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Federico Bassetti

a Dipartimento di Matematica "F. Casorati", Università di Pavia, Pavia, Italy

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Variable Time-Step Discretization of Degenerate Evolution Equations in Banach Spaces

Federico Bassetti*

Dipartimento di Matematica “F. Casorati”,
Università di Pavia, Pavia, Italy

ABSTRACT

This work is concerned with an abstract differential inclusion of the form

$$Bu' + \partial \phi(u) \ni f,$$

where $B : V \rightarrow V'$ is a linear, continuous, symmetric and monotone operator defined over a separable Banach space $V$, and $\partial \phi$ is the subdifferential of a proper, convex, l.s.c, positive real function. We consider an approximation of the previous equation by a backward Euler method with variable time-step. Under suitable hypothesis of coercivity we prove that the discrete solution converges uniformly to a strong solution of the equation, in the seminorm induced by $B$, as the maximum of the time steps goes to 0. We derive computable estimates of the discretization error, which are optimal w.r.t. the order and impose no constrains between consecutive time steps. In addition we prove some regularity and uniqueness results for the solution. Finally we extend some of the previous results to the case in which $\partial \phi$ is perturbed by a Lipschitz map.

Key Words: Backward Euler method; Degenerate evolution equations; Differential inclusion; Error estimates.

AMS Subject Classification: 35A40; 35A35; 35K65; 35K90.
INTRODUCTION

This work is concerned with an abstract differential inclusion of the form

\[
\begin{aligned}
(Bu)'(t) + \partial \phi(u(t)) &\ni f(t), \quad t \in (0, T), \\
Bu(0) &= B u_0,
\end{aligned}
\]

where \(B : V \to V'\) is a linear, continuous, symmetric, and monotone operator defined over a separable Banach space \(V\) and \(\partial \phi\) is the subdifferential of a proper, convex, and l.s.c function \(\phi : V \to [0, +\infty]\).

There is a number of interesting results about existence and regularity of solutions for equations of this sort. In most of these \(\partial \phi\) is replaced by a more general nonlinear monotone hemicontinuous operator \(A : V \to V'\) and \(V\) is reflexive. In such cases the coercivity of \(A\) is required as classical hypothesis, even if weaker assumptions involving \(A + B\) have been considered too, see (Bardos and Brézis, 1969; Brézis, 1970; Carroll and Showalter, 1976; DiBenedetto and Showalter, 1981; Kuttler, 1982; Kenneth L. Kuttler, 1985; Showalter, 1975, 1997). To obtain a solution to Eq. (1) many approaches have been used. In Bardos and Brézis (1969), Brézis (1970), Kuttler (1982), Eq. (1) is approximated by a stationary problem, using elliptic regularization or replacing the time derivative by a finite difference. In Kenneth L. Kuttler (1985), Eq. (1) is approximated by using a Galerkin method.

**Backward Euler Approximation**

In this work we consider an approximation of Eq. (1) by a backward Euler method with variable time step. First of all, inspired by the Minimizing Movements Method introduced by De Giorgi in (1993), we link this classical discretization technique with a recursive minimization problem of a suitable family of functionals defined over \(V\). Under hypothesis of weak compactness of the sublevels of these functionals we prove that the discrete solution \(U_\tau\), obtained with the backward Euler method, converges uniformly to a strong solution \(u\) of Eq. (1), in the seminorm induced by \(B\), as the maximum of the time steps \(\tau\) goes to 0. In addition we prove some regularity and uniqueness results for Eq. (1).

**Error Estimates**

Then we consider the problem of finding computable error estimates. Many error estimates have been studied for non degenerate problem, i.e., when \(V\) is an Hilbert space and \(B\) is the Riesz isomorphism between \(V\) and \(V'\). In 1971, Crandall and Ligget proved that in the case of a uniform partition, with time step \(\tau\), the approximate solution \(U_\tau\) converges uniformly to \(u\) as \(\tau \to 0\). This was extended to the case of nonuniform partition by Crandall and Evans in 1975. When the partition is uniform an \(a\ priori\) estimate of the error \(\|u - U_\tau\|\) of order \(O(\sqrt{\tau})\) has
been derived in Crandall and Liggett (1971). As for it concerns equations with a subdifferential, Baiocchi in 1989 proved an \textit{a priori} estimate of order $O(\tau)$ in the case in which $\phi$ is the sum of a positive, l.s.c., quadratic form and of the indicator of a convex set in an Hilbert space (obstacle problem). This latter result is optimal with respect to both the order and the regularity of the solution. In Rulla (1996), Savaré (1996) this result has been extended to a generic l.s.c convex function. In the more general case of angle bounded operators (possibly perturbed by a Lipschitz map), Nochetto et al. (2000) using Discrete Energy Dissipation techniques derived \textit{a posteriori} and \textit{a priori} error estimates for variable time step discretization approach. Such estimates are optimal w.r.t both order and regularity, and impose no constraints between consecutive time steps.

In the present work we resort the Energy Dissipation techniques developed in Nochetto et al. (2000) and derive computable estimates of the discretization error for degenerating Eq. (1). As it happens in non degenerate case, these estimates impose no constraints between consecutive time steps.

**Lipschitz Perturbations**

Finally we extend some of the previous results to the case in which $\partial \phi$ is perturbed by a Lipschitz map. More exactly we prove an existence theorem and derive error estimates for the Lipschitz perturbed equation. We also present an example of this kind of equation: a reaction-diffusion system of FitzHugh–Nagumo type related to the behavior of the electrical conduction of the cardiac muscle.

**Plain of the Article**

Section 1 describes the context of the results to be obtained and presents the discretization scheme. Section 2 contains the statements of convergence, regularity, uniqueness theorems and also of discretization error estimates. Section 3 presents some examples of degenerate equations. Sections 4 and 5 are dedicated to preliminary results and estimates, which play an important role in the sequel. Sections 6, 7, and 8 are dedicated to the proofs of theorems. Section 9 concerns with error estimates for the Lipschitz perturbed equation.

### 1. SETTING AND DISCRETIZATION SCHEME

**Preliminary definitions.** Let $V$ be a separable Banach space, not necessarily reflexive. We denote by $V'$ its dual space, by $\langle \cdot, \cdot \rangle$ the duality pairing between $V$ and $V'$, and by $\| \cdot \|$ and $\| \cdot \|_*$ the norms over $V$ and $V'$ respectively.

Let $B : V \to V'$ be a linear continuous operator and $b(\cdot, \cdot) : V \times V \to \mathbb{R}$ the associated bilinear form, namely:

$$b(u, v) := \langle Bu, v \rangle \leq M \| u \| \| v \| \quad \forall u, v \in V.$$  \hfill (2)
Moreover we suppose that $B$ is symmetric and nonnegative, i.e.,

$$b(u, u) \geq 0, \quad b(u, v) = b(v, u), \quad \forall u, v \in V. \quad (3)$$

We denote by $b(\cdot, \cdot)$ the quadratic form associated to $b(\cdot, \cdot)$, i.e., $b(u) := b(u, u)$. Note that $B$ may be degenerate, i.e., $\text{Ker}(B) := \{u : b(u) = 0\} \neq \{0\}$. Finally let

$$\phi : V \to [0, +\infty]$$

be a proper, convex, l.s.c function,

and $\partial \phi$ be its subdifferential, which is defined for every $w \in V$ by

$$\partial \phi(w) := \{w_\star \in V^* : \langle w_\star, v - w \rangle + \phi(w) \leq \phi(v) \quad \forall v \in V\}. \quad (5)$$

Given an initial datum $u_0 \in D(\phi)$ and a function $f \in L^2(0, T; V')$, we are interested in solving the Cauchy problem (1). We recall that a strong solution of Eq. (1) is a strongly measurable function $u : [0, T] \to V$ such that $Bu \in W^{1,1}(0, T; V')$ and the inclusion is true almost everywhere in $(0, T)$ (Def 3.1 (Brézis, 1973)).

We set, for every $f \in V'$,

$$\|f\|_{B'} := \sup_{v \in V : b(v) \neq 0} \frac{\langle f, v \rangle}{\sqrt{b(v)}}, \quad D(b^*) := \{f : \|f\|_{B'} < +\infty\}. \quad (6)$$

In Propositions 4.4 and 4.5 we will show that $\| \cdot \|_{B'}$ is an norm over $D(b^*)$ induced by a scalar product $b^*(\cdot, \cdot)$, and that $D(b^*)$ endowed with this norm is complete and continuously embedded in $V'$. We will denote by $D^*$ the Hilbert space $(D(b^*), \| \cdot \|_{B'})$.

If $f \in L^2(0, T; D^*)$, $u_0 \in D(\phi)$ and $(B + \partial \phi)(V) = D^*$ in Showalter (1997) (Theorem 6.1 Chapter IV) it is proved the existence of a solution $u$ of Eq. (1), such that $u \in L^2(0, T; D^*)$, $\phi \circ u \in L^\infty(0, T)$ and $(Bu') \in L^2(0, T; D^*)$. In the case of time depending $B(\cdot, t) : V \to V'$, with $V$ reflexive, weaker solutions in $L^2(0, T; V)$ have been derived for a general monotone and hemicontinuous (time depending) operator $A(\cdot, t) : V \to V'$, under particular assumptions of boundness and weakly coercivity for $A$, see for instance Prop. 6.2 Chap. III Showalter (1997) and Thm. 2 Brézis (1970).

Here we are mainly interested to approximation and convergence results: as a byproduct, we will also refine some regularity results of Showalter (1997).

**Definition 1.1.** Let be $\psi : V \to (-\infty, +\infty]$. We say that $\psi$ is weakly coercive if for every real number $r$ the sublevel $\{v \in V : \psi(v) \leq r\}$ is weakly compact in $V$; in particular $\psi$ is weakly l.s.c.

**Remark 1.** When $V$ is reflexive $\psi$ is weakly coercive iff $\psi$ is weakly l.s.c. and its sublevels are bounded.

Finally we recall that a convex function $\psi : V \to (-\infty, +\infty]$ is said *strictly convex* if

$$u, v \in V, \theta \in (0, 1), \quad (1 - \theta)\psi(u) + \theta\psi(v) = \psi((1 - \theta)u + \theta v) \Rightarrow u = v. \quad (7)$$
Discrete solutions. Let $P$ be a partition of the time interval $[0, T]$.

\[ P = \{0 = t_0 < t_1 < \cdots < t_N = T\}. \]  

We set $\tau_n := t_n - t_{n-1}$, $\tau := \{\tau_1, \ldots, \tau_N\}$ and $|\tau| := \max_{1 \leq n \leq N} |\tau_n|$. 

Once suitable approximations $U^n_0$ of the initial datum $u_0$ and $F^n_0$ of $f$ over $(t_{n-1}, t_n)$ are chosen, the discrete solution $\{U^n_\tau\}_{n=1}^N \subset D(\phi)$ is recursively defined by solving at every step $1 \leq n \leq N$ the differential inclusion:

\[ \frac{BU^n_\tau - BU^{n-1}_\tau}{\tau_n} + \partial\phi(U^n_\tau) \ni F^n_\tau. \]  

Minimizing movements. The backward Euler method Sch. (9) could also be considered as a recursive minimization problem. To see that, we introduce a functional which is strictly related to Eq. (9).

**Definition 1.2.** For every $u, v \in V$ and $1 \leq n \leq N$ define

\[ F(\tau, n, v, u) := \frac{1}{2\tau_n} b(v-u) + \phi(v) - \langle F^n_\tau, v \rangle \]  

Now $\{U^n_\tau\}_{n=1}^N$ can be recursively determined by solving the minimization problem:

\[ \min_{v \in V} F(\tau, n, v, U^{n-1}_\tau) = F(\tau, n, U^n_\tau, U^{n-1}_\tau). \]  

From this point of view our discretizing approach could be read in the general framework introduced by De Giorgi, who called it Minimizing Movements Method, see Ambrosio (1995), De Giorgi (1993), and Gianazza and Savaré (1994). In order to ensure the existence of a solution to Eq. (11), we will assume that:

\[ \exists \lambda > 0 : \phi(.) + \lambda b(.) \text{ is weakly coercive.} \]  

**Remark 2.** Being $\phi$ and $b$ nonnegative, Eq. (12) is fulfilled if and only if $\phi(.) + \lambda' b(.)$ is weakly coercive for each $\lambda' > 0$.

An easy consequence of hypothesis (12) and of the continuity of $b$ is the following:

**Lemma 1.3.** Under the previous assumptions there exists a sequence $\{U^n_\tau\}_{n=1}^N$ such that Eqs. (9) and (11) are fulfilled for $1 \leq n \leq N$.

**Remark 3.** In this article we will assume that Hys. (2), (3), (4), and (12) are fulfilled.

**Notation.** Let $P$ be a partition of the time interval $[0, T]$ like in Eq. (8). For any sequence $\{W^n_\tau\}_{n=0}^N$ we define the piecewise constant function $W_\tau$ and the linear interpolant $W_\tau$ on the intervals $(t_{n-1}, t_n)$ as
\[ W_n(t) := (1 - l_t(t))W^n_t + l_t(t)W^{n-1}_t, \quad \overline{W}_n(t) := W^n_t, \quad (13) \]

for \( 1 \leq n \leq N \), where \( l_t(t) = (t_n - t)/\tau_n \). In particular the linear interpolant of \( \{ U^n_t \}_{t \in P} \) is

\[ U_t(t) := \frac{t - t_{n-1}}{\tau_n} U^n_t + \frac{t_n - t}{\tau_n} U_t^{n-1}, \quad t \in (t_{n-1}, t_n]. \quad (14) \]

We also denote by \( \{ \delta W^n_t \}_{t \in P} \) the discrete derivative of the sequence \( \{ W^n_t \}_{t \in P} \), namely

\[ \delta W^n_t = \frac{W^n_t - W_t^{n-1}}{\tau_n} = W'_n(t), \quad \forall t \in (t_{n-1}, t_n]. \quad (15) \]

If we denote by \( \overline{\tau}(t) \) the piecewise constant function taking values \( \tau_n \) on \( (t_{n-1}, t_n] \), we observe that \( W_n(t) - \overline{W}_n(t) = -l_t(t)\overline{\tau}(t) W'_n(t) \), for every \( t \in (0, T) \setminus P \).

2. MAIN RESULTS

Convergence, regularity, and uniqueness. In this section we collect our main results about the convergence of the discrete solutions \( U_t \), defined by Eq. (14), to a solution of Eq. (1), and also about its uniqueness and regularity.

Let \( D_{0}^t := \overline{B}(V) \); for every \( f \in D^* \), let \( f^* \in D_0^t \) be defined by the relation \( b^*(f^*, v) = (f, v) \forall v \in V \), see Lemma 4.6.

**Theorem 2.1. (Convergence).** Let \( f \in L^2(0, T; D^*) \) and \( u_0 \in D(\phi) \). For every partition \( P \) like Eq. (8), set \( B^0 \overline{U}_t := Bu_0, \quad F_t := (1/\tau_n) \int_{t_{n-1}}^{t} f(s) \, ds \) and let \( U_t \) be defined as in Eq. (14). Then there exists a strong solution \( u \) of the differential inclusion

\[
\begin{cases}
(Bu_t(t)) + \partial \phi(u(t)) \ni f(t) & \text{a.e. in } (0, T),
\end{cases}
\]

with

\[
Bu(0) = Bu_0, \quad (16)
\]

\[
u \in L^\infty(0, T; V), \quad Bu \in H^1(0, T; D^*), \quad \phi \circ u \in AC([0, T]),
\]

\[
u(t) \in \arg\min_{v \in B(u(t))} \phi(v) \quad \forall t \in [0, T],
\]

\[
\frac{d}{dt}(\phi \circ u(t)) = b^*(f^*(t) - (Bu_t(t)), (Bu_t(t))) \quad \text{a.e. in } (0, T),
\]

such that when \( |\tau| \to 0 \)

\[
Bu_t \to Bu \quad \text{in } C^0([0, T]; D^*) \quad \text{and} \quad H^1(0, T; D^*),
\]

\[
\lim_{|\tau| \to 0} \phi(U_t(t)) = \phi(u(t)) \quad \forall t \in [0, T],
\]

and \( U_t \to u \) in the weak topology \( \sigma(L^\infty(0, T; V), L^1(0, T; V')) \) for a suitable sequence \( \tau_t \).
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Remark 4. Since $D^*$ is an Hilbert space we can introduce the Banach space $BV([0, T], D^*)$ of functions $g : [0, T] \rightarrow D^*$ with bounded total variation (Def. A.2 Brezis (1973)), that is

$$\text{Var}_n(g) := \sup_{0 = r_0 < r_1 < \cdots < r_J = T} \sum_{j=1}^J \|g(r_j) - g(r_{j-1})\|_{D^*} < +\infty.$$  

(20)

We recall that if $g \in BV([0, T], D^*)$ then $g$ is strong measurable and bounded, moreover at every point $t_0 \in [0, T)$ there exist the right limit $g^+(t_0) := \lim_{t \uparrow t_0} g(t)$, and the left limit $g^-(t_0) := \lim_{t \downarrow t_0} g(t)$ if $t_0 \in (0, T]$.

Theorem 2.2. (Regularity). Under the same assumptions of Theorem 2.1, if $f \in BV([0, T]; D^*)$ and $\phi(u_0) \cap D^* \neq \emptyset$ then $Bu \in W^{1,\infty}(0, T; D^*)$.

Theorem 2.3. (Uniqueness and pointwise convergence). Under the same assumptions of Theorem 2.1, if there exists a positive number $\lambda$ such that $\phi(\cdot) + \lambda b(\cdot)$ is strictly convex, then the solution of Eq. (16) is unique and weakly continuous, i.e., $u \in C^0([0, T]; V_w)$. Moreover $U\tau(t)$ is weakly convergent to $u(t)$ in $V$ for every $t \in [0, T]$.

A posteriori and a priori estimates. Let $u$ be a solution of the differential inclusion (16) and $U\tau$ be defined as in Eq. (14). Since we are interested in studying the rate of convergence of $U\tau$ to $u$, we define the global error:

$$E\tau = \max \left\{ \max_{0 \leq t \leq T} \sqrt{b(U\tau - u)}; \left( 2 \int_0^T \|\sigma(U\tau, u) + \sigma(u, U\tau)\| dt \right)^{1/2} \right\},$$  

(21)

where (Def. 2.3 Nochetto et al. (2000))

$$\sigma(w, v) := \phi(v) - \phi(w) - \sup_{y \in \phi(w)} \langle w, v - w \rangle.$$  

(22)

Note that $\sigma(w, v) \geq 0$ and $w_\ast \in \phi(w)$ if and only if

$$\langle w_\ast, v - w \rangle + \sigma(w, v) \leq \phi(v) - \phi(w) \quad \forall v \in D(\phi).$$  

(23)

In many situations $\sigma$ gives more information about the convergence, see e.g. in Sec. 4 of this article and Sec. 2 in Nochetto et al. (2000). Observe that when $\phi$ is strictly convex, then $\sigma(v, w) = 0 \iff v = w$, so that it provides a sort of measure of the difference between $v$ and $w$.

Adapting the techniques developed in Nochetto et al. (2000), we will prove that $E\tau$ converges to zero.

A posteriori estimates. In view of giving a posteriori error estimates, we introduce a quantity which, in some sense, measures the amount of Discrete Energy Dissipation (Def. 3.1 Nochetto et al. (2000)), more precisely.

Definition 2.4. To every discrete solution $\{U^n_\tau\}_{n=0}^N$ of Eq. (9) we associate the error estimators

$$E^n_\tau = \langle F^n_\tau - \delta B U^n_\tau, \delta U^n_\tau \rangle - \delta \phi(U^n_\tau), \quad 1 \leq n \leq N,$$

(24)

$E^n_\tau$ is defined provided $U^n_\tau$ belongs to $D(\phi)$. 
Notice that these estimators measure the discrete speed of decay of the functional $v \mapsto \mathcal{F}(\tau, n, v, U_{\tau}^{n-1})$, because

$$E_{\tau}^{n} = \left\{ \frac{1}{2\tau_n} \left[ \mathcal{F}(\tau, n, U_{\tau}^{n-1}, U_{\tau}^{n-1}) - \mathcal{F}(\tau, n, U_{\tau}^{n}, U_{\tau}^{n-1}) \right] - \frac{1}{2} b(U_{\tau}^{n}) \right\}. \quad (25)$$

The importance of $E_{\tau}^{n}$ is enlightened by the following a posteriori error estimate:

**Theorem 2.5.** Under the same assumptions of Theorem 2.1, let $u$ be a solution of Eq. (16) and $E_{\tau}$ and $E_{\tau}^{n}$ be defined as in Eqs. (21) and (24); then the following a posteriori error estimate holds

$$E_{\tau} \leq \left( \sum_{n=1}^{N} \tau_n^2 E_{\tau}^{n} \right)^{1/2} + \int_{0}^{T} \| f(t) - \mathcal{F}_{\tau}(t) \|_{\mathcal{B}'} \, dt. \quad (26)$$

**Theorem 2.6.** Under the same assumptions of Theorem 2.2 in addition we have

$$\lim_{|\tau| \to 0} \sigma(u(t), U_{\tau}(t)) = 0 \quad \forall t \in (0, T). \quad (27)$$

**A priori estimates.** Now we show that the a posteriori quantities introduced before vanish as $|\tau| \to 0$ with an optimal rate in terms of the regularity of the data.

**Theorem 2.7.** Under the same assumptions of Theorem 2.1, we have

$$\sum_{n=1}^{N} \tau_n^2 E_{\tau}^{n} \leq \left( \phi(u_0) + \frac{1}{4} \int_{0}^{T} \| f(t) \|_{\mathcal{B}'}^2 \, dt \right) |\tau|, \quad (28)$$

and the global error defined in Eq. (21) is bounded by

$$E_{\tau} \leq \sqrt{3|\tau|(\phi(u_0) + \| f \|_{L^2(0, T; \mathcal{D}')}^2)^{1/2}}. \quad (29)$$

When $f \in BV([0, T]; D')$ we can improve the order of convergence.

**Theorem 2.8.** If $U_{\tau}^0 = u_0 \in D(\partial \phi)$, $f \in BV([0, T], D')$, $F_{\tau}^n := f^+(t_n)$, and $\partial \phi(u_0) \cap D' \neq \emptyset$, then the following error estimate holds

$$E_{\tau} \leq |\tau| \left( \frac{1}{\sqrt{2}} \| f^+(0) - \partial \phi(u_0) \|_{\mathcal{B}'} + 2 Var_{\mathcal{B}'}(f) \right), \quad (30)$$

where $(f^+(0) - \partial \phi(u_0))^{\circ}$ denotes the minimal selection of $f^+(0) - \partial \phi(u_0)$ with respect to $\| \cdot \|_{\mathcal{B}'}$.

**Lipschitz perturbations.** A natural extension of problem (1) is to consider the case in which $\partial \phi$ is perturbed by a Lipschitz map. We have proved an existence theorem which is analogous to Thm. 3.17 in Brezis (1973).

**Theorem 2.9.** Under the same hypothesis of Theorem 2.1, let $\mathcal{L} : [0, T] \times V \to V'$ be a function such that $L > 0$ exists:

$$\| \mathcal{L}(v_1, t) - \mathcal{L}(v_2, t) \|_{\mathcal{B}'} \leq L \sqrt{b(v_1 - v_2)}, \quad \forall t \in [0, T], \quad \forall v_1, v_2 \in V, \quad (31)$$
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and

\[ t \mapsto \mathcal{L}(v, t) \in L^2(0, T; D^*) \quad \forall v \in V; \]  

then there exists a function \( u \in L^\infty(0, T; V), \) with \( Bu \in H^1(0, T; D^*), \)

\[
\begin{cases}
(Bu)(t) + \partial\phi(u(t)) \ni f(t) + \mathcal{L}(u(t), t) & \text{a.e. in } (0, T), \\
Bu(0) = Bv_0.
\end{cases}
\]  

In the case of time independent Lipschitz perturbations, we simply replace \( F^n \) by \( G^n := F^n + \mathcal{L}(U^n) \) in the discrete scheme (9) which can be recursively solved if every step size satisfies \( L\varepsilon_n < 1 \). We refer to the last section for the related error estimates.

3. EXAMPLES

Let \( \Omega \) be a Lipschitz bounded connected open set of \( \mathbb{R}^N \), with outward unit normal \( v \). We set \( Q := \Omega \times (0, T) \) and \( \Sigma := \partial\Omega \times (0, T). \)

1. Let \( m \in L^\infty(\Omega), \) \( m(x) \geq 0 \). We choose \( V = V := L^2(\Omega), \) and define a bilinear nonnegative symmetric continuous form as follows

\[
b(u, v) := \int_\Omega m(x)u(x)v(x) \, dx, \quad Bu := m(x)u(x)\]  

In this case \( D^* = L^2(\Omega; m^{-1} \, dx) = \{ f : f / \sqrt{m} \in L^2(\Omega) \} \). Finally we set \( \Omega_0 := \{ x \in \Omega : m(x) > 0 \} \).

Non homogeneous diffusion equation. If

\[
\phi(w) := \frac{1}{2} \int_\Omega |\nabla w(x)|^2 \, dx, \quad D(\phi) := H^1_0(\Omega),
\]  

the subdifferential of \( \phi \) is given by

\[
\partial\phi(w) = -\Delta w, \quad D(\partial\phi) = \{ w \in D(\phi) : \Delta w \in L^2(\Omega) \},
\]  

and differential inclusion (9) becomes

\[
\begin{cases}
m(x)\partial_\tau u(x, t) - \Delta u(x, t) = f(t, x) & \text{a.e. on } Q, \\
\partial_\tau u(x, t) = 0 & \text{on } \Sigma, \\
u(x, t) = u_0(x) & \text{in } \Omega_0.
\end{cases}
\]  

Otherwise it is possible to choose \( D(\phi) = H^1(\Omega) \) obtaining

\[
\begin{cases}
m(x)\partial_\tau u(x, t) - \Delta u(x, t) = f(x, t) & \text{a.e. on } Q, \\
\partial_\tau u(x, t) = 0 & \text{on } \Sigma, \\
u(x, 0) = u_0(x) & \text{on } \Omega_0.
\end{cases}
\]
In this case it is possible to prove that (Lemma 2.8 Nochetto et al. (2000))

$$\sigma(w, v) \geq \frac{1}{2} \int_{\Omega} |\nabla w - \nabla v|^2 \, dx.$$  \hspace{1cm} (39)

**Bilaplacian.** If $$\Omega$$ is $$C^{1,1}$$, and

$$\phi(w) := \frac{1}{2} \int_{\Omega} |\Delta w(x)|^2 \, dx, \quad D(\phi) := H_0^2(\Omega),$$  \hspace{1cm} (40)

then the subdifferential of $$\phi$$ is the bilaplacian

$$\partial \phi(w) := \Delta^2 w, \quad D(\partial \phi) := \{w \in D(\phi) : \Delta^2 w \in L^2(\Omega)\}.$$  \hspace{1cm} (41)

The differential inclusion (9) becomes

$$\begin{cases}
m(x)\partial_t u(x, t) - \Delta^2 u(x, t) = f(t, x) & \text{a.e. on } Q, \\
\partial_t u(x, t) = u(x, t) = 0 & \text{on } \Sigma, \\
u(x, 0) = u_0(x) & \text{on } \Omega_0.
\end{cases}$$  \hspace{1cm} (42)

In this case it is possible to prove that (Lemma 2.8, Nochetto et al. (2000))

$$\sigma(w, v) \geq \frac{1}{2} \int_{\Omega} |\Delta w - \Delta v|^2 \, dx.$$  \hspace{1cm} (43)

A non linear example is given by the p-Laplacian.

### 2. p-Laplacian

For $$p > 1$$ set

$$V := \begin{cases}
L^2(\Omega, m(x) \, dx) \cap L^{p'}(\Omega) & \text{if } 1 < p < 2, \ N \geq 2, \\
L^2(\Omega) & \text{otherwise},
\end{cases}$$  \hspace{1cm} (44)

where $$1/p^* = 1/p - 1/N$$. Let $$b(\cdot, \cdot)$$ be defined as before. If we choose

$$\phi(w) := \frac{1}{p} \int_{\Omega} |\nabla w(x)|^p \, dx, \quad D(\phi) := W^{1,p}_0(\Omega),$$  \hspace{1cm} (45)

we have

$$\partial \phi(w) := -\text{div}(\nabla w |^{p-2} \nabla w),$$

$$D(\partial \phi) := \{v \in D(\phi) : \text{div}(|\nabla v|^{p-2} \nabla v) \in L^2(\Omega; m^{-1} \, dx) + L^{(p')'}(\Omega)\},$$  \hspace{1cm} (46)

and the associated equation is

$$\begin{cases}
m(x)\partial_t u(x, t) - \text{div}(|\nabla u(x, t)|^{p-2} \nabla u(x, t)) = f(t, x) & \text{a.e. on } Q, \\
u(x, t) = 0 & \text{on } \Sigma, \\
u(x, 0) = u_0(x) & \text{on } \Omega_0.
\end{cases}$$  \hspace{1cm} (47)
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If \( p \geq 2 \) it is possible to prove (Lemma 2.11, Nochetto et al. (2000)) the existence of a constant \( c_p \) such that
\[
\sigma(w, v) \geq c_p \int_\Omega |\nabla w - \nabla v|^p \, dx
\]
(48)

3. Choose \( V := L^1(\Omega) \cap H^{-1}(\Omega) \) with \( V' := L^\infty(\Omega) + H_0^1(\Omega) \). Let \( -B : H^{-1}(\Omega) \to H_0^1(\Omega) \) the inverse of the Laplace operator subject to a homogeneous Dirichlet boundary condition, i.e.,
\[
Bu \in H_0^1(\Omega), \quad \int_\Omega \nabla Bu \nabla w \, dx = \langle u, w \rangle_{H_0^1} \quad \forall w \in H_0^1(\Omega).
\]

Finally let \( \beta : \mathbb{R} \to \mathbb{R} \) a surjective monotone function, such that \( \beta(0) = 0 \). Consider the convex primitive of \( \beta \)
\[
j(r) := \int_0^r \beta(s) \, ds \quad \forall r \in \mathbb{R}.
\]

We can introduce a l.s.c. convex functional over \( V \) as follows
\[
\phi(w) := \int \int_0 j(w(x)) \, dx, \quad D(\phi) := \{ w \in V : j(w) \in L^1(\Omega) \}.
\]
(49)

The subdifferential of \( \phi \) is given by
\[
\partial \phi(w) := \beta(w), \quad D(\partial \phi) := \{ w : \beta(w) \in H_0^1(\Omega) + L^\infty(\Omega) \}.
\]
(50)

The surjectivity of \( \beta \) yields \( \lim_{|r| \to +\infty} j(r)/r = +\infty \) and then we have that \( \phi \) is weakly coercive in \( L^1(\Omega) \) (see for instance Thm. 1.3 Sect. 1.7 Chap. VIII Ekeland and Temam (1974)), moreover \( B : V \to V' \) is continuous. So \( u \mapsto \phi(u) + \langle Bu, u \rangle \) is weakly coercive in \( V \). Differential inclusion (9) becomes the following equation
\[
\begin{aligned}
\partial_t Bu(x, t) + \beta(u(x, t)) &= f(x, t) \quad \text{a.e. in } Q, \\
\beta(u(x, t)) &= 0 \quad \text{on } \Sigma, \\
u(x, 0) &= u_0(x).
\end{aligned}
\]
(51)

That could be also read as
\[
\partial_t u - \Delta \beta(u) = g, \quad g = -\Delta f,
\]
(52)
in the distribution sense. In this case it is possible to prove (Lemma 2.11 Nochetto et al. (2000)) that
\[
\sigma(w, v) \geq \frac{1}{2} \| \beta(w) - \beta(v) \|^2_{L^2(\Omega)}.
\]
(53)

Now we give two typical examples of \( \beta \).

**Two-phase Stefan problem.** A simple choice corresponds to
\[
\beta(r) := (r - 1)^+ - r^-.
\]
(54)
Porous medium equation. In this case the typical choice is

$$
\beta(r) := \frac{|r|^{p-2}r}{p-2}.
$$

(55)

4. An application to a reaction-diffusion system. The study of reaction-diffusion systems of FitzHugh–Nagumo type related to the behavior of the electrical conduction of cardiac muscle at micro- and macroscopic level involves an interesting example of degenerate evolution equation, (see Colli Franzone and Savaré (2000)). Here we present only a simplified version (i.e., with homogeneous Dirichlet boundary condition and without the recovery variable) of the macroscopic equation, for more details and the physiological interpretation see (Colli Franzone et al., 1990; Colli Franzone and Savaré, 1996; Colli Franzone and Guerri, 1990; Keener and Sneyd, 1998).

Let $\Omega$ a Lipschitz, bounded, open set of $\mathbb{R}^N$ $0 < T < +\infty$, $Q := \Omega \times (0, T)$ and $\Sigma := \partial\Omega \times (0, T)$.

We assign two $N \times N$ symmetric matrices $M_i(x), M_e(x)$, measurably depending on $x \in \Omega$, and satisfying the usual ellipticity condition

$$
\exists \alpha, m > 0 : \quad \alpha|\xi|^2 \leq \xi^T M_i(x)\xi \leq m|\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \quad \forall x \in \Omega.
$$

(56)

We assign also a continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $F(0) = 0$ and

$$
\exists \lambda_F > 0 : \quad \frac{F(x) - F(y)}{x - y} \geq -\lambda_F \quad \forall x, y \in \mathbb{R}, x \neq y.
$$

(57)

Given $v_0(x) \in H^1(Q)$, $I_i, I_e \in L^2(Q)$ we seek a couple $(u_i, u_e) : [0, T] \rightarrow H_0^1(\Omega) \times H_0^1(\Omega)$ which solve

$$
\begin{cases}
\partial_t(u_i - u_e) + F(u_i - u_e) = \text{div}(M_i \nabla u_i) + I_i & \text{in } Q, \\
\partial_t(u_e - u_i) + F(u_i - u_e) = -\text{div}(M_e \nabla u_e) - I_e & \text{in } Q, \\
u_i(x, 0) - u_e(x, 0) = v_0(x) & \text{in } \Omega.
\end{cases}
$$

(58)

In view of giving an abstract formulation of Eq. (58) let $V := H_0^1(\Omega) \times H_0^1(\Omega)$, $u := (u_i, u_e) \in V$. If we set

$$
\langle A u, \tilde{u} \rangle := \sum_{i,e} \int_{\Omega} \nabla u_i \cdot M_{i,e} \nabla \tilde{u}_e \, dx,
$$

$$
\langle B u, \tilde{u} \rangle := \int_{\Omega} (u_i - u_e)(\tilde{u}_i - \tilde{u}_e) \, dx,
$$

$$
\langle f(t), \tilde{u} \rangle := \sum_{i,e} \int_{\Omega} I_{i,e} \tilde{u}_e \, dx,
$$

$$
\langle M u, \tilde{u} \rangle := \langle A u, \tilde{u} \rangle + \int_{\Omega} F(u_i - u_e)(\tilde{u}_i - \tilde{u}_e) \, dx,
$$

$$
D(M) := \{ u \in V : F(u_i - u_e) \in L^1(\Omega) \cap H^{-1}(\Omega) \}.
$$

(59)
the problem becomes to find \( u : [0, T] \rightarrow V \) such that

\[
\begin{align*}
(Bu)' + M(u) &= f, \quad \text{a.e. on} \quad (0, T), \\
Bu(0) &= Bu_0,
\end{align*}
\]

where \( u_0 := (u_{0,i}, u_{0,e}) \) with \( u_{0,i} - u_{0,e} = v_0 \). The non linear functional \( M \) is a Lipschitz perturbation of the subdifferential of a convex, l.s.c. function, more exactly

\[
M = \partial \phi - \lambda F.
\]

To see this, set

\[
F_\lambda(v) := F(v) + \lambda F v, \quad j(v) := \int_0^v F_\lambda(s) ds = \frac{\lambda}{2} v^2 + \int_0^v F(s) ds.
\]

Thanks to Eq. (57) \( F_\lambda \) is a monotone continuous function, then, if we set

\[
\psi(u) := \begin{cases} 
\int_\Omega j(u_i - u_e) dx & \text{if} \quad j(u_i - u_e) \in L^1(\Omega), \\
+\infty & \text{otherwise},
\end{cases}
\]

we have (see Colli Franza and Savaré (2000))

\[
1 \in \partial \psi(u) \iff \begin{cases} 
u \in D(M) \\
I_{\nu} = \int_\Omega F(u_i - u_e) (u_i - u_e) dx + \lambda F(Bu, \nu) \quad \forall \nu \in V.
\end{cases}
\]

Since \( A \) is continuous if we set \( \phi(u) = 1/2 \langle Au, u \rangle + \psi(u) \), we get Eq. (61). The weak coercivity of \( b(\cdot) + \phi(\cdot) \), follows from Eq. (56), since

\[
\sum_{i,e} \int_\Omega \nabla u_{i,e}^T M_{i,e} \nabla u_{i,e} dx \geq \alpha \sum_{i,e} \int_\Omega \nabla u_{i,e}^2.
\]

We observe that the hypothesis \( f \in L^2(0, T; D^*) \) is fulfilled if \( I_e = -I_i \). Using Lemma 2.11 (Nochetto et al. (2000)), it can be proved that

\[
\sigma(w, v) \geq \frac{\alpha}{2} \sum_{i,e} \int_\Omega |\nabla w_{i,e} - \nabla v_{i,e}|^2 dx.
\]

### 4. PRELIMINARY RESULTS

In view of proving Theorems 2.1, 2.2 and 2.3 we need some preliminary results.

**Lemma 4.1.** For every \( u, v \in V \)

\[
b(u, v) \leq \sqrt{\hat{b}(u)} \sqrt{\hat{b}(v)} \leq \frac{1}{2\varepsilon} b(u) + \frac{\varepsilon}{2} b(v) \quad \forall \varepsilon > 0;
\]

\[
\|Bu\|^2 \leq Mb(u);
\]
Proof. For every \( \lambda \in \mathbb{R} \) we have \( 0 \leq b(u - \lambda v) = b(u) + \lambda^2 b(v) - 2\lambda b(u, v) \), and then, since the discriminant should be negative, \( b^2(u, v) - b(u)b(v) \leq 0 \). Inequality (68) follows from \( \|Bu\|_v = \sup_{v \in V} \langle \langle Bu, v \rangle \rangle / \|v\| \leq \sup_{v \in V} \sqrt{b(u)/\sqrt{b(v)}}/\|v\| \), and \( \sup_{v \in V} \sqrt{b(v)} / \|v\| \leq \sqrt{M} \). \( \blacksquare \)

Definition 4.2. We define

\[
\frac{1}{2} b^*(f) := \sup_{v \in V} \left\{ \langle f, v \rangle - \frac{1}{2} b(v) \right\},
\]

(69)

for every \( f \in V' \).

We observe that \( b^*/2 \) is the conjugate function of \( b/2 \) in the sense of convex analysis.

Proposition 4.3. For every \( f \in V' \) such that \( b^*(f) < +\infty \) and every \( v \in V \)

\[
\langle f, v \rangle \leq \sqrt{b^*(f)} \sqrt{b(v)} \leq \frac{1}{2\lambda^2} b^*(f) + \frac{\lambda^2}{2} b(v) \quad \forall \lambda;
\]

(70)

\[
\sqrt{b^*(f)} = \|f\|_{b^*}.
\]

(71)

Proof. A direct consequence of definition is that \( (1/2\lambda^2)b^*(f) + (\lambda^2/2)b(v) \geq \langle f, v \rangle \) for each \( \lambda \in \mathbb{R} \). If \( b(v) \neq 0 \), choosing \( \lambda^2 = (\sqrt{b^*(f)})/(\sqrt{b(v)}) \) we get Eq. (70), otherwise we note that \( \langle f, v \rangle = 0 \). As for it concernes Eqs. (71), (70) yields \( \|f\|_{b^*} \leq \sqrt{b^*(f)} \). On the other hand for every \( \epsilon > 0 \) there exists \( u_\epsilon \in V \) such that

\[
\frac{1}{2} b^*(f) \leq \langle f, u_\epsilon \rangle - \frac{1}{2} b(u_\epsilon) + \epsilon,
\]

(72)

that implies

\[
\sqrt{b^*(f)} \sqrt{b(u_\epsilon)} \leq \langle f, u_\epsilon \rangle + \epsilon.
\]

(73)

If \( \liminf_{\epsilon \rightarrow 0} b(u_\epsilon) = c > 0 \), Eq. (73) yields

\[
\sqrt{b^*(f)} \leq \langle f, \frac{u_\epsilon}{\sqrt{b(u_\epsilon)}} \rangle + \frac{\epsilon}{c},
\]

(74)

and then \( \|f\|_{b^*} \geq \sqrt{b^*(f)} \). Otherwise if \( \liminf_{\epsilon \rightarrow 0} b(u_\epsilon) = 0 \), we deduce from Eqs. (72) and (70) that \( b^*(f) = 0 \). \( \blacksquare \)

Proposition 4.4. \( b^*(\cdot) \) and \( \|\cdot\|_{b^*} \) are l.s.c. and convex \( \|\cdot\|_{b^*} \) is a norm over \( D(b^*) \) and moreover:

\[
b^*(Bv) = b(v) \quad \forall v \in V, \]

(75)

\[
\|f\|_{b^*} \leq \sqrt{M} \|f\|_{b^*} \quad \forall f \in D(b^*),
\]

(76)

\[
2b^*(f) + 2b^*(g) = b^*(f + g) + b^*(f - g) \quad \forall f, g \in D(b^*). \]

(77)
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Proof. We prove only the identity (77). Define \( A : V \times V \to \mathbb{R} \) as \( A(u, v) := 1/2b(u) + 1/2b(v) \). On one hand we have that the conjugate of \( A \) is \( A^*(f, g) = 1/2b^*(f) + 1/2b^*(g) \), on the other, since \( A(u, v) = b((u + v)/2) + b((u - v)/2) \), we have

\[
A^*(f, g) = \sup_{(u, v) \in V \times V} \left\{ \langle f, u \rangle + \langle g, v \rangle - b\left(\frac{u + v}{2}\right) - b\left(\frac{u - v}{2}\right) \right\}
\]

\[
= \sup_{(u, v) \in V \times V} \left\{ \langle f + g, \frac{u + v}{2} \rangle + \langle f - g, \frac{u - v}{2} \rangle - b\left(\frac{u + v}{2}\right) - b\left(\frac{u - v}{2}\right) \right\}
\]

\[
= \sup_{(u, v) \in V \times V} \left\{ \langle f + g, s \rangle + \langle f - g, t \rangle - b(s) - b(t) \right\}
\]

\[
= b^*\left(\frac{f + g}{2}\right) + b^*\left(\frac{f - g}{2}\right).
\]

(78)

As previously said \( \| \cdot \|_{b^*} \) is a norm over \( D(b^*) \), in particular from Eq. (77) we deduce that \( \| \cdot \|_{b^*} \) is an Hilbert norm and a scalar product is defined by

\[
b^*(u, v) = \frac{1}{4} b^*(u + v) - \frac{1}{4} b^*(u - v).
\]

(79)

We note that

\[
b^*(Bu, Bv) = b(u, v) = \langle Bu, v \rangle \quad \forall u, v \in V.
\]

(80)

It is easy to see that \( (D(b^*), \| \cdot \|_{b^*}) \) is complete, that is

**Proposition 4.5.** The space \( D(b^*) \) endowed with \( b^*(\cdot, \cdot) \) is an Hilbert space, which we denote by \( D^* \).

**Proof.** Let \( \{f_n\} \) be a Cauchy sequence in \( (D(b^*), \| \cdot \|_{b^*}) \), then from Eq. (76) we deduce that it is a Cauchy sequence also in \( (V', \| \cdot \|_\alpha) \). Since \( (V', \| \cdot \|_\alpha) \) is complete a limit \( f \in V' \) exists. From the Cauchy condition \( \forall \varepsilon > 0 \exists \tilde{m} : \| f_n - f_m \|_{b^*} \leq \varepsilon \forall n, m \geq \tilde{m} \). Since \( \| \cdot \|_\alpha \) is l.s.c. we get

\[
\| f_n - f \|_{b^*} \leq \liminf_{m \to \infty} \| f_n - f_m \|_{b^*} \leq \varepsilon, \quad \forall n \geq \tilde{m}.
\]

(81)

from which we deduce that \( f \in D(b^*) \) and that \( f_n \to f \) in \( D^* \).

Set \( D^*_0 := \overline{B(V)}^{D^*} \). It is an Hilbert space endowed with \( b^*(\cdot, \cdot) \).

**Lemma 4.6.** For every \( f \in D^* \), there exists \( f^* \in D^*_0 \) such that

\[
\langle f, v \rangle = b^*(f^*, Bv) \quad \forall v \in V.
\]

(82)

**Proof.** We can define the linear functional on \( B(V) \) \( \langle L_f, w \rangle := \langle f, v \rangle \) \( \forall v \in B^{-1}w \). Observe that \( L_f \) is well defined since \( Bv_1 = Bv_2 = w \Rightarrow \langle f, v_1 \rangle = \langle f, v_2 \rangle \), and continuous with respect to the \( D^* \) norm, since \( |\langle L_f, w \rangle| = |\langle f, v \rangle| \leq \| f \|_{b^*} \| Bv \|_{b^*} \leq \| f \|_{b^*} \| w \|_{b^*} \). Therefore it can be extended to a continuous functional on \( D^*_0 \). \( Bu \mapsto \langle f, u \rangle \) defines a linear operator over \( D^*_0 \). By the Riesz theorem there exists
then a function \( u \in L^1(0, T) \) and a subsequence \( \{n_k\} \) exist such that for every \( g \in L^1(0, T; V) \)

\[
\lim_{k \to \infty} \int_0^T \langle g(t), u_{n_k}(t) \rangle \, dt = \int_0^T \langle g(t), u(t) \rangle \, dt.
\]  

**Proof.** Let \( \Xi^* \) be a numerable dense subset of \( L^1(0, T) \) and let \( \Xi \) be the vector space over \( \mathbb{Q} \) generated by \( \Xi^* \). Finally let \( X := \Xi \cap B(0, 1) \) where \( B(0, 1) \) is the unitary ball of \( L^1(0, T) \). Consider the sequence of applications

\[
\xi \mapsto T_n(\xi) := \int_0^T \xi u_n \, dt,
\]

from \( \Xi \) into \( V \). Since \( T_n(\xi) := \int_0^T u_n d\mu^+ - \int_0^T u_n d\mu^- \), where \( d\mu^+ = \xi^+ \, dt \) and \( d\mu^- = \xi^- \, dt \) are measures on \([0, T]\) with \( \mu^+(\{0, T\}) + \mu^-([0, T]) \leq 1 \), then for each \( \xi \in X \)

\[
T_n(\xi) \in \overline{\sigma(\{u_n(0, T)\}) - \overline{\sigma(\{u_n([0, T]\}) \subset (K - K),}
\]

see for instance Coroll. 8 Chap. II Diestel and Uhl (1977). Since \( K - K \) is weakly compact and closed, and then weakly sequentially compact, by an usual diagonal argument there exist a subsequence \( n_j \) such that for every \( \xi \in X \)

\[
\exists \lim_{j \to +\infty} T_{n_j}(\xi) := T(\xi), \quad \text{in the weak topology of } V.
\]  

(85)

It is easy to see by an homogeneity argument that Eq. (85) holds also for every \( \xi \in \Xi \) and that \( T \) is linear. We note that \( T \) is continuous, since for each \( \xi \in X \)

\[
\|T(\xi)\| = \sup_{\|y\|_V = 1} |\langle y, T(\xi) \rangle| \leq \liminf_{n \to \infty} \|T_n(\xi)\| \leq k \int_0^T |\xi| \, dt,
\]

(86)

where \( k := \sup_{n} \|[u_n]_{L^\infty(0, T; V)} \| < +\infty \), since \( K \) is bounded in \( V \). By continuity we extend \( T \) to a linear operator \( T : L^1(0, T) \to V \). Finally we note that \( T \) is weakly compact, in fact

\[
\overline{T(B(0, 1))}^w \subset K - K^w,
\]  

(87)

and \( K - K^w \) is weakly compact.

\[ f^* \in D^*_e \text{ such that } \langle Lf, w \rangle = b^*(f^*, w) \, \forall w \in D_0^*, \text{ or equivalently } b^*(f^*, Bv) = \langle f, v \rangle \, \forall v \in V. \]
Under these conditions a function \( u \in L^\infty(0, T; V) \) exists such that
\[
T(\xi) = \int_0^T \xi u \, dt,
\]
(88)
for all \( \xi \in L^1(0, T) \), see for instance Thm. 12 Sect. 2 Chap. III Diestel and Uhl (1977).

In this way we have proved that for each \( u \in L^1(0, T; V) \)
\[
\int_0^T \xi u_n \, dt \to \int_0^T \xi u \, dt.
\]
(89)

It is easy to prove Eq. (84) by using Eq. (89) and approximating any function in \( L^1(0, T; V') \) by simple functions.

5. BASIC ESTIMATES

In this section we deduce some preliminary estimates which will be useful both in proving Theorem 2.1 and in giving the rate of convergence of \( U_t \) to \( u \).

Energy estimate. Now we prove a first estimate.

**Proposition 5.1.** Let \( U_t \) be defined as in Eq. (14), with \( F^n_t := (1/\tau_n) \int_{t_{n-1}}^t f(s) \, ds \). If we set
\[
C_T := 2\phi(U^n_0) + \int_0^T \| f(t) \|^2_{\mathcal{B}} \, dt, \quad K_T := \frac{1}{2} C_T + \left( \sqrt{T C_T} + \sqrt{b(U^n_0)} \right)^2,
\]
(90)
the following inequalities hold
\[
\int_0^T b(U'_t(s)) \, ds \leq C_T,
\]
(91)
\[
\phi(U'_t(t)) + b(U'_t(t)) \leq K_T, \forall t \in [0, T].
\]
(92)

**Proof.** By hypothesis we have that
\[
\langle B U^n_t, U^n_t - v \rangle + \phi(U^n_t) - \langle F^n_t, U^n_t - v \rangle \leq \phi(v) \quad \forall v \in D(\phi).
\]
(93)
Choosing \( v = U^{n-1}_t \) in Eq. (93) we get
\[
\frac{1}{\tau_n} b(U^n_t - U^{n-1}_t) \leq \phi(U^{n-1}_t) - \phi(U^n_t) + \langle F^n_t, U^n_t - U^{n-1}_t \rangle.
\]
(94)

Due to Eq. (70), we get
\[
\tau_n b(\delta U^n_t) \leq \phi(U^{n-1}_t) - \phi(U^n_t) + \frac{1}{2} \tau_n b^*(F^n_t) + \frac{1}{2} \tau_n b(\delta U^n_t).
\]
(95)
Summing the last relation for \( n = 1, \ldots, N \) we have
\[
\sum_{n=1}^{N} \tau_n b(\delta U^n_t) \leq 2\phi(U^0_t) + \sum_{n=1}^{N} \tau_n \|F^n_t\|_{\beta^*}^2.
\] (96)

Then, since \( \sum_{n=1}^{N} \tau_n \|F^n_t\|_{\beta^*}^2 \leq \int_0^T \|f(t)\|_{\beta^*}^2 \, dt \), from Eq. (96) we obtain Eq. (91).

From Eq. (95) we also have
\[
\phi(U^n_t) \leq \phi(U^0_t) + \frac{1}{2} \sum_{k=1}^{N} \|F^n_k\|_{\beta^*}^2 \tau_k \leq \phi(U^0_t) + \frac{1}{2} \int_0^T \|f(t)\|_{\beta^*}^2 \, dt,
\] (97)

for \( 1 \leq n \leq N \). Due to convexity of \( \phi \) it is easy to check that
\[
\phi(U_t(t)) \leq \frac{1}{2} C_T \quad \forall t \in [0, T].
\] (98)

From Eq. (91) and Jensen inequality it follows that
\[
b(U_t(t) - U_t(s)) \leq (t - s)C_T \quad 0 \leq s \leq t \leq T,
\] (99)

and then, choosing \( s = 0 \),
\[
b(U_t(t)) \leq \left(\sqrt{T C_T} + \sqrt{b(U^0_t)}\right)^2 \quad \forall t \in [0, T].
\] (100)

Assertion (92) comes from Eqs. (98) and (100).

**Dissipation inequality.** Using the function \( \sigma \) defined in Eq. (22), it is possible to extract more information from Eq. (9) than merely
\[
\langle B\delta U^n_t - F^n_t, U^n_t - v \rangle \leq \phi(v) - \phi(U^n_t) \quad \forall v \in D(\phi).
\] (101)

From Eq. (9), using Eq. (23), we get
\[
\langle B\delta U^n_t - F^n_t, U^n_t - v \rangle + \phi(U^n_t) - \phi(v) + \sigma(U^n_t, v) \leq 0 \quad \forall v \in D(\phi),
\] (102)

for \( 0 \leq n \leq N \), that is stronger than Eq. (101).

Choosing \( v = U^{n-1}_t \) in Eq. (102) we obtain:
\[
\langle B\delta U^n_t - F^n_t, \delta U^n_t \rangle \tau_n + \delta \phi(U^n_t) \tau_n + \sigma(U^n_t, U^{n-1}_t) \leq 0,
\] (103)

for \( 1 \leq n \leq N \).

Due to Eq. (103) we have immediately
\[
0 \leq \sigma(U^n_t, U^{n-1}_t) \leq \tau_n \mathcal{E}_n.
\] (104)

Now we prove an important estimate. It measures the error in evaluating the solution \( u \) of Eq. (16) through its discrete version \( \bar{U}_t \).
Proposition 5.2. If \( \{U^n_v\}_{n=0}^N \) is a discrete solution of Eq. (9) then, for each \( v \in D(\phi) \),
\[
\langle BU^n_v(t) - \mathcal{F}_v(t), U^n_v(t) - v \rangle + \phi(U^n_v(t)) - \phi(v) + \sigma(U^n_v(t), v) \\
\leq l_v(t)\tilde{\tau}(t)\tilde{E}_v(t) \quad \forall t \in (0, T).
\] (105)

Proof. If we set
\[
\mathcal{R}_v(t) := \langle BU^n_v(t) - \mathcal{F}_v(t), U^n_v(t) - U_v(t) \rangle + \phi(U^n_v(t)) - \phi(U_v(t)),
\] (106)
then it is easy to check that, for each \( t \in (t_{n-1}, t_n] \),
\[
\mathcal{R}_v(t) \leq -l_v(t)\tau_v(\delta BU^n_v - F^n_v, \delta U^n_v) + \delta\phi(U^n_v)),
\] (107)
and also that
\[
\langle BU^n_v(t) - \mathcal{F}_v(t), U_v(t) - v \rangle + \phi(U_v(t)) - \phi(v) + \sigma(U_v(t), v) \leq \mathcal{R}_v(t).
\] (108)

Using Eq. (105) it is easy to obtain the following

Proposition 5.3. If \( \{U^n_v\}_{n=0}^N \) is a discrete solution of Eq. (9) then for every \( 0 \leq s \leq t \leq T \)
\[
\left| \int_s^t \langle BU^n_v(z) - \mathcal{F}_v(z), U^n_v(z) \rangle \, dz + \phi(U^n_v(t)) - \phi(U^n_v(s)) \right| \leq \int_s^t \tilde{E}_v(z) \, dz.
\] (109)

Proof. Let \( t_{k-1} < s \leq t < t_k \). Using Eq. (105) with \( v = U^n_v(s) \), since \( U^n_v(t) - U^n_v(s) = \int_s^t U^n_v(z) \, dz \) and \( BU^n_v(z) - \mathcal{F}_v(z) \) is constant over \( (t_{k-1}, t_k] \), we get
\[
\int_s^t \langle BU^n_v(z) - \mathcal{F}_v(z), U^n_v(z) \rangle \, dz + \phi(U^n_v(t)) - \phi(U^n_v(s)) \leq l_v(t)\tilde{\tau}(t)\tilde{E}_v(t).
\] (110)

Arguing in the same way, using \( U^n_v(s) \) instead of \( U^n_v(t) \) and \( v = U^n_v(t) \) in Eq. (105) we get
\[
\int_s^t \langle BU^n_v(z) - \mathcal{F}_v(z), U^n_v(z) \rangle \, dz + \phi(U^n_v(t)) - \phi(U^n_v(s)) \geq -l_v(s)\tilde{\tau}(s)\tilde{E}_v(s).
\] (111)

Finally it is easy to check that
\[
\int_{t_{k-1}}^t \langle BU^n_v(z) - \mathcal{F}_v(z), U^n_v(z) \rangle \, dz + \phi(U^n_v(t)) - \phi(U^n_v(t_{k-1})) \geq -\tau_k c^k.
\] (112)

Summing these relations for a generic couple \( (s, t) \) the conclusion follows.

Now we introduce a new partition of \([0, T]\), that is
\[
Q = \{0 = s_0 < \cdots < s_M = T\}.
\] (113)
Like we did for $\mathcal{P}$ we set $\eta_k = s_k - s_{k-1}$, $\eta = \{\eta_1, \ldots, \eta_M\}$, $|\eta| = \max_{1 \leq k \leq M} \eta_k$. Chosen suitable approximation $V^0_t$ of initial datum $u_0$ and $F_{\eta}^k$ of $f$ over $(s_{k-1}, s_k)$, we denote by $\{V_{\eta}^k\}_{k=0}^M$ the discrete solution of

$$B\dot{V}^k + \partial \phi(V^k_{\eta}) \equiv F^k_{\eta} \quad 1 \leq k \leq M. \quad (114)$$

Furthermore we define $V_{\eta}^t$, $\overline{\eta}$, $l_{\eta}$, and $\{E_{\eta}^k\}_{k=1}^M$ like we did in Eqs. (13) and (24), using $\eta_k$ and $\{V_{\eta}^M\}_{k=0}^M$ instead of $\tau_n$ and $\{U^n_t\}_{n=0}^N$. We observe that for each $v \in D(\phi)$

$$\langle BV_t - \overline{\eta}, V_t - v \rangle + \phi(V_t) - \phi(v) + \sigma(\overline{\eta}, v) \leq l_{\eta} \overline{E}.$$  

**Definition 5.4.** To every couple of discrete solutions $\{U_t^n\}$ and $\{V_{\eta}^k\}$ we associate the error

$$E_{t, \eta} := \max \left\{ \max_{0 \leq \tau \leq T} \sqrt{\langle b(U_t - V_{\eta}) \rangle}, \left(2 \int_0^T [\sigma(U_t, V_{\eta}) + \sigma(\overline{\eta}, U_t)] \, dt \right)^{1/2} \right\}. \quad (116)$$

**Proposition 5.5.** The error defined in Eq. (116) is bounded by

$$E_{t, \eta} \leq \left\{ b(U_t^0 - V_{\eta}^0) + \sum_{k=1}^M \eta_k^2 E_{\eta}^k + \sum_{n=1}^N \tau_n^2 E_{\eta}^n \right\}^{1/2} \quad + \int_0^T \| \overline{F}_t(t) - \overline{\eta}_t(t) \|_{\mathcal{B}'}, \, dt. \quad (117)$$

**Proof.** Summing Eq. (105) with $v = V_{\eta}$ and Eq. (115) with $v = U_t$ it follows that

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \langle b(U_t - V_{\eta}) \rangle + \sigma(U_t, V_{\eta}) + \sigma(\overline{\eta}, U_t) \\
\leq l_{\tau} \overline{E} + l_{\eta} \overline{E} + \| \overline{F}_t - \overline{\eta}_t \|_{\mathcal{B}'}, \sqrt{b(U_t - V_{\eta})}.
\end{align*} \quad (118)$$

Since $\int_0^T 2l_{\eta}(t) \overline{E}_t(t) \, dt = \sum_{n=1}^N \tau_n^2 E_{\eta}^n$ and $\int_0^T 2l_{\eta}(t) \overline{E}_t(t) \, dt = \sum_{k=1}^M \eta_k^2 E_{\eta}^k$, using a refined version of Gronwall’s lemma (see for instance Lemma 3.6 Nochetto et al. (2000)) we get Eq. (117). \hfill \blacksquare

### 6. PROOF OF THEOREM 2.1

We integrate Eq. (105) between $t$ and $t + h$ and obtain

$$\begin{align*}
\frac{1}{h} \int_t^{t+h} (BU_t(s), U_t(s) - v) \, ds + \frac{1}{h} \int_t^{t+h} \phi(U_t(s)) \, ds - \phi(v) \\
+ \frac{1}{h} \int_t^{t+h} (\overline{F}_t(s), U_t(s) - v) \, ds \leq \frac{|\tau|}{h} \int_t^{t+h} \overline{E}_t.
\end{align*} \quad (119)$$

for all $v \in D(\phi)$. 
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The aim of the proof is passing to the limit in $|\tau| \to 0$ and then in $h \to 0$. To do that we need some preliminary results.

**Proposition 6.1.** Under the assumptions of Theorem 2.1, $BU_\tau$ is a Cauchy sequence in $C^0([0, T]; D^*)$ and the following estimate holds:

$$E_{\tau, \eta} \leq \left((|\tau| + |\eta|)\right)^{1/2} \left(2\phi(u_0) + \frac{1}{2} \int_0^T \|f(t)\|_{D^*}^2 \, dt\right)^{1/2}$$

$$+ \|F_\tau - F_\eta\|_{L^1(0, T; D^*)},$$

for every couple $\tau$ and $\eta$, with $|\tau| \leq 1$ and $|\eta| \leq 1$. This ensures the existence of $B \in C^0([0, T]; D^*)$ such that

$$BU_\tau \to B \text{ in } C^0([0, T]; D^*).$$

**Proof.** Since $U_0^0 = V_0^0 = u$ is supposed, we get

$$\sum_{n=1}^N E_{\tau}^{\alpha} = \sum_{n=1}^N \left(\langle F_\tau^{\alpha}, \delta U_\tau^{\alpha} \rangle - \langle F_\tau^{\alpha}, \delta U^{\alpha}_0 \rangle + \phi(u_0) - \phi(U^{\alpha}_\tau)\right)$$

$$\leq \phi(u_0) + \int_0^T \langle F_\tau(t), U_\tau(t) \rangle \, dt - \int_0^T b(U_\tau(t)) \, dt,$$

and then

$$\sum_{n=1}^N E_{\tau}^{\alpha} \leq \phi(u_0) + \frac{1}{4} \int_0^T \|F_\tau(t)\|_{D^*}^2 \, dt \leq \phi(u_0) + \frac{1}{4} \int_0^T \|f(t)\|_{D^*}^2 \, dt.$$ (122)

In the same way we get

$$\sum_{k=1}^M E_{\eta}^{\beta} \leq \phi(u_0) + \frac{1}{4} \int_0^T \|F_\eta(t)\|_{D^*}^2 \, dt \leq \phi(u_0) + \frac{1}{4} \int_0^T \|f(t)\|_{D^*}^2 \, dt.$$ (123)

In view of Eq. (117), inequalities (123) and (124) yield (120). By standard approximation results we know that

$$\lim_{\tau, \eta \to 0} \int_0^T \|F_\tau(t) - F_\eta(t)\|_{D^*} \, dt = 0.$$ (125)

In this way, using also Eq. (75), we get a Cauchy estimate of $BU_\tau$ in $C^0([0, T]; D^*)$. ■

**Proposition 6.2.** Under the same hypothesis of Proposition 6.1 there exists a function $u \in L^\infty(0, T; V)$ and a sequence $\tau_i$, with $\lim_{i \to \infty} |\tau_i| = 0$, such that for every $g \in L^1(0, T; V)$

$$\lim_{i \to \infty} \int_0^T \langle g(t), U_{\tau_i}(t) \rangle \, dt = \int_0^T \langle g(t), u(t) \rangle \, dt.$$ (126)
Moreover $B(t) = Bu(t)$ almost everywhere in $[0, T]$, $Bu \in H^1(0, T; D^*)$ and

$$BU'_{\tau} \to (Bu)' \quad \text{in} \quad L^2(0, T; D^*), \quad \text{as} \quad |\tau| \to 0.$$ (127)

**Proof.** Since $\phi + b$ is weakly coercive, Eq. (92) yields $U_\tau(t) \in K$ for every $t \in [0, T]$, with $K$ weakly compact, closed and convex. Lemma 4.7 yields the existence of a sequence $U_\tau$ which fulfills Eq. (126). Furthermore, since for all $\xi \in L^1(0, T; V)$ we have $B\xi \in L^1(0, T; V')$, from Eq. (126) it follows that

$$\lim_{i \to \infty} \int_0^T \langle BU_\tau(t), \xi(t) \rangle \, dt = \int_0^T \langle Bu(t), \xi(t) \rangle \, dt.$$ (128)

We also have

$$\lim_{i \to \infty} \int_0^T \langle BU_\tau(t), \xi(t) \rangle \, dt = \int_0^T \langle B(t), \xi(t) \rangle \, dt,$$ (129)

namely

$$\int_0^T \langle B(t), \xi(t) \rangle \, dt = \int_0^T \langle Bu(t), \xi(t) \rangle \, dt.$$ (130)

This implies $B(t) = Bu(t)$ almost everywhere. Now from Eqs. (91) and (80) we get

$$\|BU'_{\tau}\|^2_{L^2(0, T; D^*)} \leq 4MC_T,$$ (131)

where $C_T = 2\phi(u_0) + \int_0^T \|f(t)\|^2_{D^*-} \, dt$. Since $D^*$ is an Hilbert space there exist a function $\beta \in L^2(0, T; D^*)$ and a sequence $u_{\tau_i}$, such that

$$BU'_{\tau_i} \to \beta = (Bu)' = B' \quad \text{in} \quad L^2(0, T; D^*) \subset L^2(0, T; V').$$ (132)

The thesis follows. ■

**Proposition 6.3.** Let $u$, $U_\tau$, and $\tau_i$ be like in Proposition 6.2. For every $v \in V$,

$$\lim_{|\tau| \to 0} \int_0^T \langle \mathcal{F}_\tau(t), U_\tau(t) - v \rangle \, dt = \int_0^T \langle f(t), u(t) - v \rangle \, dt.$$ (133)

For every $h > 0$ and every $t \in [0, T - h]$

$$\liminf_{i \to \infty} \frac{1}{h} \int_t^{t+h} \phi(U_\tau(s)) \, ds \geq \phi \left( \frac{1}{h} \int_t^{t+h} u(s) \, ds \right).$$ (134)

For every $a, b \in [0, T]$

$$\lim_{|\tau| \to 0} \int_a^b \langle BU'_{\tau}(t), U_\tau(t) - v \rangle \, dt = \int_a^b \langle (Bu)'(t), u(t) - v \rangle \, dt.$$ (135)
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Proof. We have

\[
\begin{align*}
& \left| \int_0^T (F_v(t), U(t) - v) \, dt - \int_0^T (f(t), U(t) - v) \, dt \right| \\
& \leq \| F_v - f \|_{L^p(0,T;V')} \| U - v \|_{L^q(0,T;V)} \\
& \quad + \int_0^T \| F_v(t) \|_{b_0} \| b(U(t) - u(t)) \| \, dt \\
& \leq \| F_v - f \|_{L^p(0,T;V')} \| U - v \|_{L^q(0,T;V)} \\
& \quad + \| F_v \|_{L^p(0,T;D')} \| b(U(t) - u(t)) \|_{L^q(0,T)}. \tag{136}
\end{align*}
\]

The assertion follows since \( \| F_v \|_{L^p(0,T;D')} \leq \| f \|_{L^p(0,T;D')}, \) \( \lim_{|\tau| \to 0} \| F_v - f \|_{L^p(0,T;V')} = 0, \) and also \( \lim_{|\tau| \to 0} \| b(U(t) - u(t)) \|_{L^q(0,T)} = 0. \)

Jensen inequality yields

\[
\frac{1}{h} \int_a^{a+h} \phi(U_t) \geq \phi \left( \frac{1}{h} \int_a^{a+h} U_t \right). \tag{137}
\]

Afterwards Eq. (126) leads to \( 1/h \int_a^{a+h} U_t(s) \, ds \to 1/h \int_a^{a+h} u(s) \, ds \) in \( V. \) Since \( \phi \) is a convex l.s.c. function, and then it is weakly l.s.c., Eq. (134) follows immediately.

For every \( a, b \in [0, T] \)

\[
\limsup_{|\tau| \to 0} \int_a^b \langle BU_t(t), U_t(t) - u(t) \rangle \, dt \\
\leq \limsup_{|\tau| \to 0} \int_0^T \sqrt{b(U_t(t))} \sqrt{b(U_t(t) - u(t))} = 0. \tag{138}
\]

The assertion (135) is a simple consequence of Eq. (138), since

\[
\int_a^b \langle BU_t(t), U_t(t) \rangle \, dt = \int_a^b \langle BU_t(t), U_t(t) - u(t) \rangle \, dt + \int_a^b \langle BU_t(t), u(t) \rangle \, dt. \tag{139}
\]

Proof of Theorem 2.1. Now we can proceed with the proof of Theorem 2.1.

Since \( J_t^{a+h} \leq \sum_{n=1}^N \tau_n, \) using Eq. (123), from Eq. (119) we have

\[
\frac{1}{h} \int_t^{t+h} \langle BU_t(s), U_t(s) - v \rangle \, ds + \frac{1}{h} \int_t^{t+h} \phi(U_t(s)) \, ds - \phi(v) \\
+ \frac{1}{h} \int_t^{t+h} \langle F_v(s), U_t(s) - v \rangle \, ds \leq \frac{|\tau|}{h} \left( \phi(u_0) + \frac{1}{4} \int_0^T \| f(s) \|_{b_0}^2 \, ds \right). \tag{140}
\]

Passing to the limit in \( |\tau| \to 0, \) by Eqs. (121), (133), (134), and (135), we get

\[
\frac{1}{h} \int_t^{t+h} \langle (Bu_t), u(s) - v \rangle \, ds + \phi \left( \frac{1}{h} \int_t^{t+h} u(s) \right) \, ds \leq \phi(v) + \frac{1}{h} \int_t^{t+h} \langle f(s), u(s) - v \rangle \, ds. \tag{141}
\]
Finally passing to the limit in $h$ we obtain
\[ \phi(u(t)) + ((Bu)'(t) - f(t), u(t) - v) \leq \phi(v) \quad \text{a.e. in} \quad (0, T). \] (142)

**Notation.** If $g : \mathbb{R} \to Y$, $Y$ vector space, we set, $\forall h \in \mathbb{R}, \, h \neq 0$,
\[ \Delta_h g(t) := \frac{g(t + h) - g(t)}{h}. \]

We observe that from Eq. (142) we get $\phi(u(t)) \leq \phi(v)$ \( \forall v : Bv = B(t)(= Bu(t)) \) for every $t \in [0, T] \setminus N$, where $N$ has null Lebesgue measure. If $t \in N$ we can choose
\[ u(t) \in \arg\min_{t \in \mathcal{W}(t)} \phi = \arg\min_{t \in \mathcal{W}(t)} (\phi + b), \] (143)
where $\mathcal{W}(t) := \{v : Bv = B(t)\}$. It is easy to check that such a minimum exists, in fact $\phi + b$ is convex and weakly coercive, and $\{v : Bv = B(t)\}$ is a closed convex set where $b$ is constant thanks to Eq. (80). Now we prove that $\phi \circ u$ is absolutely continuous. We claim that for every $0 \leq s < t \leq T$, we have
\[ \phi(u(t)) - \phi(u(s)) = \int_s^t b^*((Bu)'(z) - f^*(z), (Bu)'(z)) \, dz. \] (144)

First we suppose that $t_i$ is a Lebesgue point for $u$, such that $s \leq t_i$. Set $\bar{u}(z) := u(s)$ if $z \leq s$ and $\bar{u}(z) = u(z)$ if $s \leq z \leq t$. If we choose $v = \bar{u}(z - h)$ in Eq. (142), with $z$ instead of $t$, integrating between $s$ and $t_i$ and dividing by $h$ we get
\[ \int_s^{t_i} ((Bu)'(z) - f(z), \Delta_h \bar{u}(z)) \, dz \]
\[ \leq \frac{1}{h} \left( \int_s^{t_i} \phi(\bar{u}(z - h)) \, dz - \int_s^{t_i} \phi(u(z)) \, dz \right) \]
\[ = \frac{1}{h} \left( \int_{s-h}^s \phi(\bar{u}(z)) \, dz - \int_{s-h}^{t_i} \phi(\bar{u}(z)) \, dz \right) \]
\[ = \phi(u(s)) - \int_{s-h}^{t_i} \phi(u(z)) \, dz. \] (145)

which yields,
\[ \frac{1}{h} \int_{s-h}^{t_i} \phi(\bar{u}(z)) \, dz \leq \int_s^{t_i} b^*((Bu)'(z) - f^*(z), \Delta_h \bar{u}(z)) \, dz + \phi(u(s)). \] (146)

Since $Bu \in H^1(0, T; D^*)$ and $t_i$ is a point of Lebesgue for $u$, taking the limit for $h \to 0^+$ Jensen inequality yields
\[ \phi(u(t_i)) - \phi(u(s)) \leq \int_s^{t_i} b^*((Bu)'(z) - f^*(z), (Bu)'(z)) \, dz. \] (147)

If $t > s$ is not a Lebesgue point for $u$ we choose a sequence $t_i \to t$ of Lebesgue points for $u$. Since $u(t_i) \in K$, $K$ weakly sequentially compact, we have $u(t_i) \to v \in W(t)$ for a suitable subsequence $t_{i_k}$, therefore, since $\phi$ is l.s.c. and convex,
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\[ \phi(v) \leq \liminf_{k \to +\infty} \phi(u(t_i)). \]  

On the other hand \( \phi(u(t)) \leq \phi(v) \), so that passing to the limit for \( k \to \infty \) in Eq. (147) we get

\[ \phi(u(t)) - \phi(u(s)) \leq \phi(v) - \phi(u(s)) \leq \int_s^t b^*(Bu)'(z) - f^*(z), (Bu)'(z) \, dz. \]  \( \text{(148)} \)

We argue in the same way, setting \( \tilde{u}(z) := u(t) \) if \( z \geq t \) and \( \tilde{u}(z) = u(z) \) if \( s \leq z \leq t \) and using \( \tilde{u}(z + h) \) instead of \( \tilde{u}(z - h) \), to get, \( \forall s < t \in [0, T] \),

\[ \phi(u(t)) - \phi(u(s)) \geq \int_s^t b^*(Bu)'(z) - f^*(z), (Bu)'(z) \, dz. \]  \( \text{(149)} \)

The identity (144) yields that \( \phi \circ u \in AC([0, T]). \)

Now we prove that

\[ \lim_{|t| \to 0} \phi(U_t(t)) = \phi(u(t)) \quad \forall t \in [0, T]. \]  \( \text{(150)} \)

From Eq. (109) it is easy to get for every \( t \in (0, T] \)

\[ \limsup_{|t| \to 0} \left\{ \int_0^t \| Bu_t'(z) \|_{L^p}^2 \, dz + \phi(U_t(t)) \right\} \]

\[ \leq \limsup_{|t| \to 0} \left\{ \phi(u(0)) + \int_0^t b^*\left( F_t'(z), Bu_t'(z) \right) \, dz + \int_0^t E_t(z) \, dz \right\} \]

\[ \leq \phi(u(0)) + \int_0^t b^*\left( f^*(z), (Bu)'(z) \right) \, dz \]

\[ = \int_0^t \| Bu_t'(z) \|_{L^p}^2 \, dz + \phi(u(t)). \]  \( \text{(151)} \)

Since \( \liminf_{|t| \to 0} \int_0^t \| Bu_t'(z) \|_{L^p}^2 \, dz \geq \int_0^t \| (Bu)'(z) \|_{L^p}^2 \, dz \) we have that

\[ \limsup_{|t| \to 0} \phi(U_t(t)) \leq \phi(u(t)) \quad \forall t \in (0, T]. \]  \( \text{(152)} \)

We claim that \( \liminf_{|t| \to 0} \phi(U_t(t)) \geq \phi(u(t)), \) and then

\[ \lim_{|t| \to 0} \phi(U_t(t)) = \phi(u(t)). \]  \( \text{(153)} \)

By absurdum let \( t \) such that \( \lim_{|t| \to \infty} \phi(U_t(t)) < \phi(u(t)) \). Since \( U_{t\delta}(t) \in K \) with \( K \) weakly sequentially compact, there exists further \( U_{t\delta}(t) \to v \in W(t) \) and then since \( \phi \) is weakly l.s.c. \( \phi(v) < \phi(u(t)) \). Using Eq. (153) in Eq. (151) we get

\[ \limsup_{|t| \to 0} \int_0^t \| Bu_t'(z) \|_{L^p}^2 \, dz \leq \int_0^t \| (Bu)'(z) \|_{L^p}^2 \, dz, \]  \( \text{(154)} \)

and then, since \( Bu_t' \to (Bu)' \) in \( L^2(0, T; D^+) \), we obtain the strong convergence too. ■
7. PROOFS OF THEOREMS 2.2, 2.3, AND 2.6

Since $b^*(\cdot, \cdot)$ is the scalar product of $D^*$ and $Bu \in H^1(0, T; D^*)$, we have almost everywhere in $(0, T)$

$$\frac{1}{2} \frac{d}{dt} b(u(t)) = \frac{1}{2} \frac{d}{dt} b^*(Bu(t)) = b^*((Bu)'(t), Bu(t)) = \langle (Bu)'(t), u(t) \rangle$$

(155)

**Proof of Theorem 2.2.** Let $h > 0$, let $u$ be a solution of Eq. (16) and $\xi \in \partial \phi(u_0) \cap D^*$ such that $\|f^+(0) - \xi\|_{b^*} = \|f^+(0) - \partial \phi(u_0)^*\|_{b^*}$. We set

$$\tilde{u}(t) := \begin{cases} u(t) & 0 < t \leq T, \\ u_0 & -h \leq t \leq 0, \end{cases}$$

(156)

and

$$\tilde{f}(t) := \begin{cases} f(t) & 0 < t \leq T, \\ \xi & -h \leq t \leq 0. \end{cases}$$

(157)

Note that $(Bu)'(t) = 0 \quad \forall -h \leq t < 0$. Since $\tilde{f}(t) \in (Bu)'(t) + \partial \phi(\tilde{u}(t))$ and $\tilde{f}(t + h) \in (Bu)'(t + h) + \partial \phi(\tilde{u}(t + h))$ almost everywhere in $[-h, T - h]$, using Eq. (155) and the monotonicity of $\partial \phi$, we have almost everywhere in $[-h, T - h]$,

$$\frac{d}{dt} b(\tilde{u}(t + h) - \tilde{u}(t)) \leq \langle \tilde{f}(t + h) - \tilde{f}(t), \tilde{u}(t + h) - \tilde{u}(t) \rangle,$$

(158)

then

$$\frac{d}{dt} b(\Delta_h \tilde{u}(t)) \leq \| \Delta_h \tilde{f}(t) \|_{b^*} \sqrt{b(\Delta_h \tilde{u}(t))}.$$  

(159)

Using the Gronwall’s Lemma we get

$$\sup_{t \in [-h, T - h]} \| \Delta_h B \tilde{u}(t) \|_{b^*} = \sup_{t \in [-h, T - h]} \sqrt{b(\Delta_h \tilde{u}(t))} \leq \sqrt{b(\Delta_h \tilde{u}(-h))} + \int_{-h}^{T-h} \| \Delta_h \tilde{f}(s) \|_{b^*} \, ds.$$  

(160)

Now we observe that $b(\Delta_h \tilde{u}(-h)) = b((\tilde{u}(-h) - \tilde{u}(0))/(h)) = 0$, and that

$$\int_{-h}^{T-h} \| \Delta_h \tilde{f}(s) \|_{b^*} \, ds \leq (\text{Var}_{b^*}(f) + \| f^+(0) - \xi \|_{b^*}).$$

The thesis follows passing to the limit for $h \to 0.$

In view of proving Theorem 2.3 we need the following.

**Lemma 7.1.** Let $\lambda \geq 0$ be such that $\phi(\cdot) + \lambda b(\cdot)$ is strictly convex. If $u, v \in D(\partial \phi)$ with $Bu = Bv$ and $\partial \phi(u) \cap \partial \phi(v) \neq \emptyset$, then $u = v.$
Proof. Let \( \xi \in V' \) be such that \( \xi \in \partial \phi(u) \cap \partial \phi(v) \), and let \( \eta = Bu = Bv \). By hypothesis we have

\[
\xi + 2\lambda \eta \in \left( \partial \phi(u) + 2\lambda Bu \right) \cap \left( \partial \phi(v) + 2\lambda Bv \right).
\]

It follows that \( u \) and \( v \) are two minima for

\[
w \mapsto \phi(w) + \lambda b(w) - \langle \xi + 2\lambda \eta, w \rangle,
\]

which is a strictly convex function. Since a strictly convex function cannot have two distinct minima, it follows that \( u = v \).

Proof of Theorem 2.3. Let \( u_1 \) and \( u_2 \) be two solutions of Eq. (16). Using the monotonicity of \( \partial \phi \) and Eq. (155) we get

\[
\frac{d}{dt} b(u_1(t) - u_2(t)) = \langle (Bu_1)'(t) - (Bu_2)'(t), u_1(t) - u_2(t) \rangle \leq 0.
\]

Then, being \( t \mapsto b(u_1(t) - u_2(t)) \) a nonnegative and non increasing function which is zero in zero, we deduce that \( b(u_1(t) - u_2(t)) = 0 \) a.e in \((0, T)\). This implies that

\[
Bu_1(t) = Bu_2(t) \quad \text{a.e in } (0, T).
\]

In particular

\[
(Bu_1)'(t) = (Bu_2)'(t) \quad \text{a.e in } (0, T).
\]

and then \( \partial \phi(u_1) \cap \partial \phi(u_2) \neq \emptyset \). Lemma 7.1 yields \( u_1(t) = u_2(t) \) a.e. in \((0, T)\).

Now we prove that \( u \in C^1([0, T]; V') \). We recall that we can choose (a version of) \( u \) such that

\[
u(t) \in \arg\inf_{v : Bv = B(t)} \phi, \quad \forall t \in [0, T],
\]

but in this case, since \( \phi + b \) is strictly convex, the minimum is achieved in a unique point. Let \( t \in [0, T) \); by absurdum suppose that there exists a sequence \( t_n \to t \), \( t_n \geq t \), such that \( u(t_n) \neq u(t) \) for every subsequence \( t_n \). Since \( u(t_n) \in K \), with \( K \) weakly sequentially compact, \( u(t_n) \rightharpoonup v \) for a suitable \( t_n \) and \( v \in W(t) \). Then

\[
\phi(v) \leq \liminf_{i \to +\infty} \phi(u(t_n(i))) \leq \liminf_{i \to +\infty} \left( \phi(u(t)) + \int_t^{t_n} b^*((Bu)'(z) - f^*(z), (Bu)'(z)) \, dz \right) \leq \phi(u(t)),
\]

for all \( t \in [0, T) \).
which implies \( v = u(t) \). We have proved that \( u \) is right weakly continuous. Arguing in the same way, we obtain that \( u \) is left weakly continuous too, and then that \( u \) is weakly continuous.

Finally let \( t \in [0, T] \), as usual \( U_n(t) \to v \in W(t) \) for a suitable subsequence \( \tau_k \), and then \( \phi(v) \leq \liminf_{k \to \infty} \phi(U_{\tau_k}(t)) = \phi(u(t)) \). Since there is only one minimum point, \( u(t) = v \). It follows that \( U_n(t) \to u(t) \) for every \( t \in [0, T] \) as \( |\tau| \to 0 \).

Proof of Theorem 2.5. We have already proved that \( \phi(U_n(t)) \to \phi(u(t)) \). Observe that \( \sqrt{b^*(Bu)(t)-f^*(t)} \leq C \) for each \( t \in [0, T] \), then the thesis follows from Theorem 2.1, since

\[
0 \leq \sigma(u(t), U_n(t)) \leq \phi(U_n(t)) - \phi(u(t)) + \langle (Bu)(t) - f(t), U_n(t) - u(t) \rangle \\
\leq \phi(U_n(t)) - \phi(u(t)) + b^*(Bu)(t) - f^*(t), BU_n(t) - Bu(t) \\
\leq \phi(U_n(t)) - \phi(u(t)) + C\sqrt{b(U_n(t) - u(t))}.
\]

(167)

8. PROOFS OF THEOREMS 2.5, 2.7, AND 2.8

Proof of Theorem 2.5. If \( u \) is a solution of Eq. (16) then for each \( v \in V \) and every \( t \in [0, T] \)

\[
\langle (Bu)(t) - f(t), u(t) - v \rangle + \phi(u(t)) - \phi(v) + \sigma(u, v) \leq 0.
\]

(168)

Summing Eqs. (105) and (168) with \( v = u \) and \( v = U_n \) respectively and using Eq. (155), we get

\[
\frac{d}{dt} b(u(t) - U_n(t)) + 2 \left( \sigma(u(t), U_n(t)) + \sigma(U_n, u) \right) \\
\leq 2l_t \overline{\varepsilon}_t + 2 \langle f(t) - \overline{F}_n(t), u(t) - U_n(t) \rangle.
\]

(169)

Reasoning like we did for Eq. (117), we conclude.

Proof of Theorem 2.7. Using Eq. (123) we get Eq. (28). In view of Eq. (26) it is enough to bound \( \int_0^T \| f(t) - \overline{F}_n(t) \|_{b'} dt \). Since the supposed regularity for \( f \) does not give any order of convergence, we argue as follows, for all \( 1 \leq n \leq N \) (see Nochetto et al. (2000)). Being \( b^*(\cdot, \cdot) \) a scalar product, we have

\[
\int_{t_{n-1}}^{t_n} \| f(t) - \overline{F}_n(t) \|_{b'}^2 dt = \int_{t_{n-1}}^{t_n} \| f(t) \|_{b'}^2 dt - \int_{t_{n-1}}^{t_n} \| \overline{F}_n(t) \|_{b'}^2 dt.
\]

(170)

Given an integrable function \( w : [0, T] \to V \), let \( \overline{w} \) be the piecewise average of \( w \) over \( \mathcal{P} \), namely \( R_n w = (1/\tau_n) \int_{t_{n-1}}^{t_n} w(t) dt \). It is easy to see that

\[
\int_0^T \langle f(t) - \overline{F}_n(t), \overline{w}(t) - U_n(t) \rangle dt = 0.
\]

(171)
Some tedious calculations yield

\[
\left( \int_0^{\tau_n} b(u(t) - U_\tau(t) - \mathcal{R}u(t) + \mathcal{R}U_\tau(t)) \, dt \right)^{1/2}
\]

\[
\leq \left( \int_0^{\tau_n} \| Bu(t) - \mathcal{R}Bu(t) \|_{L^2} \, dt \right)^{1/2} + \left( \int_0^{\tau_n} b(U_\tau(t) - \mathcal{R}U_\tau(t)) \, dt \right)^{1/2}
\]

\[
\leq \frac{|\tau_n|}{\sqrt{6}} \left( \int_0^{\tau_n} \|(Bu)'(t)\|_{L^2} \, dt \right)^{1/2} + \frac{|\tau_n|}{2\sqrt{3}} \left( \int_0^{\tau_n} b(U_\tau(t)) \, dt \right)^{1/2}.
\]

(172)

For all \(1 \leq n \leq N\) we set

\[
e(t_n) := b(u(t_n) - U_\tau(t_n)) + 2 \int_0^{\tau_n} (\sigma(u(t), U(t)) + \sigma(U_\tau(t), u(t))).
\]

Integrating Eq. (169) and using Eq. (171) we get

\[
e(t_n) \leq \int_0^{\tau_n} 2l \mathfrak{E}_n \, dt + 2 \int_0^{\tau_n} (f - \mathcal{F}_n, u - U_\tau + \mathcal{R}u + \mathcal{R}U_\tau) \, dt
\]

\[
\leq \int_0^{\tau_n} 2l \mathfrak{E}_n \, dt + 2 \kappa_0 (f - \mathcal{F}_n) \, dt
\]

\[
\times \left( \int_0^{\tau_n} b(u - U_\tau - \mathcal{R}u + \mathcal{R}U_\tau) \, dt \right)^{1/2}.
\]

(174)

Since \(BU'_\tau \to (Bu)'\) in \(L^2(0, T; D^r)\) and

\[
\| Bu'_\tau \|_{L^2(0, T; D^r)} \leq 2\phi(u_0) + \| f \|_{L^2(0, T; D^r)},
\]

we have

\[
\|(Bu)' \|_{L^2(0, T; D^r)} \leq 2\phi(u_0) + \| f \|_{L^2(0, T; D^r)}.
\]

(175)

Then using Eqs. (122), (170), and (172) we obtain

\[
e(t_n) \leq |\tau_n| \left\{ \phi(u_0) - \phi(U_\tau^n) - \| Bu'_\tau \|_{L^2(0, t_n; D^r)} + \| BU'_\tau \|_{L^2(0, t_n; D^r)} \right\}
\]

\[
+ \| f - \mathcal{F}_n \|_{L^2(0, t_n; D^r)} \left[ \frac{\sqrt{2}}{\sqrt{3}} \| (Bu)' \|_{L^2(0, t_n; D^r)} + \frac{1}{\sqrt{3}} \| BU'_\tau \|_{L^2(0, t_n; D^r)} \right]
\]

\[
\leq |\tau_n| \left\{ 2\phi(u_0) - \phi(U_\tau^n) + \frac{1}{2} \| f \|_{L^2(0, T; D^r)} \right\}.
\]

(177)
Now applying Eq. (26) from $t_{n-1}$ to $t_n$ and also Eq. (123) we get
\[
\max_{t \in (t_{n-1}, t_n)} b(u(t) - U^*_t(t)) \leq \left( (e(t_{n-1}) + \tau_n^2 \xi_0^*)^{1/2} + \int_{t_{n-1}}^{t_n} \| f - \overline{f}_t \|_{\mathcal{B}_1} \, dt \right)^2
\]
\[
\leq \frac{3}{2} (e(t_{n-1}) + \tau \left( \phi(U^n_{n-1}) - \phi(U^n_t) + \frac{1}{4} \tau_0 \| F^n_t \|_{\mathcal{B}_1}^2 \right))
\]
\[
+ 3 \tau_n \int_{t_{n-1}}^{t_n} \| f - \overline{f}_t \|_{\mathcal{B}_1}^2 \, dt.
\] (178)

In view of Eq. (177) with $t_{n-1}$ in place of $t_n$, we conclude
\[
\max_{t \in (t_{n-1}, t_n)} b(u(t) - U^*_t(t)) \leq 3 |\tau| \left( \phi(u_0) - \frac{1}{2} \phi(U^n_t) + \| f \|_{L^2(0, T; D')}^2 \right).
\] (179)

In view of proving Theorem 2.8 we need a discrete version of Gronwall’s Lemma, see Lemma 3.11 Nochetto et al. (2000).

**Lemma 8.1.** Let $\{ a_n \}_{n=0}^N$ and $\{ b_n, c_n, d_n \}_{n=1}^N$ be non negative numbers. If
\[
2a_n(a_n - a_{n-1}) + b_n^2 \leq c_n^2 + 2a_n d_n \quad \forall 1 \leq n \leq N,
\] (180)
then
\[
\max_{1 \leq n \leq N} a_n \leq \left( a_0 + \sum_{n=1}^N c_n \right)^{1/2} + \sqrt{2} \sum_{n=1}^N d_n,
\] (181)
\[
\left( \sum_{n=1}^N b_n^2 \right)^{1/2} \leq \left( a_0 + \sum_{n=1}^N c_n \right)^{1/2} + \sqrt{2} \sum_{n=1}^N d_n.
\] (182)

**Definition 8.2.** We define (see Def. 3.8 Nochetto et al. (2000))
\[
\mathcal{D}_1 := \varepsilon_1^1,
\]
\[
\mathcal{D}_n := \langle \delta F^n_t - \delta^2 B U^n_t, \delta U^n_t \rangle_{\mathcal{B}_1} \quad 2 \leq n \leq N.
\] (183)

Summing Eq. (103) with $v = U^n_t$, $v = U^{n-1}_t$ respectively, we get
\[
\langle \delta B U^n_t - \delta^2 F^n_t \tau_n - \delta^2 B U^{n-1}_t, U^n_t - U^{n-1}_t \rangle + \sigma(U^n_t, U^{n-1}_t)
\]
\[
+ \sigma(U^{n-1}_t, U^n_t) \leq 0 \quad 2 \leq n \leq N,
\] (184)
from which it follows that $\mathcal{D}_n \geq 0$.

In the next Proposition we give two important estimates.

**Proposition 8.3.** Let $U^n_t$ be a solution of Eq. (9), with $f \in BV([0, T]; D')$ and $F^n_t := f^*(t_n)$. If $\partial \phi(U^n_t) \cap D(b^n) \neq \emptyset$, we have
\[
\left( \sum_{n=1}^N \mathcal{D}_n \right)^{1/2} \leq \frac{1}{\sqrt{2}} \|(F^0 - \partial \phi(U^0_t)) \|^2_{\mathcal{B}_1} + \sum_{n=1}^N \| F^n_t - F^{n-1}_t \|_{\mathcal{B}_1},
\] (185)
where \((F^0 - \partial \phi(u_0))^\circ\) denotes the minimal selection of \(F^0 - \partial \phi(u_0)\) with respect to \(\| \cdot \|_{\overline{B}}\). Moreover

\[\mathcal{E}^n_t \leq D_n, \quad 1 \leq n \leq N.\]  \hfill (186)

**Proof.** We have for \(2 \leq n \leq N\)

\[D_n + \sqrt{b(\delta U^n_t)}(\sqrt{b(\delta U^n_t)} - \sqrt{b(\delta U^{n-1}_t)})) \]

\[= \tau_n(\delta F^n_t, \delta U^n_t) + b(\delta U^{n-1}_t, \delta U^n_t) - b(\delta U^n_t b(\delta U^{n-1}_t)) \]

\[\leq \tau_n(\delta F^n_t, \delta U^n_t) + b(\delta U^{n-1}_t, \delta U^n_t) - b(\delta U^{n-1}_t, \delta U^n_t).\]  \hfill (187)

and then

\[2D_n + 2\sqrt{b(\delta U^n_t)}(\sqrt{b(\delta U^n_t)} - \sqrt{b(\delta U^{n-1}_t)})) \leq 2\| F^n_t - F^{n-1}_t \|_{\overline{B}} \sqrt{b(\delta U^n_t)}.\]  \hfill (188)

From \(\phi(U^n_t) - \phi(U^1_t) \leq (\partial \phi(U^n_t), U^n_0 - U^1_0)\) we obtain

\[-\delta \phi(U^n_t) \leq (\delta \phi(U^n_t), \delta U^n_t),\]  \hfill (189)

and then with simple calculations

\[2\mathcal{E}^n + 2\sqrt{b(\delta U^n_t)}(\sqrt{b(\delta U^n_t)} - \| (F^n_t - \partial \phi(U^n_t))^\circ \|_{\overline{B}}) \leq 2\| F^n_t - F^0_t \|_{\overline{B}} \sqrt{b(\delta U^n_t)}.\]  \hfill (190)

The estimate (185) comes from Eq. (188) using Gronwall’s like Lemma 8.1 with

\[a_0 = \| (F^0_t - \partial \phi(U^n_t))^\circ \|_{\overline{B}}, \quad a_n = \sqrt{b(\delta U^n_t)} \text{ for } 1 \leq n \leq N, \quad b_2 = 2D_n, \quad c_n = 0, \quad \text{and } d_n = \| F^n_t - F^{n-1}_t \|_{\overline{B}}.\]

As for it concerns Eq. (186), writing Eq. (102) for \(U^n_{t-1}\) with \(v = U^n_t\), we get:

\[\langle \delta B U^n_{t-1} - F^n_t - 1, -\delta U^n_t \rangle \tau + \phi(U^n_{t-1}) - \phi(U^n_t) + \sigma(U^n_t, U^n_{t-1}) \leq 0,\]  \hfill (191)

which is

\[-\delta \phi(U^n_t) \tau \leq \langle \delta B U^n_{t-1} - F^n_t - 1, \delta U^n_t \rangle \tau - \sigma(U^n_t, U^n_{t-1});\]  \hfill (192)

finally

\[\tau_n \mathcal{E}^n = \tau_n(F^n_t - \delta B U^n_t, \delta U^n_t) - \tau_n \delta \phi(U^n_t) \]

\[\leq \tau_n(F^n_t - \delta B U^n_t, \delta U^n_t) + \tau_n(\delta B U^n_{t-1} - F^n_t, \delta U^n_t) + \sigma(U^n_t, U^n_{t-1}) \]

\[= \tau_n^2(\delta F^n_t - \delta B U^n_t, \delta U^n_t) - \sigma(U^n_t, U^n_{t-1}) \]

\[= \tau_n D_n - \sigma(U^n_t, U^n_{t-1}).\]  \hfill (193)
Proof of Theorem 2.8. In view of Eqs. (186) and (185), estimate (30) follows from Eq. (26) and the properties \( \int_0^T \| f(t) - F_t \|_{b'} \, dt \leq |t| \text{Var}_{b'}(f) \) and \( \sum_{k=0}^N \| F_t^k - F_t^{k-1} \|_{b'} \leq \text{Var}_{b'}(f) \).

\[ 9. \, \text{LIPSCHITZ PERTURBATION} \]

Proof of Theorem 2.9. Let \( u_0(t) = u_0 \). For each \( k \in \mathbb{N} \) let \( u_k : [0, T] \to V \) be the solution of

\[
\begin{aligned}
(Bu_k)'(t) + \partial \phi(u_k(t)) &\in f(t) + \mathcal{L}(u_{k-1}(t), t) \quad \text{a.e. in } [0, T], \\
Bu_k(0) &= Bu_0.
\end{aligned}
\]

(194)

Arguing as in the proof of Theorem 2.2, we get

\[
\begin{aligned}
\frac{d}{dt} (Bu_k(t) - u_{k-1}(t)) &\leq (\mathcal{L}(u_{k-1}, t) - \mathcal{L}(u_{k-2}, t), u_k(t) - u_{k-1}(t)) \\
&\leq L \sqrt{b^*(Bu_{k-1}(t) - Bu_{k-2}(t))} \sqrt{b(u_k(t) - u_{k-1}(t))},
\end{aligned}
\]

(195)

that is

\[
\frac{d}{dt} \left\| u_k(t) - u_{k-1}(t) \right\|_{b'} \leq L \left\| Bu_{k-1}(t) - Bu_{k-2}(t) \right\|_{b'} \left\| Bu_k(t) - Bu_{k-1}(t) \right\|_{b'}.
\]

(196)

Using a variant of the classical Gronwall’s Lemma, see Lemma A.5, Brezis (1973), we deduce that for each \( 0 \leq t \leq T \)

\[
\sup_{s \in [0, t]} \left\| Bu_k(s) - Bu_{k-1}(s) \right\|_{b'} \leq L \int_0^t \left\| Bu_{k-1}(s) - Bu_{k-2}(s) \right\|_{b'} \, ds,
\]

(197)

and then

\[
\left\| Bu_k - Bu_{k-1} \right\|_{C^0([0, T], D')} \leq \frac{(LT)^k}{k!} \left\| Bu_1 - Bu_0 \right\|_{C^0([0, T], D')}.
\]

(198)

which implies \( \sum_{k=0}^{+\infty} \left\| Bu_k - Bu_{k-1} \right\|_{C^0([0, T], D')} < +\infty \). Therefore \( Bu_k \) converges to a function \( Bu \) in \( C^0([0, T], D') \). It is easy to verify that for each \( k \in \mathbb{N} \) \( \left\| (Bu_k) \right\|_{L^2([0, T], D')} \leq C^* < +\infty \) and also that \( \phi(u_k(t)) + b(u_k(t)) \leq K^* < +\infty \) for every \( t \in [0, T] \). It is possible to conclude adapting the proof of Theorem 2.1.

We conclude giving two error estimates in the case of time independent Lipschitz perturbations. Let \( \mathcal{L} : V \to V' \) be such that:

\[
\left\| \mathcal{L}(v_1) - \mathcal{L}(v_2) \right\|_{b'} \leq L \sqrt{b(v_1 - v_2)}, \quad \forall v_1, v_2 \in V.
\]

(199)

Note. A simple example of such a perturbation is \( \mathcal{L}(u) := LBu \).
Variable Time-Step Discretization of Degenerate Evolution Equations

Definition 9.1. Set

\[ D_n^a := (F_n^1 + \mathcal{L}U_n^1 - \mathcal{L}U_n^0 - \delta U_n^1) - b(\delta U_n^1), \]
\[ D_n^a := (\delta G_n^a - \delta^2 BU_n^a, \delta U_n^a) \tau_n \quad 2 \leq n \leq N, \]

where \( G_n^a := F_n^a + \mathcal{L}U_n^a \).

Theorem 9.2. Under the same assumptions of Theorem 2.1, let \( \mathcal{L} \) be as before, let \( u \) be a solution of Eq. (33) and \( \{U_n^a\}_{n=0}^N \) be a solution of Eq. (9) with \( L \tau_n < 1 \) for all \( 1 \leq n \leq N \), then

\[
\max_{t \in [0, T]} e^{-Lt} \sqrt{b(u(t) - U_n(t))} \leq \left( \sum_{n=1}^{N} c_n^2 D_n^a \right)^{1/2} + \|f - F_n\|_{L(0, T; D')} + \frac{L}{2} \sum_{n=1}^{N} c_n^2 \sqrt{b(\delta U_n^a)}. \tag{201}
\]

If \( f \in BV([0, T]; D') \), \( F_n^a := f^+(t_n^+), U_n^0 := u_0 \in D(\phi) \) and \( D(b^*) \cap D(\phi) \neq \emptyset \) then

\[
\max_{t \in [0, T]} \sqrt{b(u(t) - U_n(t))} \leq |\tau|e^{LT} \left( \frac{1}{\sqrt{2}} C^* \|f^+(0) - \delta \phi(u_0)\|_{b^*} + (1 + C^*) \text{Var}_b(f) \right), \tag{202}
\]

where \( C_{\tau, T} := e^{L|\tau|(1+LT)}/(1-L|\tau|) \) and \( C^* := C_{\tau, T}(1 + LT/\sqrt{2}) \).

We need a variant of Lemma 8.1, see Lemma 4.9 Nochetto et al. (2000).

Lemma 9.3. Let \( \{a_n\}_{n=0}^N \) and \( \{b_n, c_n, d_n\}_{n=1}^N \) be non negative numbers, let \( -1 < \lambda_n \leq 0 \) be given coefficients, and set

\[ \lambda := \min_{1 \leq n \leq N} \lambda_n, \quad \tilde{s}_n := \frac{\tilde{s}_n + \tilde{s}_{n-1}}{2}, \quad \tilde{s}_0 := 0. \]

If

\[
2a_n(a_n - a_{n-1}) + b_n^2 + 2\lambda_n d_n^2 \leq \frac{c_n^2}{\lambda_n} + 2a_n d_n, \quad \forall 1 \leq n \leq N, \tag{203}
\]

then we have

\[
\max \left( \max_{1 \leq n \leq N} e^{\lambda_n/(1+\lambda_n)} a_n, \left( \sum_{n=1}^{N} e^{2\tilde{s}_1/(1+\lambda_n)} b_n^2 \right)^{1/2} \right) \]

\[
\leq \left( c_0^2 + \sum_{n=1}^{N} e^{2\tilde{s}_n} c_n^2 \right)^{1/2} + \sqrt{2} \sum_{n=1}^{N} e^{\lambda_n} d_n. \tag{204}
\]
Proof of Theorem 9.2. Let \( g(t) := f(t) + L(u(t)) \). We argue as in the proof of 
Theorem 2.8 to get

\[
\frac{1}{2} \frac{d}{dt} b(u - U_t) \leq \ell \tau \overline{D}^\gamma + (g - \overline{g}(u - U_t)) \\
\leq \ell \tau \overline{D}^\gamma + (\| \mathcal{F}_t \|_{b^*} + L \ell \tau \| B U_t^* \|_{b^*}) \sqrt{b(u - U_t)} + L b(u - U_t),
\]  
(205)

Using Gronwall’s Lemma we deduce

\[
\max_{t \in [0,T]} e^{-L \tau} \sqrt{b(u(t) - U_t(t))} \\
\leq \left( b(u_0 - U_t^0) + \sum_{n=1}^N \tau_n^2 e^{-2L \tau_n} D_{n}^\gamma \right)^{1/2} + \int_0^T e^{-L \tau} \| \mathcal{F}_t \|_{b^*} \, dt \\
+ \frac{L}{2} \sum_{n=1}^N \tau_n^2 e^{-2L \tau_n} \sqrt{b(\delta U_t^n)},
\]  
(206)

which yields Eq. (201). We have also

\[
2 D_{n}^\gamma + 2 \sqrt{b(\delta U_t^n)} (\sqrt{b(\delta U_t^n)} - \sqrt{b(\delta U_t^{n-1})}) - L \tau_n b(\delta U_t^n) \\
\leq 2 \| F_t^n - F_t^{n-1} \|_{b^*} \sqrt{b(\delta U_t^n)} \quad 2 \leq n \leq N,
\]  
(207)

and moreover

\[
2 D_{1}^\gamma + 2 b(\delta U_t^1) - \|(F_t^0 - \partial \phi(u_0))^\gamma\|_{b^*} \sqrt{b(\delta U_t^1)} - L \tau_n b(\delta U_t^n) \\
\leq 2 \| F_t^1 - F_t^0 \|_{b^*} \sqrt{b(\delta U_t^1)}.
\]  
(208)

We observe that \( \|(F_t^0 - \partial \phi(u_0))^\gamma\|_{b^*} < +\infty \) since by hypothesis we have \( D(b^*) \cap \partial \phi(u_0) \neq \emptyset \).

Using Lemma 9.3 with \( a_0 := \|(F_0^0 - \partial \phi(u_0))^\gamma\|_{b^*} \), \( a_n := \sqrt{b(\delta U_t^n)} \) \( 1 \leq n \leq N \), \( b_n := 2 D_{n}^\gamma \), \( c_n := 0 \), \( d_n := \| F_t^n - F_t^{n-1} \|_{b^*} \), and \( \lambda_n := -L \tau_n \), and given that \( t_n/(1 - L \tau) \leq t_{n-1} + \tau (1 + L T)/(1 - L T) \), we obtain

\[
\max_{1 \leq n \leq N} e^{-L \tau_n} \sqrt{b(\delta U_t^n)} (2 \sum_{n=1}^N e^{-2L \tau_n} D_{n}^\gamma)^{1/2} \\
\leq C_{\tau, T} \left( \left( \left( (F_0^0 - \partial \phi(u_0))^\gamma \right) \right)^{1/2} + \sqrt{2} \sum_{n=1}^N e^{-L \tau_n} \| F_t^n - F_t^{n-1} \|_{b^*} \right).
\]  
(209)
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REFERENCES


