galois theory tells us that A corresponds to the finite G-module $F(A) = Hom_k(A, k_{sep})$. Thus given a discrete G-module M, we define

$$\mathcal{F}(A) = Hom_G(F(A), M)$$

To see that $\mathcal{F}$ is a sheaf, we only have to check the two cases in 6.1. A Zariski open covering of a collection of points $\prod_{i=1}^n k_i$ is just the inclusion of a subset of the points $\prod_{1 \leq i_j \leq n} k_{i_j} \hookrightarrow \prod_{i=1}^n k_i$.
Thus as $F(\prod k_i) = \prod F(k_i)$, and so $\mathcal{F}(\prod k_i) = \coprod \mathcal{F}(k_i)$, the sheaf conditon for Zariski open coverings is satisfied.

In checking the second condition, we may assume that V and U are given by single points, i.e. finite separable extensions L and K of k, and indeed that L/K is Galois. Then if $H = Gal(k_{sep}/k')$, then $F(k') = G/H$, and so $\mathcal{F}(k') = M^H$, the elements of M fixed by H. Then

$$\mathcal{F}(K) \to \mathcal{F}(L) \to \mathcal{F}(L \otimes_K L)$$

is exact, as $\mathcal{F}(K) = \mathcal{F}(L)^{Gal(L/K)}$, and $\mathcal{F}(L \otimes_K L) = \mathcal{F}(L) \times \mathcal{F}(L)$. □

Remark 7.3. *The calculation shows that a point $x = \mathrm{Spec} k$ has trivial etale cohomology groups iff k is separable. Thus it is the geometric point that plays the role of a point for the etale topology.*

8 Cohomology of curves

This section is devoted to computing the cohomology of curves, which forms the basis for calculation of etale cohomology of higher dimensional varieties as we can often reduce to this case by the various methods of devissage ([Arcata] Chap 3 Sect 4).

Specifically we show for a smooth connected complete curve X over an algebraically closed field of characteristic coprime to n

Theorem 8.1.

$$H^q(X, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & q = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & q = 1 \\ \mathbb{Z}/n\mathbb{Z} & q = 2 \\ 0 & q > 2 \end{cases}$$

This agrees with the corresponding result for complex varieties.

As we may expect from Section 2, the calculation rests on the Kummer sequence (6.8)

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

We first compute the cohomology of the curve X with co-efficients in $\mathbb{G}_m$

Proposition 8.2. *For a connected smooth curve X over an algebraically closed field k*

$$H^q(X, \mathbb{G}_m) = \begin{cases} k^* & q = 0 \\ Pic(X) & q = 1 \\ 0 & q > 1 \end{cases}$$

To show that the cohomology vanishes in dimensions at least 2, we look at the cohomology of the generic point.

Lemma 8.3. $H^q(\mathrm{Spec}(k(X)), \mathbb{G}_m) = 0$ *for* $q > 0$

Proof. This follows immediately from Tsen's theorem as $H^q(\mathrm{Spec} k(X), \mathbb{G}_m) = H^q(k(X), k(X)^*)$ □

Proposition 8.4. $H^q(X, \mathbb{G}_m) = 0$ *for* $q > 1$

Proof. If we write $X = X \setminus x \cup \mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$ as the "union" of the strictly local ring at a point with the variety with that point removed, then we expect their "intersection" $\tilde{X} = X \setminus x \times_X \mathrm{Spec}(\mathcal{O}_{X,\bar{x}}) = \mathrm{Spec}(Frac(\mathcal{O}_{X,\bar{x}}))$, to have zero cohomology in dimensions > 0. This again follows from Tsen's theorem (11.3).

Then as $\mathcal{O}_{X,\bar{x}}$ is also acyclic for the etale topology (5.8), we obtain from the Mayer-Vietoris sequence

$$H^{q-1}(\tilde{X}) \to H^q(X) \to H^q(X \setminus x) \oplus H^q(\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})) \to H^q(\tilde{X})$$

an isomorphism $H^q(X) \cong H^q(X \setminus x)$ for all $q > 1$. Thus by successively deleting points, we obtain in the limit $H^q(X) = H^q(\mathrm{Spec}(k(X)) = 0$.

Alternatively we can consider the pushforward of the sheaf $\mathbb{G}_m$ on the generic point via the inclusion $j : \mathrm{Spec}(k(X)) \to X$.

Then by the Leray spectral sequence (11.4)

$$H^q(X, j_*\mathbb{G}_m) = H^q(\mathrm{Spec}(k(X), \mathbb{G}_m) = 0$$

for $q > 0$

As X is smooth, we have the Weil-divisor exact sequence $0 \to \mathbb{G}_m \to j_*\mathbb{G}_m \to Div \to 0$, where Div is the sheaf of Weil divisors on X.

But $H^q(X, Div) = 0$, as $Div = \oplus_{codim(x)=1}(i_x)_*\mathbb{Z}_x$ is a skyscraper sheaf - we know that the finite morphisms $(i_x)_*$ are acyclic by 6.12.

Thus the result now follows from the long exact sequence associated to the Weil-divisor sequence. □

Remark 8.5. *The first part of the long exact sequence in the above proof gives us*

$$0 \to H^0(X, \mathbb{G}_m) \to k(X)^* \to Div(X) \to H^1(X, \mathbb{G}_m) \to 0$$

so $H^0(X, \mathbb{G}_m) = k^*$ and $H^1(X, \mathbb{G}_m) = Pic(X)$, so we have proved the proposition 8.2 about the cohomology of curves with coefficients in $\mathbb{G}_m$.

Now we consider the long exact sequence associated to the Kummer sequence. From the Proposition, we must have $H^q(X, \mu_n) = 0$ for $q > 2$

The start of the LES looks like

$$0 \to \mu_n(k) \to k^* \xrightarrow{n} k^* \to \dots$$

and as k is algebraically closed, the n-th power map is surjective. Thus it only remains to look at

$$0 \to H^1(X, \mu_n) \to Pic(X) \xrightarrow{n} Pic(X) \to H^2(X, \mu_n) \to 0$$

Now the degree map on line bundles gives us the following exact sequence

$$0 \to Pic^0(X) \to Pic(X) \xrightarrow{deg} \mathbb{Z} \to 0$$

and $Pic^0(X)$, the group of line bundles of degree 0, is isomorphic to the Jacobian variety $J(X)$. Thus we have the commutative diagram

$$\begin{array}{ccccc} J(X) & \longrightarrow & Pic(X) & \xrightarrow{deg} & \mathbb{Z} \\ \big\downarrow{\scriptstyle n} & & \big\downarrow{\scriptstyle n} & & \big\downarrow{\scriptstyle n} \\ J(X) & \longrightarrow & Pic(X) & \xrightarrow{deg} & \mathbb{Z} \end{array}$$

But we know that

$$ker(J(X) \xrightarrow{n} J(X)) = J_n(X), \text{the n-torsion points of J(X)}$$
$$coker(J(X) \xrightarrow{n} J(X)) = 0$$
$$ker(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = 0$$
$$coker(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$$

so

$$H^0(X, \mu_n) = \mu_n(X)$$
$$H^1(X, \mu_n) = ker(Pic(X) \xrightarrow{n} Pic(X)) = J_n(X)$$
$$H^2(X, \mu_n) = coker(Pic(X) \xrightarrow{n} Pic(X)) = \mathbb{Z}/n\mathbb{Z}$$

and after choosing a root of unity, i.e. an isomorphism $\mu_n(k) \cong \mathbb{Z}/n\mathbb{Z}$, $J_n(X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$, where g is the genus of the curve X.

9 Proper Base Change Theorem

In this section we prove the proper base change theorem for the etale topology. Amongst other applications, it allows us to show that there is a well-defined notion of cohomology with compact support.

The theorem states that the higher direct images of a proper map with torsion coefficients commutes with base change.

Theorem 9.1 (Proper Base Change Theorem)**.** *Suppose $f : X \to S$ is proper, and $g : T \to S$ any morphism.*

$$\begin{array}{ccc} X \times_S T & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

Then for any torsion sheaf F, the natural mapping $g^*(R^q f_* \mathcal{F}) \to R^q f'_*(g'^* \mathcal{F})$ of sheaves on T is an isomorphism.

Proof. We first construct the base change homomorphism. As g^* and g_* are adjoint, it suffices to construct

$$R^q f_* \to g_* R^q f'_* g^*$$

We have a map

$$R^q f_* \to R^q f_* g'_* g'^*$$

induced by the adjunction mapping $id \to g'_* g'^*$. But then as g_* is exact, we have a map

$$R^q f_* g'_* g'^* \to R^q(g_* f'_*) g'^* \to g_* R^q f'^* g'^*$$

and the required homomorphism is the composite of these two functors.

A map on sheaves is an isomorphism iff it is so on stalks. Suppose t maps to s, so that g' induces an isomorphism on the fibres $X_t \to X_s$. Then by theorem 9.2 below, we have

$$(R^q f_* \mathcal{F})_s \cong H^q(X_s, \mathcal{F}) \cong H^q(X_t, g^* \mathcal{F}) \cong (R^q f'^*(g'^* \mathcal{F}))_t$$

so it then it suffices to verify the theorem in the case that the base change morphism g is a map between geometric points, i.e. $T = \mathrm{Spec} K \to \mathrm{Spec} k = S$ where K/k is a finite algebraic extension of a separable field k.

But then K/k is a purely inseparable extension, for which f_* defines an equivalence of categories between the sheaves on $\mathrm{Spec} K$ and $\mathrm{Spec} k$ (see theorem 3.12 in [FK]). □

Theorem 9.2. *Let $f : X \to S$ be a proper morphism and $\mathcal{F}$ be a torsion sheaf. Let s be a geometric point of S and $X_s = X \times_S \mathrm{Spec} k(s)$ be the geometric fibre of f at s. Then for all $q > 0$, we have an isomorphism*

$$(R^q f_* \mathcal{F})_s \cong H^q(X_s, \mathcal{F})$$

We have seen earlier (6.11) that the fibres of the higher direct images of a sheaf over a geometric point are given by $(R^q f_* \mathcal{F})_s = H^q(X \otimes_S \mathcal{O}_{S,s}, \mathcal{F})$. Thus we may assume that S is the strictly local ring at s, so we can reduce to the following case:

Theorem 9.3. *Let A be a strictly Henselian ring and $S = \mathrm{Spec}(A)$. Let $f : X \to S$ be proper and X_0 be the fibre of f over the closed point $\mathrm{Spec}(k)$. Then for all torsion sheaves $\mathcal{F}$ and all $q \geq 0$, the natural map*

$$H^q(X, \mathcal{F}) \to H^q(X_0, \mathcal{F})$$

is an isomorphism.

We need to work in a category of sheaves which is stable under base change. This motivates the following inductive definition.

Definition 9.4. *A sheaf $\mathcal{F}$ is on a scheme X is constructible if it is constructible on $X \backslash U$ and locally constant on some Zariski open subset U, i.e. the is a covering $(U_i \to U)_i$ such that $\mathcal{F}|U_i$ is constant with finite fibres.*

Remark 9.5. *We see that constructible sheaves have finite fibres, and are indeed precisely the sheaves representable by quasi-finite schemes. They form an Abelian category over X, and are stable under extension and direct and inverse images.*

Lemma 9.6. *Any torsion sheaf is the inductive limit of constructible sheaves.*

Remark 9.7. *As cohomology commutes with direct limits, proving the theorem for constructible sheaves will, by passing to the limit, prove it for all torsion sheaves.*

Remark 9.8. *A sheaf is constructible if there is a finite family of* finite *morphisms $p_i : X_i \to X$ and constant torsion sheaves C_i on X_i such that $\mathcal{F} \to \prod p_i C_i$ is a momomorphism. Thus as the p_i are acyclic (6.12), it suffices to consider the constant torsion sheaves $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$*

After an inductive argument (see [FK] Chap 6), it suffices to show bijecivity for $q = 0$ and surjectivity for $q > 0$

Theorem 9.9. *The homomorphism $H^q(X, \mathbb{Z}/n\mathbb{Z}) = H^q(X_0, \mathbb{Z}/n\mathbb{Z})$ is bijective for $q = 0$, and surjective for $q > 0$*

Proof. By fibreing by hyperplane sections, we may assume that the fibre X_0 has dimension ≤ 1. We treat the cases $q = 0, 1, 2$ separately. In each case we look to compare a property of X with X_0.

Case q=0

This says the number of connected components of X and the closed fibre $X_0 = X \otimes_A A/m$ are equal. We deduce this from the Zariski connectedness theorem 11.5

Consider the Stein factorisation of f, $X \to S' \to S$, where $S' = SpecA'$ and A' is a *finite* A-algebra. Then (5.9) as A is Henselian, A' is the product of local A-algebras A'_i corresponding to the points i of the fibre S'_s

Write $S'_i = SpecA'_i$ and $X_i = X \times_S S'_i$. Then X_i is proper over S_i, so any component of X_i meets the fibre $(X_i)_i$. But by Zariski's connectedness theroem, these fibres are connected. Thus the X_i are the connected components of X and they correspond bijectively to the $(X_i)_i$, the connected components of X_0

Case q=1

We must show every etale covering $X'_0 \to X_0$ extends to an etale covering $X' \to X$.

As the etale coverings of a scheme do not depend on the nilpotents, we see that X'_0 extends (uniquely) to a covering X'_n any infinitesimal neighbourhood $X_n = X \otimes_A A/m^{n+1}$. Then my Grothendieck's existence theorem for formal schemes, we get an etale covering of the scheme $X \otimes_A \hat{A}$.

Then we can apply Artin's approximation theorem (11.7) to the functor F : (A-algebras) $\to$ (Sets), which assigns to an A-algebra B the set of isomorphism classes of etale coverings of $X \otimes_A B$. Thus there exists an etale covering of X which agrees over $X_0 = X \otimes_A A/m$ with the covering of $X \otimes_A \hat{A}$.

(We may apply 11.7 here as the Henselisation of a finitely generated excellent algebra over an excellent discrete valuation ring is isomorphic to the strict Henselisation of a finitely generated algebra over $\mathbb{Z}$, and every strictly Henselian ring is the direct limit of such rings.)

Case q=2
From the Kummer sequence 6.8, we have the commutative diagram

$$\begin{array}{ccc} Pic(X) & \longrightarrow & H^2(X,\mu_n) \\ \downarrow & & \downarrow \\ Pic(X_0) & \longrightarrow & H^2(X_0,\mu_n) \end{array}$$

The Kummer sequence for X_0 continues

$$Pic(X_0) \to H^2(X_0,\mu_n) \to H^2(X_0,\mathbb{G}_m) \xrightarrow{n} H^2(X_0,\mathbb{G}_m)$$

But as X_0 is a curve, $H^2(X_0,\mathbb{G}_m)$ is 0 (8.2) if k has characteristic 0, and a p-group in characteristic p ([FK] 5.2), so in any case the multiplication by n map is injective, as p and n are coprime. Thus the bottom line of the commutative diagram is surjective.

Thus we're done if we can show that $Pic(X) \to Pic(X_0)$ is surjective, i.e. any Cartier divisor D_0 on X_0 can be extended to a divisor D on X. As X_0 is a curve, a divisor is the sum of divisors supported at a point, so we can assume D_0 is supported at a point x with local equation $t_0 \in \mathcal{O}_{X_0,x}$.

By lifting the local equation t_0 to t in a sufficiently small neighbourhood U of x in X, we can suppose that the equation $Y = \{t = 0\}$ meets only meets X_0 at x.

But as A is Henselian, any other connected component of Y does not meet X_0. Thus by taking U sufficiently small, we can assume that Y is closed in X and so $D = div(t)$ lifts D_0.

□

Corollary 9.10. *Let X be a complete variety over a separably closed field k, and k' be a separably closed extension of k. Then the cohomology of X and $X' = X \otimes_k k'$ agree.*

Proof. Take $S = \mathrm{Spec}k$ in the above theorem. Then $f : X \to S$ is proper, and Spec$k$ and Speck' are both geometric points of S. Thus as the stalk of a geometric point depends only on the residue field,

$$H^q(X,\mathcal{F}) = (R^q f_*\mathcal{F})_{\mathrm{Spec}k} = (R^q f_*\mathcal{F})_{\mathrm{Spec}k'} = H^q(X',\mathcal{F})$$

□

Definition 9.11. *Let $\mathcal{F}$ be a torsion sheaf on a scheme X, and $j : X \to \bar{X}$ an open immersion into a complete scheme $\bar{X}$. Denoting by $j_!\mathcal{F}$ the extension by zero to $\bar{X}$, we define the cohomology groups with compact support $H^*_c(X,\mathcal{F})$ as*

$$H^q_c(X,\mathcal{F}) = H^q(\bar{X}, j_!\mathcal{F})$$

It is a theorem of Nagata that such a compactification exists. We now show the definition is independent of the compactification chosen.

Suppose $j_1 : X \to \bar{X_1}$ and $j_2 : X \to \bar{X_2}$ are two such compactifications. Then the closure of the image of the diagonal mapping $X \xrightarrow{(j_1,j_2)} \bar{X_1} \times \bar{X_2}$ is also a compactification of X, whose restrictions to X_1 and X_2 are proper maps. Thus we may assume we have a proper map $f : \bar{X_2} \to \bar{X_1}$.

We have the Leray spectral sequence

$$H^q(\bar{X_2}, R^r f_*(j_2)_! \mathcal{F}) \to H^{q+r}(\bar{X_1}, (j_1)_! \mathcal{F})$$

Thus it suffices to show $f_*(j_2)_! \mathcal{F} = (j_1)_!$ and $R^r f_*(j_2)_! \mathcal{F} = 0$ for $r > 0$. By the proper base change theorem, we can check this on each fibre of f. But on X, f is an isomorphism, and on $\bar{X_1} \backslash X$, $(j_2)_! \mathcal{F}$ on zero on the fibres of f, so the conditions are satisfied on both sets.

Theorem 9.12 (Finiteness)**.** *Let X be a complete variety over a separably closed field k, and $\mathcal{F}$ a constructible sheaf. Then the cohomology groups $H^q(X, \mathcal{F})$ are finite.*

Proof. As X is complete, any map from $f : X \to s$ to a (geometric) point is proper. But any constructible sheaf on a point is just a finite abelian group, so as $H^q(X, \mathcal{F}) = R^q f_* \mathcal{F}$ (9.2) we must show that the higher direct images $R^q f_* \mathcal{F}$ are constructible.

As in the proof of 9.2, we can assume the fibres of f are smooth complete curves and $/F = \mathbb{Z}/n\mathbb{Z}$. By replacing S with an open set, we may assume that f is smooth so the curves have the same genus. But then by the proper base change theorem, the fibres of $R^q f_* \mathcal{F}$ are all the same, given by the cohomology of curves (8.2). □

Theorem 9.13 (Vanishing)**.** *Let X be a complete variety over a separably closed field k, and $\mathcal{F}$ a constructible sheaf. Then $H^q(X, \mathcal{F}) = 0$ for $q > 2(dim X)$.*

Proof. As above, finding an appropriate fibration we can reduce to the case of curves, and take $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$ where we have already seen the result. □

Remark 9.14. *There are corresponding finiteness and vanishing results for cohomology with compact support for varieties that are not necessarily complete.*

Theorem 9.15 (Poincare Duality)**.** *Suppose X is a smooth variety of dimension d, and $\mathcal{F}$ a constructible sheaf. Then there is a perfect pairing*

$$H^q_c(X, \mathcal{F}) \otimes H^{2d-q}(X, \mathrm{Hom}(\mathcal{F}, \mathbb{Z}/n\mathbb{Z})) \to \mathbb{Z}/n\mathbb{Z}$$

Proof. The strategy is to reduce to the case of smooth complete connected curves by finding an appropriate fibration. As in the proof of the proper base change theroem, we may assume that $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$. Then the only interesting case is $q = 1$, and as $H^1(X, \mu_n) = J_n(X)$, this is just the Weil pairing on the Jacobian $J_n(X) \otimes J_n(X) \to \mu_n$

□

10 l-Adic Cohomology

So far we have only dealt with etale cohomology in torsion sheaves, and indeed the cohomology with coefficients in nontorsion sheaves often does not agree with what we expect from the complex topology. However, a Weil cohomology has coefficient in a field of characteristic zero. As we saw in Section 2, the solution lies in using the torsion sheaves to approximate the integral cohomology

Definition 10.1. $H^q(X, \mathbb{Z}_\ell) = \varinjlim_{n} H^q(X, \mathbb{Z}/\ell^n\mathbb{Z})$

Remark 10.2. *This is* not *the same as the etale cohomology with co-efficients in the constant sheaf* $\mathbb{Z}_\ell$ *(cohomology does not commute with projective limits. Indeed the are no homomorphisms* $\pi_1^{et}(X) \to \mathbb{Z}_\ell$ *if* $\mathbb{Z}_\ell$ *has the discrete topology, whereas there are if, as in the definition, we use the natural topology of the* ℓ*-adic integers.*

We then get the ℓ-adic cohomology groups by introducing denominators.

Definition 10.3. $H^q(X, \mathbb{Q}_\ell) = H^q(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

The main theorems for ℓ-adic cohomology then follow from the corresponding theorems for constructible sheaves in etale cohomology.

11 Appendix

In this appendix I have collected a few of the theorems quoted at some point is the main text.

Theorem 11.1 (Riemann existence theorem)**.** *For any scheme* X *any finite unramified analytic covering of* X_{an} *is algebraic.*

Theorem 11.2 (Tsen's theorem)**.** *A field* K *of transcendence degree 1 over an algebraically closed field is quasi-algebraically closed*

Proof. See [Dan] 4.5.2 □

This has an important corollary in terms of the Galois cohomology of finite Galois extensions of the function field of a variety which is used in the text to calculate the cohomology of points.

Corollary 11.3. *Let* $K = k(X)$ *be the function field of a variety* X*, and* L/K *a finite Galois extension. Then for all* $q > 0$*,*

$$H^q(Gal(L/K)), L^*) = 0$$

Theorem 11.4 (The Leray Spectral Sequence)**.** *For any morphism* $g : Y \to X$ *of schemes and sheaf* $\mathcal{F}$ *on* Y*, there is a spectral sequence*

$$H^r(X, R^s g_* \mathcal{F}) \Rightarrow H^{r+s}(Y, \mathcal{F})$$

Proof. [Dan] 1.4.3 □

Theorem 11.5 (Zariski's connectedness theorem). *Let $f : X \to Y$ be a proper morphism. Then $f_*\mathcal{O}_X$ is a finite $\mathcal{O}_Y$-algebra, and setting $Y' = \mathrm{Spec} f_*\mathcal{O}_X$ we have the* Stein factorisation $X \xrightarrow{f} Y' \xrightarrow{g} Y$ *where f is finite and g proper.*

Further $f'_*\mathcal{O}_X = \mathcal{O}_Y$, and the fibres of f' are connected and non-empty.

Proof. See [Ill] theorem 2.12 □

The connectedness theorem has the following equivalent reformulation.

Theorem 11.6 (Zariski's Main Theorem). *Suppose $f : X \to Y$ is quasi-finite. Then f factors as $X \to Z \hookrightarrow Y$ into a finite morphism followed by an open inclusion*

Proof. See [Ill] corollary 2.17 □

Theorem 11.7 (Artin's Approximation Theorem). *Let R be a field or an excellent discrete valuation ring, and A be the Henselisation of an R-algebra of finite type at a prime ideal, and m be the maximal ideal of A.*

Let F : (A-algebras) $\to$ (Sets) be a functor locally of finite presentation associating to an A-algebra B the set of isomorphism classes of a given structure over B. Then given any integer $n \geq 0$, and any element $\bar{\zeta} \in F(\hat{A})$, there exists an element $\zeta \in F(A)$ inducing the same structure on A/m^{n+1}

Proof. Theorem 1.12 in [Artin]. □

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