FINITE ELEMENT APPROXIMATION OF MAXWELL'S EIGENPROBLEM

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We consider the penalized formulation of Maxwell's eigenproblem. It is well-known that nodal finite elements fails in approximating this problem in the case of non-convex polyhedral (or polygonal) domains. Here we introduce two non standard finite element methods: the first one is based on nonconforming elements, while the second one on a projection procedure. We shall show that the nonconforming method is not consistent with this problem. The projected method instead gives good numerical results.

1 Introduction

Let us consider the Maxwell's cavity eigenproblem:

$$\begin{cases} \underline{\mathrm{curl}}\left(\mu^{-1}\,\mathrm{rot}\,\underline{u}\right) = \lambda\varepsilon\underline{u} & \text{in } \Omega \\ \mathrm{div}(\varepsilon\underline{u}) = 0 & \text{in } \Omega \\ \underline{u}\cdot\underline{t} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\underline{t}$ is the counterclockwise oriented tangent versor to the boundary $\partial\Omega$.

The unknowns λ and $\underline{u}$ represent respectively the angular frequency and the electric field phasor; μ and ε are known and stand respectively for the magnetic permeability and the electric permittivity. For simplicity, we shall consider from now on $\mu = \varepsilon = 1$. Here Ω is a two-dimensional polygon which in particular may be non-convex.

One of the main difficulties in discretizing the eigenvalue problem associated with Maxwell's equations is to deal with the divergence-free constraint. Over the years many attempts to circumvent this trouble have been proposed; among other strategies, several authors have suggested the following penalization procedure ($s > 0$ is a suitably chosen parameter):

$$\begin{array}{ll} \text{find } \lambda \in \mathbb{R} \text{ and } \underline{u} \not\equiv 0 \text{ such that} & \\ \underline{\mathrm{curl}}\, \mathrm{rot}\, \underline{u} - s\, \underline{\nabla}\, \mathrm{div}\, \underline{u} = \lambda \underline{u} & \text{in } \Omega \\ \underline{u} \times \underline{n} = 0 & \text{on } \partial\Omega \\ \mathrm{div}\, \underline{u} = 0 & \text{on } \partial\Omega, \end{array} \tag{2}$$

It was soon realized that this idea gives good results on convex domains, while in non-convex domains it turned out to be not completely satisfactory [1].

In the next section, we associate with (2) a variational formulation; a crucial role will be played by the functional space in which the eigenfunctions are looked for. Note that the natural energy norm associated with problem (2) is $(||\,\underline{\mathrm{curl}}(\cdot)||_0^2 + ||\,\mathrm{div}(\cdot)||_0^2)^{1/2}$, which is equivalent to the norm of $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ thanks to the boundary conditions.

Recently Costabel and Dauge [2] studied the eigensolutions of (2) depending on the chosen functional setting, thus explaining the reason why the approximating eigensolutions of the penalized problem do not behave correctly if the domain contains a reentrant corner. In fact in this case a vector field which has both divergence and rotational bounded in $L^2(\Omega)$ does not necessarily have the gradient bounded in $L^2(\Omega)$, that is $H^1(\Omega) \cap H_0(\mathrm{rot};\Omega)$ is a proper subspace of $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$. Moreover, $H^1(\Omega) \cap H_0(\mathrm{rot};\Omega)$ is closed in $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ with respect to the energy norm. Therefore if we consider a finite element space which is contained in H^1, then we cannot approximate those "singular" elements of $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ which do not have the gradient bounded in $L^2(\Omega)$.

On the other hand a piecewise polynomial vector field which has both divergence and rotational in $L^2(\Omega)$ must belong to $H^1(\Omega)$, because its normal and tangential components are continuous along the interelement sides.

Therefore we have to consider non-standard finite element methods to discretize problem (2). With the aim of weakening in some sense the continuity constraint along the interelement boundaries, we propose two different schemes: the first one is based on nonconforming piecewise linear polynomials while the second one on a projection of the curl and the divergence. We shall show that the nonconforming method is not consistent with the problem, while the projection method gives quite good results.

In the following section we introduce the variational formulation associated with (2). The next two sections are devoted to the description and the analysis of the nonconforming finite element method and the projection method respectively.

2 Setting of the problem

We are interested in the approximation of problem (1) via the penalization formulation (2). We shall deal with the following variational formulation of problem (2). Given a Hilbert space V, which will be fixed later on, our problem reads:

$$\begin{aligned}&\text{find } \lambda \in \mathbb{R} \text{ and } \underline{u} \in V, \text{ with } \underline{u} \neq 0 \text{ such that}\\ &(\underline{\operatorname{curl}}\,\underline{u}, \underline{\operatorname{curl}}\,\underline{v}) + s(\operatorname{div}\underline{u}, \operatorname{div}\underline{v}) = \lambda(\underline{u}, \underline{v}) \quad \forall \underline{v} \in V,\end{aligned} \tag{3}$$

where $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$.

The natural functional setting is $V = H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$. Indeed, in [2] it has been proved that with this choice the eigenvalues of (3) tend to those of (1) as s increases. In particular the eigensolutions of (3) can be separated in two families: *the first one* contains eigenfunctions with zero divergence, the values of their corresponding frequencies are independent of s and coincide with those of the eigensolutions of (1); while the eigenvalues of *the second one* increase linearly as s tends to infinity.

If the domain Ω is convex it is well-known that the space $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ coincides with $H^1(\Omega) \cap H_0(\mathrm{rot};\Omega)$ and that the corresponding norms are equivalent. Moreover in [3] it has been proved that the bilinear form in the left-hand side of (3) is coercive in $H^1(\Omega) \cap H_0(\mathrm{rot};\Omega)$ also if the domain is non-convex. Hence one might think to choose $V = H^1(\Omega) \cap H_0(\mathrm{rot};\Omega)$ in (3). However this choice, as shown in [2], leads to a problem which is completely different from (1) when the domain is not regular. This is due to the fact that $H^1(\Omega) \cap H_0(\mathrm{rot};\Omega)$ is closed in $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$.

This fact shows why no standard finite element approximation of (3) can give reasonable results when the space $H^1(\Omega) \cap H_0(\mathrm{rot};\Omega)$ is a proper subspace of $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ (and this is the case, for instance, when Ω has a reentrant corner). In fact it is not difficult to prove that any conforming finite element subspace of $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ necessarily belongs to $H^1(\Omega)$ too.

These remarks open the question about the construction of non-standard finite elements which are able to approximate a singular function of $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ which is not in $H^1(\Omega)$.

3 A nonconforming method

Let Ω be a polygon and let $\mathcal{T}_h$ be a regular finite element partition of it in triangles. We define the finite element space V_h we shall deal with in the following way: an element $\underline{v}_h$ of V_h restricted to a triangle of $\mathcal{T}_h$ is a linear polynomial in each component and $\underline{v}_h$ is continuous at the midpoints of the interelement sides. Moreover the boundary conditions are prescribed on the tangential component of $\underline{v}_h$ at the midpoints of the boundary edges.

As it is usual with nonconforming discretizations, since V_h is not contained in $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$, we use a discrete version of (3) which is obtained triangle by triangle as follows:

$$
\begin{aligned}
&\text{find } \lambda_h \in \mathbb{R} \text{ and } \underline{u}_h \in V_h \text{ with } \underline{u}_h \neq 0 \text{ such that}\\
&\textstyle\sum_{T\in\mathcal{T}_h} \{(\underline{\mathrm{curl}}\,\underline{u}_h, \underline{\mathrm{curl}}\,\underline{v}_h)_T + s(\mathrm{div}\,\underline{u}_h, \mathrm{div}\,\underline{v}_h)_T\} = \lambda_h(\underline{u}_h, \underline{v}_h) \quad \forall \underline{v}_h \in V_h,
\end{aligned}
\tag{4}
$$

where $(\varphi,\psi)_T = \int_T \varphi\psi$.

In order to make the notation easier, we call $a(\underline{u},\underline{v})$ and $a_h(\underline{u}_h,\underline{v}_h)$ the left-hand sides of (3) and (4), respectively.

Next we define:

$$
|\underline{u}|_h^2 = a_h(\underline{u},\underline{u}), \qquad ||\underline{u}||_h^2 = ||\underline{u}||_0^2 + |\underline{u}|_h^2. \tag{5}
$$

Notice that if $\underline{u}$ belongs to V then $a_h(\underline{u},\underline{u})$ coincides with $a(\underline{u},\underline{u})$ and, thanks to the boundary conditions, $|\underline{u}|_h$ is a norm equivalent to $||\underline{u}||_V$.

On the other hand the following proposition, which can be easily proved by induction, shows that (5) defines only a seminorm on V_h.

Proposition 1 *Let $\mathcal{T}_h$ be a triangulation of Ω which contains n_t triangles and denote by $I\!K_h$ the kernel of $a_h(\cdot,\cdot)$, that is*

$$
I\!K_h = \{\underline{v}_h \in V_h : a_h(\underline{v}_h,\underline{v}_h) = 0\}. \tag{6}
$$

Then the dimension of $I\!K_h$ is $n_t + 1$.

An immediate consequence of the previous proposition is that the zero frequency is a solution of problem (4) with multiplicity n_t+1, the corresponding eigenspace being $I\!K_h$.

Let $\tilde{V}$ be the orthogonal complement (with respect to $L^2(\Omega)$) of $I\!K_h$ in V_h, that is

$$
V_h = I\!K_h \oplus \tilde{V}, \quad \tilde{V} \perp I\!K_h. \tag{7}
$$

The following two proposition can be proved relating the elements in $\tilde{V}$ with the Raviart-Thomas elements of lowest degree (approximating $H(\mathrm{div};\Omega)$) and their rotated counterpart (approximating $H_0(\mathrm{rot};\Omega)$). The first one states the coerciveness of the discrete bilinear form a_h on $\tilde{V}$, while the second one is an approximation property of the space $\tilde{V}$ with respect to the seminorm (5).

Proposition 2 *There exists $\gamma > 0$ independent of h such that for every $\underline{v}_h \in \tilde{V}$ the following ellipticity property holds true:*

$$
a_h(\underline{v}_h,\underline{v}_h) \geq \gamma ||\underline{v}_h||_h^2. \tag{8}
$$

Proposition 3 *Let us suppose that* $\operatorname{div} \underline{u}$ *and* $\underline{\operatorname{curl}}\, \underline{u}$ *belong to* $H^1(\Omega)$. *Then there exists* $\underline{v}_h \in \tilde{V}$ *such that*

$$|\underline{u} - \underline{v}_h|_h \leq Ch \left(||\operatorname{div} \underline{u}||_1^2 + ||\operatorname{rot} \underline{u}||_1^2 \right)^{1/2} . \tag{9}$$

Hence if we consider problem (4) with V_h substituted by $\tilde{V}$, it admits only solutions with nonvanishing frequency which should approximate the eigensolutions of problem (3).

All these properties induce us to think that the nonconforming finite element method could avoid the troubles related to the approximation of singular functions (see the previous section). Indeed the qualitative behavior of the computed eigenvalues on an L-shaped domain agrees with the one observed for the continuous problem (see fig. 1). On the other hand fig. 2 shows that the computed eigensolution are not well approximated, in particular the divergence free constraint induced by the penalization forces the field to be aligned with the sides of the triangles of the mesh.

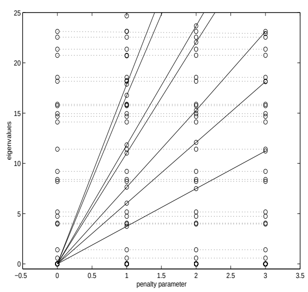

Fig.1 Eigenvalues on the L-shaped domain

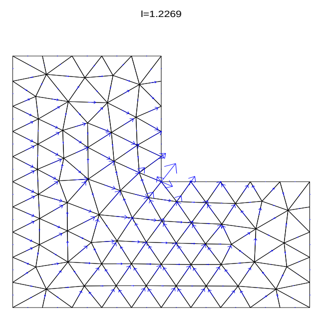

Fig.2 Computed singular eigensolution.

A deeper analysis of the approximation properties for the space $\tilde{V}$ shows that in general there is no convergence in $L^2(\Omega)$.

Let us consider a square domain of side 1 subdivided into n^2 equal subsquares each of them divided into two triangles. The side of each subsquare is $h = 1/n$. Then let us consider $\underline{u} \in H_0(\operatorname{rot}; \Omega) \cap H(\operatorname{div}; \Omega)$ such that $\underline{u} = (u, 0)$ with the first component u which is constant with respect to the variable x. Hence it is clear that $\operatorname{div} \underline{u} = 0$. Figs. 3 and 4 show the projections $\underline{u}_0$ and $\underline{u}_1$ of $\underline{u}$ onto $\mathbb{K}_h$ and $\tilde{V}$, respectively, in the case of $\underline{u} = (\sin \pi y, 0)$.

Since the basis functions of the nonconforming space are orthogonal, it is not difficult to calculate the norm of the projections and then pass to the

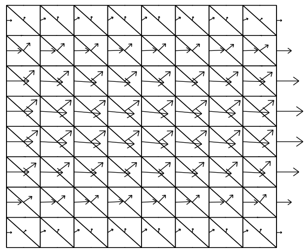

Fig.3 Projection onto $I\!K_h$

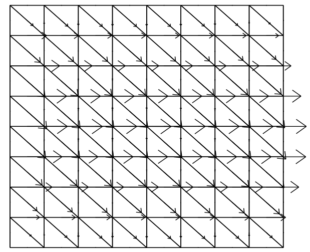

Fig.4 Projection onto $\tilde{V}$

limit as h goes to zero. This computation gives:

$$\begin{array}{l} \lim_{h\to o} ||\underline{u}_0||_0^2 = \frac{1}{2}||\underline{u}||_0^2 \\ \lim_{h\to o} ||\underline{u}_1||_0^2 = \frac{1}{2}||\underline{u}||_0^2. \end{array} \tag{10}$$

The limits in (10) show that the subspace $\tilde{V}$ of nonconforming finite elements cannot approximate a smooth function with respect to the $L^2(\Omega)$-norm.

4 A projection method

Another way to weaken the continuity conditions required to finite elements belonging to $H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega)$ is to consider a mixed formulation of problem (3), which has ben introduced for fluid-structure interaction problems in [4,5] and analyzed also in [6].

Let us introduce the following new variables

$$p = -\,\underline{\mathrm{curl}}\,\underline{u} \in L^2_0(\Omega), \quad r = -s\,\mathrm{div}\,\underline{u} \in L^2(\Omega), \tag{11}$$

where $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$ of the elements with zero mean value. Then problem (3) can be rewritten in the following way:

$$\begin{array}{l} \text{Find } \lambda \in \mathbb{R} \text{ such that there exist } (p,r) \in L^2_0(\Omega) \times L^2(\Omega) \text{ and} \\ \underline{u} \in H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega) \text{ with } \underline{u} \neq 0 : \\ \quad (p,q) + (\mathrm{rot}\,\underline{u}, q) = 0 \quad \forall q \in L^2_0(\Omega) \\ \quad \dfrac{1}{s}(r,t) + (\mathrm{div}\,\underline{u}, t) = 0 \quad \forall t \in L^2(\Omega) \\ \quad (\mathrm{rot}\,\underline{v}, p) + (\mathrm{div}\,\underline{v}, r) = -\lambda(\underline{u},\underline{v}) \quad \forall \underline{v} \in H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega). \end{array} \tag{12}$$

Let us consider a quadrilateral mesh and the following finite dimensional sub-

Table 1. Discrete eigenvalues computed by nonconforming elements on different meshes

exact	unstrucured	crisscross	8 x 8	4 x 8	4 x 12	4 x 16
1.00000	1.99344	1.99570	1.58462	1.54908	1.52755	1.51688
1.00000	2.01716	1.99570	2.77038	2.76339	2.79978	2.81840
2.00000	4.04132	3.98283	3.74459	3.88888	4.05837	4.15651
4.00000	7.81135	7.93045	8.16883	7.15018	6.63358	6.39262
4.00000	8.08403	7.93045	8.45338	8.25400	8.48643	8.61389
5.00000	9.88606	9.89190	9.06584	9.17021	9.45341	9.56733
5.00000	10.1795	9.89190	12.1035	11.3006	11.2833	11.2689
8.00000	15.9128	15.7237	14.0959	14.0350	14.5355	14.1565
9.00000	17.0469	17.6406	17.8526	15.4414	14.5829	14.8470
9.00000	17.4825	17.6406	19.6844	16.3875	16.4799	16.5313

spaces:

$$\begin{aligned}\Sigma_h &= \{\underline{v} \in H_0(\mathrm{rot};\Omega) \cap H(\mathrm{div};\Omega) : \\ &\quad \underline{v} \text{ is a continuous piecewise biquadratic vector field}\}, \\ Q_h &= \{q_h \in L^2_0(\Omega) : q_h \text{ is piecewise linear}\}, \\ T_h &= \{t_h \in L^2(\Omega) : t_h \text{ is piecewise linear}\}.\end{aligned} \tag{13}$$

Then the discrete counterpart of problem (12) reads:

$$\begin{aligned}&\text{Find } \lambda_h \in \mathbb{R} \text{ such that exist}(p_h, r_h) \in Q_h \times T_h \text{ and} \\ &\underline{u}_h \in \Sigma_h \text{ with } \underline{u}_h \neq 0 : \\ &\quad (p_h, q_h) + (\mathrm{rot}\, \underline{u}_h, q_h) = 0 \quad \forall q_h \in Q_h \\ &\quad \frac{1}{s}(r_h, t_h) + (\mathrm{div}\, \underline{u}_h, t_h) = 0 \quad \forall t_h \in T_h \\ &\quad (\mathrm{rot}\, \underline{v}_h, p_h) + (\mathrm{div}\, \underline{v}_h, r_h) = -\lambda_h(\underline{u}_h, \underline{v}_h) \quad \forall \underline{v}_h \in \Sigma_h.\end{aligned} \tag{14}$$

We observe that, with this choice, $\lambda_h = 0$ may be a spurious solution of (14), corresponding to those $\underline{u}_h$, such that $(\mathrm{div}\, \underline{u}_h, q_h) + (\mathrm{rot}\, \underline{u}_h, t_h) = 0$ for all $q_h \in Q_h$ and $t_h \in T_h$. Let us denote by $\mathbb{K}_h$ this discrete nullspace. Let us define the space Σ^2_h obtained by projecting Σ_h onto the orthogonal space of $\mathbb{K}_h$ in Σ_h and consider problem (14) with $\Sigma_h = \Sigma^2_h$. It follows that the resulting eigenvalues are strictly positive and coincides with the nonzero ones obtained with Σ_h.

The discretization scheme we have introduced in (14) can be interpreted as a reduced integration procedure of (3). With the introduction of $L^2(\Omega)$-projection operators $\mathbf{P}_1$ and $\mathbf{P}_2$ into the finite element subspaces Q_h and T_h of $L^2_0(\Omega)$ and $L^2(\Omega)$ respectively, we can obtain from the first two equations in (14) that $p_h = -\mathbf{P}_1 \operatorname{rot} \underline{u}_h$ and $r_h = -s\mathbf{P}_2 \operatorname{div} \underline{u}_h$, then substituting these relations in the last equation we get:

$$\begin{array}{l}\text{find } \lambda_h \in \mathbb{R}, \text{ such that there exists } \underline{u}_h \in \Sigma_h, \text{ with } \underline{u}_h \neq 0: \\ (\mathbf{P}_1 \operatorname{rot} \underline{u}_h, \mathbf{P}_1 \operatorname{rot} \underline{v}_h) + s(\mathbf{P}_2 \operatorname{div} \underline{u}_h, \mathbf{P}_2 \operatorname{div} \underline{v}_h) = \lambda_h(\underline{u}_h, \underline{v}_h) \ \forall \underline{v}_h \in \Sigma_h.\end{array} \quad (15)$$

We refer to [7] for a more complete analysis of this method in the case of fluid-structure interaction system. To end this section we report only the following picture with some numerical results which confirm the good behavior of the method. Figs. 5 and 6 show how the the eigenvalues computed with this projection method depend on the penalization parameter s and the first singular eigensolution.

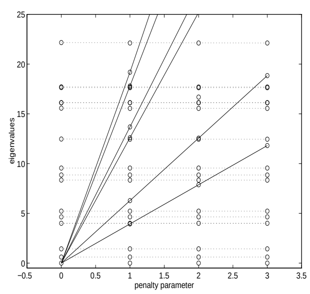

Fig.5 Eigenvalues on the L-shaped domain

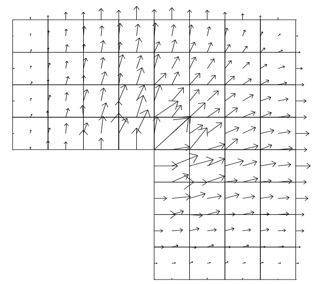
Fig.6 Computed singular eigensolution.

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