On the time harmonic Maxwell equations in general domains

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1 Introduction

In this paper we shall consider the finite element approximation to the Maxwell equations. In Section 2 the convergence theory for the finite element approximation of the time harmonic Maxwell equations is recalled. Although the short proof we present has never been written in one single paper, it is essentially known and it is based on the results of [15, 9, 4, 5]. The first analysis of this problem has been given in [16] under certain restrictions on the domain, on the coefficients and on the mesh sequence.

The analysis relies on the convergence of the discrete Maxwell eigenmodes towards the continuous ones. In Section 3 simple test cases are studied in order to compare the accuracy of the edge element method to a penalized approach which makes use of standard nodal elements. Standard penalization with nodal elements is very efficient with smooth or convex domains (and regular coefficients), while it is known to produce very bad results in presence of singularities (see [12]). For this reason, we consider a method, which has been introduced in [3] for a different problem and used for the approximation of Maxwell system in [8].

2 Finite elements for the time harmonic Maxwell equations

Let us consider the time-harmonic Maxwell equations: given ω suitably chosen in $\mathbb{R}$ and $\mathbf{f} : \Omega \to \mathbb{R}^3$ such that $\operatorname{div} \mathbf{f} = 0$, we look for $\mathbf{u} : \Omega \to \mathbb{R}^3$ such that

$$\begin{aligned} &\operatorname{curl}\left(\mu^{-1} \operatorname{curl} \mathbf{u}\right) - \omega^2 \varepsilon \mathbf{u} = \mathbf{f} && \text{in } \Omega, \\ &\operatorname{div}(\varepsilon \mathbf{u}) = 0 && \text{in } \Omega, \\ &\mathbf{u} \times \mathbf{n} = 0 && \text{on } \partial\Omega. \end{aligned} \qquad (1)$$

Here Ω is a bounded Lipschitz polyhedron in $\mathbb{R}^3$, $\partial\Omega$ its boundary, μ and ε are the usual electromagnetic tensors; the hypotheses on them are quite general. Notice that problem (1) is solvable only when ω is not a Maxwell eigenvalue (see Theorem 1 below).

Let us recall the definition of some functional spaces which will be useful in the following:

$$\begin{aligned} H(\mathrm{curl};\Omega) &= \{\mathbf{v}\in L^2(\Omega)^3 \mid \mathrm{curl}\,\mathbf{v}\in L^2(\Omega)^3\}\\ H_0(\mathrm{curl};\Omega) &= \{\mathbf{v}\in H(\mathrm{curl};\Omega) \mid \mathbf{v}\times\mathbf{n}=0 \text{ on } \partial\Omega\}\\ H(\mathrm{div};\Omega;\varepsilon) &= \{\mathbf{v}\in L^2(\Omega)^3 \mid \mathrm{div}(\varepsilon\mathbf{v})\in L^2(\Omega)\}\\ H(\mathrm{div}^0;\Omega;\varepsilon) &= \{\mathbf{v}\in H(\mathrm{div};\Omega;\varepsilon) \mid \mathrm{div}(\varepsilon\mathbf{v})=0 \text{ in } \Omega\}\\ H_0(\mathrm{div}^0;\Omega;\varepsilon) &= \{\mathbf{v}\in H(\mathrm{div}^0;\Omega;\varepsilon) \mid \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\} \end{aligned} \tag{2}$$

For all $\mathbf{u}\in H_0(\mathrm{curl};\Omega)$, $\boldsymbol{\varphi}\in H(\mathrm{div};\Omega;\varepsilon)$ and $q\in H^1_0(\Omega)$ we define the norms:

$$\begin{aligned} \|\mathbf{u}\|_{\mathrm{curl}} &= \left(\|\mathbf{u}\|_0^2+\|\,\mathrm{curl}\,\mathbf{u}\|_0^2\right)^{\frac12},\\ \|\boldsymbol{\varphi}\|_{\mathrm{div}} &= \left(\|\boldsymbol{\varphi}\|_0^2+\|\,\mathrm{div}(\varepsilon\boldsymbol{\varphi})\|_0^2\right)^{\frac12},\\ \|q\|_1 &= \left(\|q\|_0^2+\|\,\mathrm{grad}\,q\|_0^2\right)^{\frac12}. \end{aligned} \tag{3}$$

Then the variational formulation of problem (1) can be written as follows:

$$\begin{aligned} &\text{Find } (\mathbf{u},p)\in H_0(\mathrm{curl};\Omega)\times H^1_0(\Omega) \text{ such that}\\ &\begin{cases} (\mu^{-1}\,\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{v})-\omega^2(\varepsilon\mathbf{u},\mathbf{v})+(\varepsilon\mathbf{v},\mathrm{grad}\,p)=(\mathbf{f},\mathbf{v}) & \forall\mathbf{v}\in H_0(\mathrm{curl};\Omega)\\ (\varepsilon\mathbf{u},\mathrm{grad}\,q)=0 & \forall q\in H^1_0(\Omega). \end{cases} \end{aligned} \tag{4}$$

Our analysis is based on some results on the following eigenvalue problem:

$$\begin{aligned} &\text{find } \lambda\in\mathbb{R} \text{ such that } \exists\mathbf{w}\in H_0(\mathrm{curl};\Omega) \text{ with } \mathbf{w}\neq 0:\\ &(\mu^{-1}\,\mathrm{curl}\,\mathbf{w},\mathrm{curl}\,\mathbf{v})=\lambda(\varepsilon\mathbf{w},\mathbf{v}) \qquad \forall\mathbf{v}\in H_0(\mathrm{curl};\Omega) \end{aligned} \tag{5}$$

It is well known that the eigenproblem (5) admits the eigenvalue $\lambda_0=0$ and a nondecreasing sequence of positive eigenvalues λ_i, for $i=1,2,\dots$, going to infinity. The eigenspace associated to $\lambda_0=0$ is infinite dimensional and coincides with the subset of $H_0(\mathrm{curl};\Omega)$ of the vector functions with vanishing curl, that is $\mathrm{grad}(H^1_0(\Omega))$. The positive eigenvalues λ_i have finite multiplicity and the associated eigenfunctions are divergence free. Using the Fredholm alternative theorem, one can prove the following theorem:

Theorem 1. *Let λ_i, for $i=1,2,\cdots$ be the positive eigenvalues of problem (5). Then, if $\omega^2\neq\lambda_i$, for all $i=1,2,\cdots$, there exists a unique solution to problem (4), with the following stability estimate:*

$$\left(\|\mathbf{u}\|^2_{\mathrm{curl}}+\|p\|_1^2\right)^{\frac12}\le \max\left(1+\omega^2,\ \frac{1+\lambda_i}{|\lambda_i-\omega^2|},\ i=1,2,\cdots\right)\|\mathbf{f}\|_0. \tag{6}$$

The proof of Th. 1 is given in [15].

Let us introduce finite dimensional subspaces $E_h\subseteq H_0(\mathrm{curl};\Omega)$ and $Q_h\subseteq H^1_0(\Omega)$. Then the discretization of problem (4) reads:

Find $(\mathbf{u}_h, p_h) \in E_h \times Q_h$ such that

$$\begin{cases} (\mu^{-1}\operatorname{curl}\mathbf{u}_h, \operatorname{curl}\mathbf{v}_h) - \omega^2(\varepsilon\mathbf{u}_h, \mathbf{v}_h) + (\varepsilon\mathbf{v}_h, \operatorname{grad} p_h) = (\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in E_h \\ (\varepsilon\mathbf{u}_h, \operatorname{grad} q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{7}$$

Following [15], the same analysis, used for the continuous problem (4) (see Th. 1), can be performed also for the discrete one (7), provided the discrete spaces satisfy the following condition:

$$q \in Q_h \iff \operatorname{grad} q \in E_h. \tag{8}$$

The following convergence estimate holds

$$\|\mathbf{u} - \mathbf{u}_h\|^2_{\mathrm{curl}} + \|p - p_h\|_1^2 \le \beta^2 \inf_{(\mathbf{v}_h, q_h) \in E_h \times Q_h} (\|\mathbf{u} - \mathbf{v}_h\|^2_{\mathrm{curl}} + \|p - q_h\|_1^2) \tag{9}$$

with the stability constant given by:

$$\beta \le 1 + \max\left(1 + \omega^2; \frac{1 + \lambda_{ih}}{|\lambda_{ih} - \omega^2|},\ i = 1, 2, \cdots\right). \tag{10}$$

Here λ_{ih} denotes the discrete eigenvalues resulting from the approximation of problem (5):

$$\begin{array}{l} \text{find } \lambda_h \in \mathbb{R} \text{ such that } \exists \mathbf{w}_h \in E_h \text{ with } \mathbf{w}_h \neq 0: \\ (\mu^{-1}\operatorname{curl}\mathbf{w}_h, \operatorname{curl}\mathbf{v}_h) = \lambda_h(\varepsilon\mathbf{w}_h, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in E_h. \end{array} \tag{11}$$

In order to conclude the above analysis we need to know that the discrete eigenvalues converge to the continuous ones. This guarantees that, if $\omega^2 \neq \lambda_i$ then, for h sufficiently small, also $\omega^2 \neq \lambda_{ih}$ for all i. The convergence of the eigensolutions of problem (11) to the continuous ones has been considered in [9]. The analysis is based on the study of the following auxiliary eigenproblem in mixed form:

find $\lambda \in \mathbb{R}$ such that $\exists (\mathbf{w}, \boldsymbol{\varphi}) \in H_0(\mathrm{curl}; \Omega) \times H_0(\mathrm{div}^0; \Omega; \mu^{\frac{1}{2}})$ with $\mathbf{w} \neq 0$:

$$\begin{cases} (\varepsilon\mathbf{w}, \mathbf{v}) + (\mu^{-1/2}\operatorname{curl}\mathbf{v}, \boldsymbol{\varphi}) = 0 \ \forall \mathbf{v} \in H_0(\mathrm{curl}; \Omega) \\ (\mu^{-1/2}\operatorname{curl}\mathbf{w}, \boldsymbol{\psi}) = \lambda(\boldsymbol{\varphi}, \boldsymbol{\psi}) \qquad \forall \boldsymbol{\psi} \in H_0(\mathrm{div}^0; \Omega; \mu^{\frac{1}{2}}). \end{cases} \tag{12}$$

The pair $(\lambda, \mathbf{w}) \in \mathbb{R} \times H_0(\mathrm{curl}; \Omega)$ is an eigensolution of problem (5), with $\lambda \neq 0$, if and only if, there exists $\boldsymbol{\varphi} \in H_0(\mathrm{div}^0; \Omega; \mu^{\frac{1}{2}})$ such that $(\lambda, \mathbf{w}, \boldsymbol{\varphi}) \in \mathbb{R} \times H_0(\mathrm{curl}; \Omega) \times H_0(\mathrm{div}^0; \Omega; \mu^{\frac{1}{2}})$ is an eigensolution of (12).

Let F_h denote the following finite dimensional subspace of $H_0(\mathrm{div}^0; \Omega; \mu^{\frac{1}{2}})$

$$F_h = \mu^{-\frac{1}{2}} \operatorname{curl}(E_h). \tag{13}$$

Consider the following mixed discrete eigenproblem:

$$\begin{array}{l} \text{find } \lambda_h \text{ such that } \exists (\mathbf{w}_h, \boldsymbol{\varphi}_h) \in E_h \times \mathbf{F}_h \text{ with } \mathbf{w}_h \neq 0: \\ \begin{cases} (\varepsilon\mathbf{w}_h, \mathbf{v}_h) + (\mu^{-1/2}\operatorname{curl}\mathbf{v}_h, \boldsymbol{\varphi}_h) = 0 \ \forall \mathbf{v}_h \in E_h \\ (\mu^{-1/2}\operatorname{curl}\mathbf{w}_h, \boldsymbol{\psi}_h) = \lambda_h(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) \quad \forall \boldsymbol{\psi}_h \in F_h. \end{cases} \end{array} \tag{14}$$

Let us denote by E and F the range of the operators which associate to every element $\mathbf{f} \in H_0(\mathrm{div}^0; \Omega; \mu^{\frac{1}{2}})$ the first and the second component, respectively, of the solution of the source problem corresponding to the eigenproblem (12).

Moreover, the discrete kernel $\mathbb{K}$ is defined by

$$\mathbb{K} = \{\mathbf{v}_h \in E_h : (\mu^{-1/2} \operatorname{curl} \mathbf{v}_h, \boldsymbol{\psi}_h) = 0 \ \forall \boldsymbol{\psi}_h \in F_h\}. \tag{15}$$

The following hypotheses have been introduced in [6] in an abstract framework in order to obtain the convergence of solutions of eigenproblems in mixed form.

H1 The *weak approximability* of F is satisfied if there exists $\omega_1(h)$ tending to zero as h goes to zero such that for every $\boldsymbol{\varphi} \in F$ and for every $\mathbf{v}_h \in \mathbb{K}$

$$(\mu^{-1/2} \operatorname{curl} \mathbf{v}_h, \boldsymbol{\varphi}_h) \leq \omega_1(h) \|\mathbf{v}_h\|_0 \|\boldsymbol{\varphi}_h\|_F. \tag{16}$$

H2 The *strong approximability* of F is satisfied if there exists $\omega_2(h)$ tending to zero as h goes to zero such that for every $\boldsymbol{\varphi} \in F$ there exists $\boldsymbol{\varphi}^I \in F_h$ such that

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^I\|_{\mathrm{div}} \leq \omega_2(h) \|\boldsymbol{\varphi}\|_F. \tag{17}$$

H3 $\Pi_h : E \to E_h$ is called *Fortin operator* if it satisfies

$$\begin{aligned} &(\mu^{-1/2} \operatorname{curl}(\mathbf{w} - \Pi_h \mathbf{w}), \boldsymbol{\psi}_h) = 0 \ \forall \mathbf{w} \in E, \ \forall \boldsymbol{\psi}_h \in F_h \\ &\|\Pi_h \mathbf{w}\|_{\mathrm{curl}} \leq C \|\mathbf{w}\|_E \qquad\qquad \forall \mathbf{w} \in E. \end{aligned} \tag{18}$$

H4 A Fortin operator Π_h *converges uniformly to the identity* if there exists $\omega_3(h)$ tending to zero as h goes to zero such that

$$\|\mathbf{w} - \Pi_h \mathbf{w}\|_0 \leq \omega_3(h) \|\mathbf{w}\|_E \quad \forall \mathbf{w} \in E. \tag{19}$$

In [7, 6] it has been proved that the assumptions H1-H4 imply that the discrete eigenvalues of problem (14) converge to the continuous ones, that is

$$\begin{aligned} &\forall \varepsilon, \ \forall N \in \mathbb{N} \quad \exists h_0 \text{ such that } \forall h \leq h_0 \\ &\max_{i=1,\cdots,m(N)} |\lambda_i - \lambda_{ih}| \leq \varepsilon, \\ &\hat{\delta}(\mathcal{E}_N, \mathcal{E}_{Nh}) \leq \varepsilon \end{aligned}$$

where $\hat{\delta}(A, B)$, for A and B linear subspaces of $L^2(\Omega)$, denotes the gap between A and B. For $N \in \mathbb{N}$, $m(N)$ denotes the number of the eigenvalues less or equal to $\lambda_{m(N)}$, and $\mathcal{E}_N$ denotes the subspace of $L^2(\Omega)$ of dimension $m(N)$ spanned by the eigenfunctions associated to the eigenvalues λ_i with $i = 1, \cdots, m(N)$. Similarly, $\mathcal{E}_{Nh}$ stands for the subspace of $\mathbf{F}_h$ of dimension $m(N)$ spanned by the discrete eigenfunctions associated to λ_{ih} for $i = 1, \cdots, m(N)$.

Hence it remains to prove the validity of the hypotheses H1-H4. The paper [4] deals with the check of these assumptions. The main result regards hypothesis H3, which has been shown under two assumptions: the *commuting diagram property* of the finite element spaces and the following *regularity assumption*. Assume that Ω, μ and ε are such that for some $s > 1/2$ it holds

$$E \subseteq H^s(\Omega)^3 \qquad F \subseteq H^s(\Omega)^3, \tag{20}$$

with continuous embeddings.

This is the main assumption on the coefficients and on the domain. We point out that the data of most applications meet this requirement [11]. In addition to that, (20) is the minimal hypothesis for the standard edge element interpolant to be defined [1].

The convergence result is summarized in the following statement.

Theorem 2. *If Q_h, E_h and F_h enjoy the commuting diagram property and if hypotheses (20) are fulfilled, then, for h small enough, there exists a unique solution of problem (7) which satisfies the error estimate (9).*

To conclude, let us recall that on tetrahedra, the first and second type Nédélec finite elements [17, 18] as well as the Demkowicz–Vardapetyan elements [14] has been proved to satisfy the commuting diagram property. The analysis of the general hexahedral case is not so immediate as for the tetrahedral one. In particular, it is apparent that for the second type Nédélec elements the diagram cannot be written. On the other hand, Demkowicz–Vardapetyan and first type Nédélec elements satisfy the commuting diagram property on meshes of parallelepipeds.

Remark 1. The regularity hypotheses of Theorem 2 can be weakened according to the results presented in [11].

Remark 2. The result stated in Theorem 2 has been obtained also in [10] as a consequence of a more general theory.

3 Approximation of eigenvalues: edge elements versus penalized nodal elements

In this section we shall consider the finite element approximation of the Maxwell eigenvalue problem. For the sake of simplicity our computations are carried out in two space dimensions. Given a polygonal domain Ω, the problem under consideration reads: find λ and $\mathbf{u}$ such that

$$\begin{aligned} &\operatorname{curl} \mu^{-1} \operatorname{rot} \mathbf{u} = \lambda \varepsilon \mathbf{u} && \text{in } \Omega, \\ &\mathbf{u} \times \mathbf{n} = 0 && \text{on } \partial\Omega. \end{aligned} \tag{21}$$

In the following examples, μ and ε will be set equal to the identity matrix.

We shall compare two possible discretizations of (21). The first one consists of the standard edge element method, which has been proved to provide optimal results in [9, 4]. We consider the sequence V_h of lowest order triangular edge element spaces (with zero boundary condition on the tangential component) and, for a given h, we solve the discrete generalized eigenvalue problem

$$\begin{aligned}&\text{find } \lambda_h \text{ and } \mathbf{u}_h \in V_h \text{ such that } \mathbf{u}_h \not\equiv \mathbf{0} \text{ and}\\&(\operatorname{rot}\mathbf{u}_h, \operatorname{rot}\mathbf{v}) = \lambda_h(\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in V_h.\end{aligned} \tag{22}$$

It is known that the first N discrete eigenvalues λ_h of problem (22) are zero, where N is the number of internal vertices of the mesh and that the remaining positive eigenvalues provide optimal approximations to the first eigenvalues of (21) (sorted as an increasing sequence).

Our second discretization is the following one. Given a mesh of quadrilaterals, consider the space W_h of continuous piecewise biquadratic vectorfields with vanishing tangential component on the boundary and define $\mathbf{P}_h$ as the L^2 projection onto the space of discontinuous piecewise linear elements; then solve

$$\begin{aligned}&\text{find } \lambda_h \text{ and } \mathbf{u}_h \in W_h \text{ such that } \mathbf{u}_h \not\equiv \mathbf{0} \text{ and}\\&(\mathbf{P}_h \operatorname{rot}\mathbf{u}_h, \mathbf{P}_h \operatorname{rot}\mathbf{v}) + s(\mathbf{P}_h \operatorname{div}\mathbf{u}_h, \mathbf{P}_h \operatorname{div}\mathbf{v}) = \lambda_h(\mathbf{u}_h, \mathbf{v}) \quad \forall \mathbf{v} \in W_h,\end{aligned} \tag{23}$$

where s is a positive parameter to be chosen in a suitable way (we do not detail the choice of the parameter s, which can be done easily; in general s need not to be large if one is interested in the approximation of a limited number of eigenmodes). We shall refer to this method as the $Q_2 - P_1 - P_1$ element. The projection $\mathbf{P}_h$ is needed when the domain is not smooth or convex in order to capture the possible singularities of the eigenfunctions. There is a wide literature on the use of penalty (or regularized) formulations for the approximation of problem (21); the interested reader can refer to the papers [12, 13, 8], for instance. While the use of the projection $\mathbf{P}_h$ makes it possible to approximate a singular solution, on the other hand it introduces a number of zero frequencies in the spectrum of problem (23). For this reason, the qualitative behavior of the approximations provided by the two methods defined in (22) and (23) are comparable: both present a number of vanishing eigenvalues (and this number grows up as h goes to zero) and perform well for smooth solution where an optimal convergence can be observed. On the other hand the first method is a first order scheme, while the second one is quadratic.

The aim of our computations is to compare these methods for the approximation of problems which present singularities. To this aim, we shall consider an L-shaped cavity and different sequences of meshes. For the edge element approximation, we introduce the mesh sequences illustrated in Fig. 1: the first one consists of a standard uniform mesh, while the second one is a locally refined mesh in the vicinity of the reentrant corner. For the second scheme, we use the meshes shown in Fig. 2: here again we consider uniform or locally refined meshes.

We start with the representation of the convergence history for the three lowest eigenvalues. When uniform meshes are used, we plot the logarithm

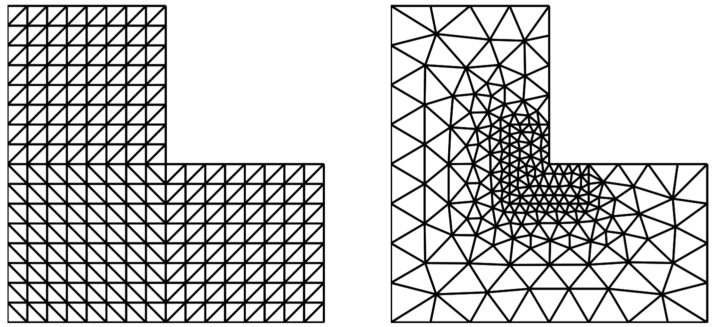

Fig. 1. Triangular meshes of the L-shaped cavity

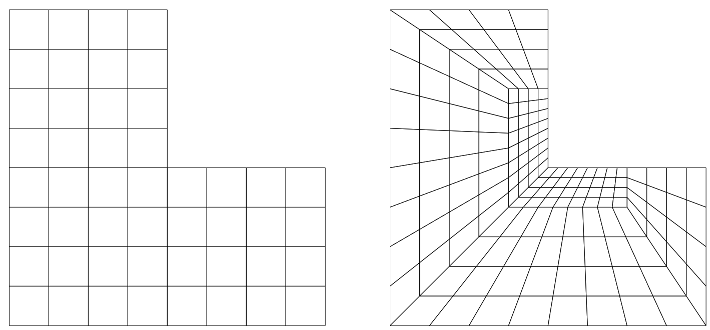

Fig. 2. Quadrilateral meshes of the L-shaped cavity

of the meshsize on the x-axis, while in case of refined mesh we consider the logarithm of the degrees of freedom number. Our first results are shown in Fig. 3, where the edge elements on uniform meshes are used. The other tests, represented in Figg. 4, 5 and 6, respectively, show the results obtained with edge elements on locally refined meshes, and the $Q_2 - P_1 - P_1$ element on uniform or refined meshes. It is apparent that the edge element perform better on locally refined meshes, while the $Q_2 - P_1 - P_1$ method provides better results on uniform meshes. This might seem a strange behavior, but can be interpreted with the remarks made in [2], where it is observed that the $Q_2 - P_1 - P_1$ method is not optimal with distorted meshes.

Finally, in Figg. 7 and 8, we compare directly the two methods for the computation of the first and second eigenvalue, respectively. We point out that the first eigenvalue is associated with a *singular* eigenfunction, while the second one corresponds to a *regular* one. The results are easily understood: in the singular case the methods perform in a similar way, the rate of convergence being driven by the regularity of the eigenfunction; on the other hand, the higher accuracy of the $Q_2 - P_1 - P_1$ method with respect

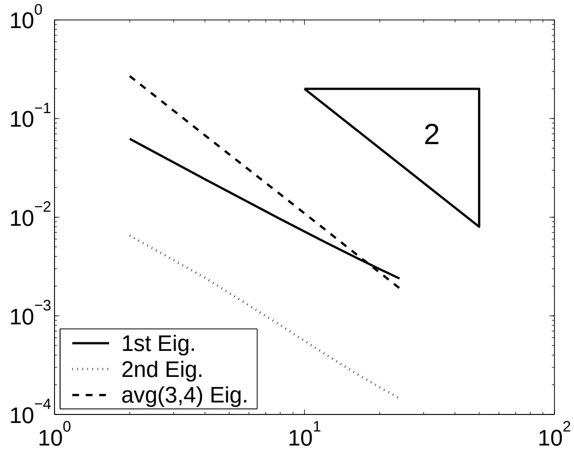

Fig. 3. Edge element on uniform meshes

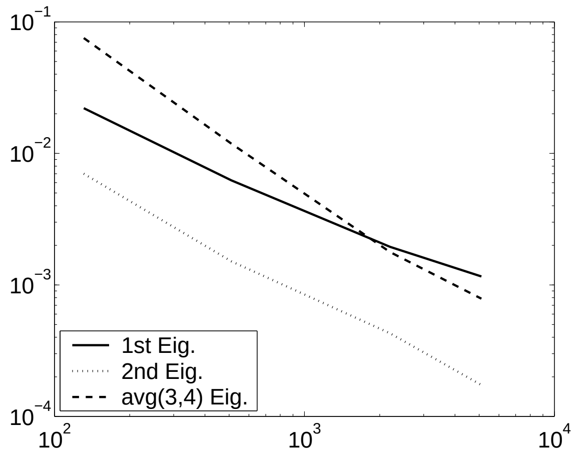

Fig. 4. Edge element on locally refined meshes

to the edge element scheme can be clearly seen when the regularity of the eigenfunction does not matter.

Acknowledgments

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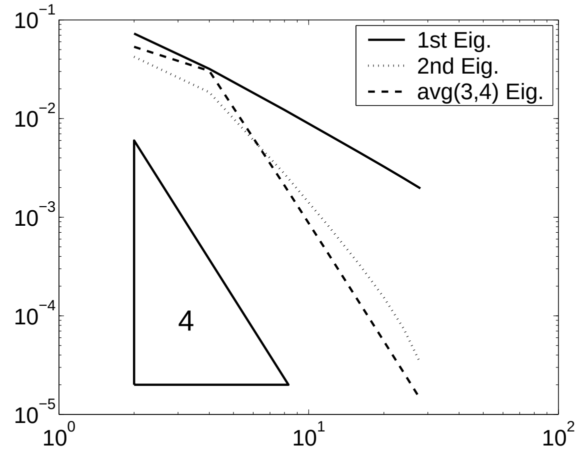

Fig. 5. $Q_2 - P_1 - P_1$ element on uniform meshes

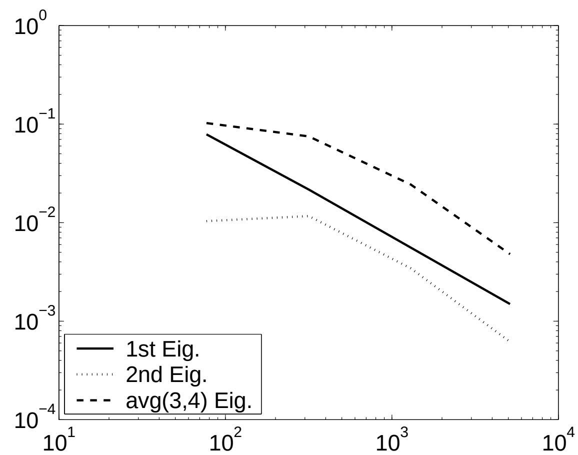

Fig. 6. $Q_2 - P_1 - P_1$ element on locally refined meshes

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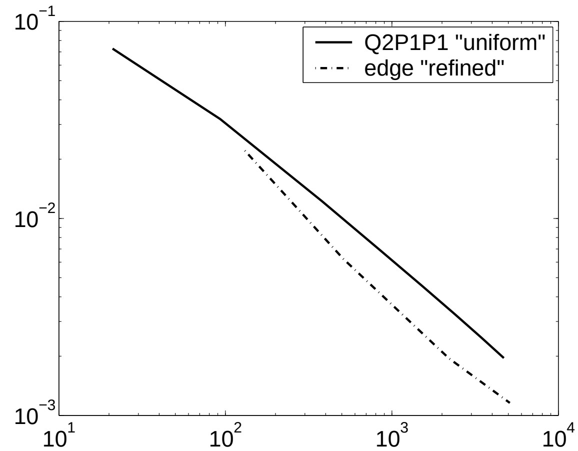

Fig. 7. Comparison of $Q_2 - P_1 - P_1$ and edge elements: first *singular* mode

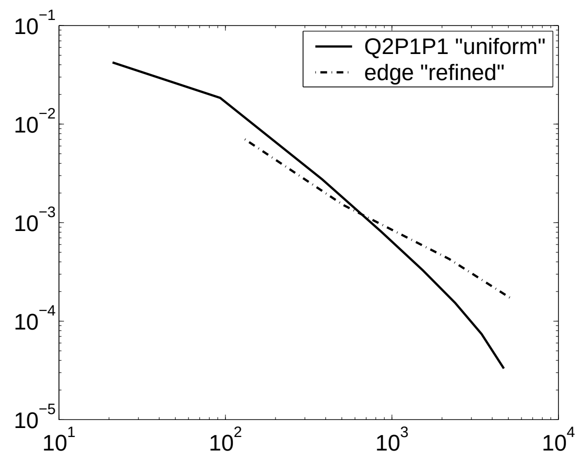

Fig. 8. Comparison of $Q_2 - P_1 - P_1$ and edge elements: second *regular* mode

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