A NOTE ON THE DE RHAM COMPLEX AND A DISCRETE COMPACTNESS PROPERTY

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ABSTRACT. The aim of this paper is to review the mathematical analysis of the eigenvalue problem associated with the Maxwell's system. Our analysis is quite general and can be applied to several families of edge finite element methods. Moreover we discuss the links between different conditions that guarantee the good approximations of the eigensolutions. In particular we prove that the commutativity of the de Rham complex implies the discrete compactness introduced by Kikuchi in [12].

1. INTRODUCTION

The approximation of the eigenvalue problem associated with the Maxwell's system has been the object of a wide study by engineers and mathematicians. However only recently the mathematical analysis of this problem has been carried out in a rigorous way.

In this paper we link in the spirit of [2, 3, 12] the de Rham complex and the discrete compactness property for finite element spaces. Indeed we show that several finite element methods proposed in the literature to face the approximation of the Maxwell's system satisfy the discrete compactness property introduced by Kikuchi in [12]. In particular basically all of the so called "edge" finite element spaces are well-suited for the approximation of the problem.

Results in this direction have been obtained in [5] and [3] where the analysis of any order tetrahedral Nédélec elements [15] has been proposed. Demkowicz, Monk, and Vardapetyan [14] proved that the discrete compactness introduced in [12] is satisfied by any order Nédélec elements. In order to do that, they use an inverse inequality; therefore thay have to assume the mesh to be quasiuniform (actually the quasiuniformity could be weakened in a sense described in the paper which does not allow however the mesh to be refined locally in a geometric way). Caorsi, Fernandes and Raffetto [9] studied the implications between discrete compactness property and the convergence of the approximation. Finally, let us quote that recently Kikuchi [13] proved the discrete compactness property for any order Nédélec edge element using the results presented in [3] and [1] and hence without any uniformity assumption on the mesh.

The outline of the paper is as follows. In the next section we present the eigenvalue problem and the functional spaces we are dealing with. In Section 3 we define the de Rham diagram property and recall the discrete compactness property. Finally, in Section 4 we state and prove our new result linking de Rham diagram and discrete compactness property.

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2. The Maxwell's cavity eigenproblem

Let Ω be a simply connected polyhedron in $\mathbb{R}^3$ (for more general situations see [5]). The eigenvalue problem we shall deal with reads as follows:

$$\text{find } (\lambda, \underline{p}) \text{ with } \underline{p} \not\equiv 0 \text{ s.t.}\qquad (1)$$
$$\begin{cases} \operatorname{curl}\operatorname{curl}\underline{p} = \lambda \underline{p} & \text{in } \Omega \\ \operatorname{div}\underline{p} = 0 & \text{in } \Omega \\ \underline{p} \times \underline{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the problem is symmetric we can confine ourselves to real eigenpairs. It is well known that problem (1) is compact and zero is not a frequency.

A variational formulation of (1) reads as follows. Consider the space

$$\begin{aligned} X = \ & H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega) \\ = \ & \{\underline{q} \in L^2(\Omega)^3 : \operatorname{curl}\underline{q} \in L^2(\Omega)^3,\ \operatorname{div}\underline{q} = 0,\ \underline{q}\times\underline{n} = 0 \text{ on } \partial\Omega\} \end{aligned}\qquad (2)$$

and

$$\begin{aligned} &\text{find } (\lambda, \underline{p}) \in \mathbb{R} \times X \text{ with } \underline{p} \not\equiv 0 \text{ s.t.} \\ &(\operatorname{curl}\underline{p}, \operatorname{curl}\underline{q}) = \lambda(\underline{p}, \underline{q}) \quad \forall \underline{q} \in X, \end{aligned}\qquad (3)$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)^3$.

This is not however a good formulation as far as the numerical approximation is concerned. Indeed a conforming approximation of (3) requires the divergence free constraint to be imposed exactly in the approximate scheme. This can be done (studies in this direction have been done for the Stokes problem, for instance), but in general it may be complicated and one of the most widely used strategy to approximate (1) is to consider the following variational formulation.

Consider the space

$$\begin{aligned} Q = \ & H_0(\mathrm{curl};\Omega) \\ = \ & \{\underline{q} \in L^2(\Omega)^3 : \operatorname{curl}\underline{q} \in L^2(\Omega)^3,\ \underline{q}\times\underline{n} = 0 \text{ on } \partial\Omega\} \end{aligned}\qquad (4)$$

and

$$\begin{aligned} &\text{find } (\lambda, \underline{p}) \in \mathbb{R} \times Q \text{ with } \underline{p} \not\equiv 0 \text{ s.t.} \\ &(\operatorname{curl}\underline{p}, \operatorname{curl}\underline{q}) = \lambda(\underline{p}, \underline{q}) \quad \forall \underline{q} \in Q. \end{aligned}\qquad (5)$$

It turns out that (5) is no longer equivalent to (1). However it is well known that the only difference, as far as the eigenvalues are concerned, is that in (5) a zero frequency is added which is associated with the infinite dimensional eigenspace $\operatorname{grad} H_0^1(\Omega)$. What is usually done in the numerical approximation of (1) is to discretize (5) and to discard the zero eigenvalues. So that the discretization of (5) reads (here Q_h is a finite element approximation of Q):

$$\begin{aligned} &\text{find } (\lambda_h, \underline{p}_h) \in \mathbb{R} \times Q_h \text{ with } \underline{p}_h \not\equiv 0 \text{ s.t.} \\ &(\operatorname{curl}\underline{p}_h, \operatorname{curl}\underline{q}) = \lambda_h(\underline{p}_h, \underline{q}) \quad \forall \underline{q} \in Q_h. \end{aligned}\qquad (6)$$

It became apparent soon that the discrete space has to be chosen very carefully in order to avoid the presence of spurious modes which could be of two different kinds. The first very annoying possibility is that the zero frequency spreads out polluting the whole spectrum. The second family of spurious modes has nothing to do with the zero frequency, but is generated by the numerical method. For a discussion about this point and an example of this second possibility we refer to [5]. In [4] a proof of the existence of spurious eigenvalues of the second kind is given.

3. Preliminary definitions

Following [2] we introduce the following notation.

$$
\begin{array}{ll}
 & W = H^1_0(\Omega) = \{w \in L^2(\Omega) : \operatorname{grad} w \in L^2(\Omega),\ w = 0 \text{ on } \partial\Omega\}, \\
(7) & Q = H_0(\operatorname{curl};\Omega), \\
 & V = H_0(\operatorname{div};\Omega) = \{\underline{v} \in L^2(\Omega)^3 : \operatorname{div} \underline{v} \in L^2(\Omega),\ \underline{v}\cdot\underline{n} = 0 \text{ on } \partial\Omega\}, \\
 & S = L^2(\Omega).
\end{array}
$$

The exactness of the following sequence expresses the de Rham complex diagram.

$$
0 \to W \xrightarrow{\operatorname{grad}} Q \xrightarrow{\operatorname{curl}} V \xrightarrow{\operatorname{div}} S/\mathbb{R} \to 0 \qquad (8)
$$

Given a regular sequence of meshes $\mathcal{T}_h$, let us now introduce finite element subspaces W_h, Q_h, V_h, and S_h. If we are using standard piecewise polynomials it is clear that the continuity requirements in order to have a conforming approximation are the following ($\underline{n}$ denotes the normal of the interfaces between two elements):

$$
\begin{array}{ll}
w \in W_h : & w \text{ continuous}, \\
\underline{q} \in Q_h : & \underline{q} \times \underline{n} \text{ continuous}, \\
\underline{v} \in V_h : & \underline{v} \cdot \underline{n} \text{ continuous}, \\
s \in S_h : & s \text{ may be discontinuous}.
\end{array} \qquad (9)
$$

Let us denote by $\tilde{W}$, $\tilde{Q}$, $\tilde{V}$, and $\tilde{S}$ smooth dense subspaces of W, Q, V, and S, respectively, that allow the definition of suitable interpolation operators

$$
\begin{array}{l}
\pi_h^W : \tilde{W} \to W_h, \\
\pi_h^Q : \tilde{Q} \to Q_h, \\
\pi_h^V : \tilde{V} \to V_h, \\
\pi_h^S : \tilde{S} \to S_h.
\end{array} \qquad (10)
$$

We introduce now the following diagram

$$
\begin{array}{cccccccccc}
0 \to & \tilde{W} & \xrightarrow{\operatorname{grad}} & \tilde{Q} & \xrightarrow{\operatorname{curl}} & \tilde{V} & \xrightarrow{\operatorname{div}} & \tilde{S}/\mathbb{R} & \to 0 \\
 & \downarrow \pi_h^W & & \downarrow \pi_h^Q & & \downarrow \pi_h^V & & \downarrow \pi_h^S & \\
0 \to & W_h & \xrightarrow{\operatorname{grad}} & Q_h & \xrightarrow{\operatorname{curl}} & V_h & \xrightarrow{\operatorname{div}} & S_h/\mathbb{R} & \to 0
\end{array} \qquad (11)
$$

and the following definition.

Definition 1. *We say that the* commuting diagram property *is fulfilled if*

- *the two rows of (11) are exact*
- *the whole diagram (11) commutes.*

Remark 1. Let us emphasize that not only the exactness of the second row in (11) is important, but also the fact that the horizontal arrows commute with the vertical ones turns out to be a fundamental property for the approximation of the eigenvalues of (1).

Remark 2. There are basically two known families of finite element spaces that meet the commuting diagram property. The first one is the Raviart–Nédélec–Thomas one [17, 15, 6, 2] (first and second kind on tetrahedra and first kind on hexahedra) known also as face-edge element family. The second one has been introduced more recently by Demkowicz and Vardapetyan (see [18, 11]) and turns out to be a generalization of the two-dimensional mixed BDM element [7] and of the Nédélec element of the second kind [16] for tetrahedra or of the first kind [15] for hexahedra. The second family seems to be more flexible, since it allows local refinements in p and h.

Let us now fix our attention to the spaces W_h and Q_h and recall the definition of discrete compactness [12, 10].

Definition 2. *We say that the pair* (W_h, Q_h) *satisfies the* discrete compactness property *if given a sequence* $\underline{q}_h \in Q_h$ *such that*

- $(\operatorname{grad} w_h, \underline{q}_h) = 0 \quad \forall w_h \in W_h$
- $||\underline{q}_h||_Q \leq 1$

then there exist a subsequence $\underline{q}_{h_n}$ *converging strongly in* $L^2(\Omega)$.

Remark 3. It is clear that the corresponding property for (W_h, Q_h) replaced by (W, Q) holds true due to the compact embedding

$$X \hookrightarrow L^2(\Omega) \tag{12}$$

Remark 4. Using the exactness of the second row in (11), the first hypothesis on $\{\underline{q}_h\}$ in the previous definition says that $\underline{q}_h$ is orthogonal to the kernel of the curl operator acting from Q_h to V_h.

4. De Rham diagram and discrete compactness

We shall make use of the following mixed problem

$$\text{given } \underline{g} \in V, \text{ find } (\underline{p}, \underline{u}) \in Q \times \operatorname{curl}(Q) \text{ such that}$$
$$\begin{cases} (\underline{p}, \underline{q}) + (\operatorname{curl} \underline{q}, \underline{u}) = 0 & \forall \underline{q} \in Q \\ (\operatorname{curl} \underline{p}, \underline{v}) = (\underline{g}, \underline{v}) & \forall \underline{v} \in \operatorname{curl}(Q) \end{cases} \tag{13}$$

and the corresponding approximation

$$\text{given } \underline{g} \in V, \text{ find } (\underline{p}_h, \underline{u}_h) \in Q_h \times \operatorname{curl}(Q_h) \text{ such that}$$
$$\begin{cases} (\underline{p}_h, \underline{q}_h) + (\operatorname{curl} \underline{q}_h, \underline{u}_h) = 0 & \forall \underline{q}_h \in Q_h \\ (\operatorname{curl} \underline{p}_h, \underline{v}_h) = (\underline{g}, \underline{v}_h) & \forall \underline{v}_h \in \operatorname{curl}(Q_h). \end{cases} \tag{14}$$

Problem (13) has been intensively studied in these last years. It turns out that it is well posed and that its approximation (14) is stable as a mixed problem [8] provided the commuting diagram property is satisfied. Moreover, in [2] the following important estimate has been proven (actually, the inequality has been proved assuming Ω convex, but it can be easily extended to the more general situation).

Proposition 1. *Suppose that the commuting diagram property (see definition 1) is satisfied and that* $\tilde{Q}$ *can be chosen such that if* $\underline{p}$ *solves (13) then it belongs to* $\tilde{Q}$*. Then* if $\underline{g}$ belongs to V_h

$$||\underline{p} - \underline{p}_h||_0 \leq ||\underline{p} - \pi_h^Q \underline{p}||_0. \tag{15}$$

We are now in position to state and prove the new result concerning the discrete compactness property.

Theorem 1. *Let* W_h, Q_h, V_h, *and* S_h *be such that the commuting diagram property is satisfied (see definition 1). Assume moreover that* $\tilde{Q}$ *can be chosen such that if* $\underline{p}$ *solves (13) then it belongs to* $\tilde{Q}$ *with the a priori bound*

$$||\underline{p}||_{\tilde{Q}} \leq C||\underline{g}||_V \tag{16}$$

and that the interpolation operator π_h^Q approximates the identity in norm, that is

$$||\underline{p} - \pi_h^Q \underline{p}||_0 \leq \rho(h)||\underline{p}||_{\tilde{Q}}, \tag{17}$$

with $\rho(h)$ going to zero as h tends to zero. Then (W_h, Q_h) satisfies the discrete compactness property (see definition 2).

Proof. Let $\{\underline{q}_h\}$ be a sequence in Q_h as in definition 2. We call $\underline{p}(h)$ the first component of the solution to problem (13) with $\underline{g} = \operatorname{curl} \underline{q}_h$ (the notation emphasizes the dependence on h). From the a priori estimate for problem (13) it follows

$$||\underline{p}(h)||_{\tilde{Q}} \leq C||\underline{g}||_V = C||\operatorname{curl} \underline{q}_h||_V = C||\operatorname{curl} \underline{q}_h||_0 \leq C||\underline{q}_h||_Q \leq C. \tag{18}$$

Moreover the first equation of (13) says that $\operatorname{div} \underline{p}(h) = 0$. It follows that $\{\underline{p}(h)\}$ is bounded uniformly in $X = H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\Omega)$. Due to the compactness of X in $L^2(\Omega)$ there exists a subsequence $\{\underline{p}(h_n)\}$ converging strongly to some $\underline{p} \in L^2(\Omega)$, that is

$$\underline{p}(h_n) \to \underline{p} \quad \text{in } L^2(\Omega). \tag{19}$$

We will show that $\{\underline{q}_{h_n}\}$ converges strongly to $\underline{p}$ in $L^2(\Omega)$.

For simplicity, let us denote the subsequence h_n still by h. We observe that there exists $\underline{u}_h \in \operatorname{curl}(Q_h)$ such that $(\underline{p}_h, \underline{u}_h)$ is the solution of (14) with $g = \operatorname{curl} \underline{p}_h$ and $\underline{p}_h = \underline{q}_h$. Indeed the second equation is trivial; given $\underline{p}_h = \underline{q}_h$ the first equation provides the definition of $\underline{u}_h$ as follows:

$$\begin{aligned} &1) \quad \text{find } \underline{\alpha}_h \in Q_h \text{ such that} \\ &\qquad (\operatorname{curl} \underline{\alpha}_h, \operatorname{curl} \underline{\beta}_h) = -(\underline{q}_h, \underline{\beta}_h) \quad \forall \underline{\beta}_h \in Q_h, \\ &2) \quad \text{define } \underline{u}_h = \operatorname{curl} \underline{\alpha}_h. \end{aligned} \tag{20}$$

It turns out that everything in (20) makes sense. In particular the equation involved in the first step is solvable because of the compatibility $(\underline{q}_h, \operatorname{grad} w_h) = 0 \ \forall w_h \in W_h$. Actually, $\underline{\alpha}_h$ is defined up to the gradient of an element in W_h which however does not influence the definition of $\underline{u}_h$.

In conclusion, $\underline{p}(h)$ solves (13) with $\underline{g} = \operatorname{curl} \underline{q}_h$ and $\underline{q}_h$ solves (14) with the same datum. Therefore we can use (15), (17), and the definition of $\underline{p}(h)$ and $\underline{q}_h$ to get

$$||\underline{p}(h) - \underline{q}_h||_0 \leq ||\underline{p}(h) - \pi_h^Q \underline{p}(h)||_0 \leq \rho(h)||\underline{p}(h)||_{\tilde{Q}} \leq C\rho(h)||\operatorname{curl} \underline{q}_h||_V \leq C\rho(h). \tag{21}$$

Putting together (19) and (21) and using the triangular inequality we obtain the convergence of $\underline{q}_h$ to $\underline{p}$ in $L^2(\Omega)$. □

Remark 5. The proof above contains the main argument showing the relationship between commuting diagram and the discrete compact property. However it cannot be applied directly to the edge element families. Indeed the a priori estimate (16) in general does not hold true. In fact, if $\underline{g}$ is only in V in general the first component $\underline{p}$ of the solution of problem (13) does not meet the requirement stated in [1] for the interpolant π_h^Q to be well-defined. In particular we would need $\operatorname{curl} \underline{p}$ to be in $L^p(\Omega)$ $(p > 2)$, while it only belongs to $L^2(\Omega)$.

On the other hand it is clear that hypothesis (16) is used only to get the estimate (18). Actually, it turns out that the bound (18) is true in general even if the a priori estimate (16) may be false. This can be easily seen observing that in this case $\underline{g}$ is regular enough in order to define $\pi_h^Q \underline{p}(h)$; then the same argument as in [1] (p. 856), which has been already used in this framework in [3] and [13], can be applied to this situation.

We state the related result in the following corollary. Note that the edge finite elements of second type on hexahedra [16] do not satisfy the commuting diagram property; hence they have not been included in the present result. On the other hand, to the best author's knowledge, there is no numerical evidence in the literature about the convergence of this element when applied to eigenvalue problems.

Corollary 1. *The edge finite element spaces of the Raviart–Nédélec–Thomas (first and second type on tetrahedra and first kind on hexahedra) and of the Demkowicz–Vardapetyan family satisfy the discrete compactness property.*

References

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potential in three-dimensional nonsmooth domains*, Math. Methods Appl. Sci., 21 (1998), pp. 823–864.

[2] D. N. Arnold, R. S. Falk, and R. Winther, *Multigrid in H(div) and H(curl)*, Numer. Math., (1998). To appear.

[3] D. Boffi, *Fortin operator and discrete compactness for edge elements*. Numer. Math., to appear, 1998.

[4] D. Boffi, F. Brezzi, and L. Gastaldi, *On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form*, Math. Comp., (1998). To appear.

[5] D. Boffi, P. Fernandes, L. Gastaldi, and I. Perugia, *Computational models of electromagnetic resonators: analysis of edge element approximation*, SIAM J. Numer. Anal., 36 (1999), pp. 1264–1290.

[6] A. Bossavit, *Un noveau point de vue sur les éléments finis mixtes*, Matapli (bulletin de la Société de Mathématiques Appliquées et Industrielles), (1989), pp. 23–35.

[7] F. Brezzi, J. Douglas, and L. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[8] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.

[9] S. Caorsi, P. Fernandes, and M. Raffetto, *On the convergence of galerkin finite element approximations of electromagnetic eigenproblems*. preprint.

[10] F. Chatelin, *Spectral approximation of linear operators*, Academic Press, Inc., 1983.

[11] L. Demkowicz, P. Monk, L. Vardapetyan, and W. Rachowicz, *De Rham diagram for hp finite element spaces*. TICAM Report 99-6.

[12] F. Kikuchi, *On a discrete compactness property for the Nédélec finite elements*, J. Fac. Sci., Univ. Tokyo, Sect. I A, 36 (1989), pp. 479–490.

[13] ———, *Theoretical analysis of Nédélec edge elements*, in Proc. of Comp. Eng. Conference, vol. 4, Tokyo, Japan, May 26-28 1999.

[14] P. Monk and L. Demkowicz, *Discrete compactness and the approximation of Maxwell's equations in* $\mathbb{R}^3$, 1999. Math. Comp., to appear.

[15] J. C. Nédélec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[16] ———, *A new family of mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 50 (1986), pp. 57–81.

[17] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems*, in Mathematical Aspects of the Finite Element Method, I. Galligani and E. Magenes, eds., vol. 606 of Lecture Notes in Math., New York, 1977, Springer-Verlag, pp. 292–315.

[18] L. Vardapetyan and L. Demkovicz, *hp-adaptive finite elements in electromagnetics*, Comp. Meth. Appl. Mech. Eng., 169 (1999), pp. 331–344.

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