ANALYSIS OF FINITE ELEMENT APPROXIMATION OF EVOLUTION PROBLEMS IN MIXED FORM

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Abstract. This paper deals with the finite element approximation of evolution problems in mixed form. Following [8], we handle separately two types of problems. A model for the first case is the heat equation in mixed form, while the time dependent Stokes problem fits within the second one. For either case, we give sufficient conditions for a good approximation in the natural functional spaces. The results are not obvious in the first situation. In this case, the well-known conditions for the well-posedness and convergence of the corresponding steady problem are not sufficient for the good approximation of the time dependent problem. This issue is demonstrated with a numerical (counter-) example and justified analytically.

Key words. mixed finite element, evolution problem, Stokes problem, Laplace problem in mixed form

AMS subject classifications. 65N30, 65M60

1. Introduction. Mixed finite elements are often used in engineering applications and their analysis has been considered in several papers, starting from the 70's (see [10, 3, 13]), mainly for the approximation of steady source problems. For the reader's convenience, we recall here what we mean by a general standard mixed problem. We consider a pair of Hilbert spaces Φ and Ξ and denote by Φ' and Ξ' their dual spaces. Given two data $f \in \Phi'$ and $g \in \Xi'$, a general (steady) mixed problem reads: find $\psi \in \Phi$ and $\chi \in \Xi$ such that

$$
\begin{aligned}
a(\psi, \varphi) + b(\varphi, \chi) &= {}_{\Phi'}\!< f, \chi >_{\Phi} & \forall \varphi \in \Phi, \\
b(\psi, \xi) &= {}_{\Xi'}\!< g, \xi >_{\Xi} & \forall \xi \in \Xi,
\end{aligned}
\tag{1}
$$

where $a : \Phi \times \Phi \to \mathbb{R}$ and $b : \Phi \times \Xi \to \mathbb{R}$ are continuous bilinear forms.

When a finite element approximation to problem (1) is considered, it is well-known that the necessary and sufficient condition for the well-posedness, stability and convergence of the scheme (for any given data) is that two inf-sup constants are bounded below away from zero independently of the meshsize parameter.

In the 80's the use of mixed finite elements has been considered also for the approximation of eigenvalue problems (see [18, 4]) and only fairly recently it has been understood (see [8, 7]) that the inf-sup conditions are not the main assumptions in this context. It has been observed that, in most cases, we can distinguish between two families of mixed problems, depending on the role played by the two equations which define the mixed problem itself and that suitable conditions have to be considered in either case for the good convergence of the computed eigenvalues. Namely, we can consider $\binom{f}{0}$-type problems, when in (1) $g = 0$ and $\binom{0}{g}$-type problems when the opposite situation occurs, i.e., $f = 0$. For instance, the Stokes problem belongs to the first family and the standard mixed formulation for Laplace problem to the second one.

In this paper, we want to consider the finite element approximation of evolution problems in mixed form. The mathematical literature on this field is mainly related to the approximation of the heat equation by means of Raviart-Thomas elements (see, e.g., [16]); mixed finite element schemes have been used extensively for the approximation of evolution problems, in particular in fluid-dynamic applications (see [14, 15]).

It is not an unexpected result, that the theoretical analysis of such approximations strongly relies on the behavior of the corresponding eigenvalue approximations. For this reason, we consider separately the $\binom{0}{g}$- and $\binom{f}{0}$-type formulations. Actually, it is not completely true that, for instance, in the mixed form of the heat equation f has to vanish, being possibly related to some nonhomogeneous boundary conditions. However, in this paper, we shall consider truly $\binom{0}{g}$ and $\binom{f}{0}$ problems, only; in the case of the heat equation, for instance, we can reduce the problem into this form via a suitable extension of the boundary trace.

The outline of the paper is as follows. In the next section we recall some known results about the standard Galerkin space semidiscretization of parabolic problems. In Section 3 we introduce the problems we are dealing with, and present some examples. We are going to use a different notation than the one introduced in (1). In particular, we want to adapt our notation to the mixed Laplace equation for the $\binom{0}{g}$-type system and to the Stokes problem when dealing with the $\binom{f}{0}$ problem. In Section 4, we report on some numerical experiments. In particular, we construct test cases and approximating spaces in such a way

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that the inf-sup conditions hold true (hence the corresponding steady problems are well approximated) but for which we observe that the evolution problem is not well approximated. These results show the need for an accurate analysis of the evolution case, which is not a straightforward extension of the standard steady state analysis. In the next two sections, where the main results of this paper are stated and proved, we give sufficient conditions for the convergence of the approximation of evolution problems of $\binom{0}{g}$ and $\binom{f}{0}$ type, respectively. The last section contains additional remarks related to the counterexample presented in Section 4. It follows, in particular, that, if the data are smooth enough, the solution can be accurately approximated even with the scheme used for our counterexample.

2. Galerkin semidiscretization of parabolic problems. In this section we collect some known results on the semidiscrete approximation to parabolic problems with the aim of introducing some basic estimates and of comparing them with those obtained in the case of mixed formulations. The interested reader is referred to [23], e.g., for a more detailed analysis of the problem under consideration in this section.

We consider two Hilbert spaces V and H, $V \subseteq H$, V dense in H. We identify H with its dual space H'. Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear form, satisfying the coercivity condition: there exist $\alpha > 0$ and $\mu \geq 0$ such that $a(v,v) + \mu||v||_H^2 \geq \alpha||v||_V^2 \ \forall v \in V$. The variational formulation of the parabolic problem (denoting by $(\cdot, \cdot)$ the scalar product in H) reads: given $T > 0$, $f :]0, T[\to V'$ and $u_0 \in H$, for almost every $t \in]0, T[$ find $u(t) \in V$ such that

$$\frac{d}{dt}(u(t), v) + a(u(t), v) = {}_{V'}< f(t), v >_V \quad \forall v \in V; \qquad u(0) = u_0. \tag{2}$$

The following existence and uniqueness theorem for problem (2) is well-known (see, e.g. [17]):

THEOREM 1. *Assume that the bilinear form a is continuous and coercive on $V \times V$. Then given $f \in L^2(]0, T[; V')$ and $u_0 \in H$, there exists a unique solution $u \in L^2(]0, T[; V) \cap C^0([0, T]; H)$ to (2), with $\partial u/\partial t \in L^2(]0, T[; V')$. Moreover, the following energy estimate holds true:*

$$\max_{t \in [0,T]} ||u(t)||_H^2 + \alpha \int_0^T ||u||_V^2 \, dt \leq ||u_0||_H^2 + C \int_0^T ||f||_{V'}^2 \, dt.$$

A suitable shift reduces our problem to the case $\mu = 0$; for this reason we consider this case in detail.

If the bilinear form a is symmetric and the embedding of V in H is compact, then, for every $f \in L^2(]0, T[; H)$, the solution to (2) can be represented with the following series:

$$u(t) = \sum_{i=1}^{\infty} \left((u_0, w_i)e^{-\lambda_i t} + \int_0^t (f(s), w_i)e^{-\lambda_i(t-s)} \, ds \right) w_i. \tag{3}$$

Here, $\lambda_i \in \mathbb{R}$ and $w_i \in V$, with $w_i \neq 0$, are eigenvalues and eigenvectors of the bilinear form a, that is, for each i they satisfy $a(w_i, v) = \lambda_i(w_i, v) \quad \forall v \in V$.

EXAMPLE 1 (The heat equation). *The standard example is given by the heat equation: Ω is a polygon in $\mathbb{R}^2$ or a Lipschitz polyhedron in $\mathbb{R}^3$, $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $a(u,v) = \int_\Omega \text{grad}\, u \cdot \text{grad}\, v \, dx$. Clearly, in this case, the coercivity assumption is valid with $\mu = 0$ for the Poincaré inequality.*

Approximating the space V by a finite dimensional subspace V_h provides a space semidiscretization of the variational formulation (2). Given $f \in L^2(]0, T[; H)$ and $u_{0,h} \in V_h$, for each $t \in [0, T]$ find $u_h(t) \in V_h$ such that

$$\frac{d}{dt}(u_h(t), v) + a(u_h(t), v) = {}_{V'}< f(t), v >_V \quad \forall v \in V_h; \qquad u_h(0) = u_{0,h}. \tag{4}$$

Subtracting (4) from (2) we obtain the error equation

$$\frac{d}{dt}(u(t) - u_h(t), v) + a(u(t) - u_h(t), v) = 0 \quad \forall v \in V_h.$$

Then we can derive the following estimate for all $v_h \in V_h$:

$$\max_{t \in [0,T]} ||u(t) - u_h(t)||_H^2 + \alpha \int_0^T ||u(t) - u_h(t)||_V^2 \leq ||u_0 - u_{0,h}||_H^2 + $$
$$\int_0^T \left(\left(\frac{\partial}{\partial t}(u(t) - u_h(t)), u(t) - v_h \right) + a(u(t) - u_h(t), u(t) - v_h) \right) dt. \tag{5}$$

Let us try to get some error estimates from equation (5) in the case of the finite element approximation of the heat equation (see Example 1). If we take $v_h = u^I(t)$, the interpolant of $u(t)$ in V_h, then we get

$$\max_{t \in [0,T]} ||u(t) - u_h(t)||_H^2 + \alpha \int_0^T ||u(t) - u_h(t)||_V^2 \le ||u_0 - u_{0,h}||_H^2 +$$

$$\int_0^T \left(\left(\left\| \frac{\partial u}{\partial t}(t) \right\|_H + \left\| \frac{\partial u_h}{\partial t}(t) \right\|_H \right) ||u(t) - u^I(t)||_H + C ||u(t) - u^I(t)||_V^2 \right) dt.$$

If V_h is the space of continuous piecewise linear functions, and $u_{0,h}$ is the interpolated of u_0, then we have the optimal convergence (with the obvious modifications in the case of nonconvex domains):

$$||u - u_h||_{L^\infty(L^2)} + \alpha ||u - u_h||_{L^2(H^1)} \le Ch \left(||u_0||_{H^1} + ||f||_{L^2(L^2)} \right). \tag{6}$$

Unfortunately, the generalization of (6) to higher order approximations is not optimal; when k-th order elements are used, (5) gives only $O(h^{(k+1)/2})$. On the other hand, we can take in (5) $v_h = Pu(t)$, the L^2-projection of $u(t)$ into V_h. Now the first term in the integral on the right hand side of (5) reads

$$\left(\frac{\partial}{\partial t}u(t) - \frac{\partial}{\partial t}u_h(t), u(t) - Pu(t) \right) = \left(\frac{\partial}{\partial t}u(t) - P\frac{\partial}{\partial t}u(t), u(t) - Pu(t) \right) = \frac{1}{2}\frac{d}{dt}||u(t) - Pu(t)||_H^2,$$

since $\partial u_h(t)/\partial t \in V_h$, P is the L^2 projection onto V_h, and P commutes with the time derivative. Moreover, the estimate of the second term in the integral on the right hand side of (5) involves the term $||u(t) - Pu(t)||_{H^1}$. If V_h is the space of continuous piecewise polynomials of degree k, and if we assume that the mesh is such that we can use an *inverse estimate*, then we can obtain

$$||u - u_h||_{L^\infty(L^2)} + \alpha ||u - u_h||_{L^2(H^1)} \le Ch^k \left(||u_0||_{H^k} + ||u||_{L^\infty(H^k)} + ||u||_{L^2(H^{k+1})} \right). \tag{7}$$

If, instead of using (5), we introduce the *elliptic projection* operator Π defined as follows for each $w \in V$

$$\Pi w \in V_h, \quad a(\Pi w, v) = a(w, v) \quad \forall v \in V_h,$$

then the following error estimate holds true:

$$\max_{t \in [0,T]} ||u(t) - u_h(t)||_H^2 + \alpha \int_0^T ||u(t) - u_h(t)||_V^2 \, dt \le ||u_0 - u_{0,h}||_H^2 +$$

$$\max_{t \in [0,T]} ||u(t) - \Pi u(t)||_H^2 + C \int_0^T \left(\left\| \frac{\partial}{\partial t}\left(u(t) - \Pi u(t) \right) \right\|_{V'}^2 + ||u(t) - \Pi u(t)||_V^2 \right) dt.$$

In the case of the heat equation, the last equation, together with the usual error estimates for elliptic problems, leads to an estimate similar to (7) but without using the inverse inequality (see, e.g., [23, 20]).

3. Setting of the problem.

3.1. $\binom{0}{g}$-type problems. We consider two Hilbert spaces Σ and V and a third Hilbert space H (which will be identified with its dual space H') such that the following standard inclusions hold with dense and continuous embedding $V \subseteq H \simeq H' \subseteq V'$. When referring to the inner product of H, we shall omit the reference to the space; in the main application we have in mind, H is simply L^2. A model $\binom{0}{g}$-type evolution problem reads: given $T > 0$, $g :]0, T[\to V'$, and $u_0 \in H$, for almost every $t \in]0, T[$ find $\sigma(t) \in \Sigma$ and $u(t) \in V$ such that

$$a(\sigma(t), \tau) + b(\tau, u(t)) = 0 \qquad\qquad \forall \tau \in \Sigma,$$

$$b(\sigma(t), v) - \frac{d}{dt}(u(t), v) = -_{V'}\!< g(t), v >_V \qquad \forall v \in V, \tag{8}$$

$$u(0) = u_0,$$

where the first two equations are defined in the sense of distributions in $]0, T[$. We make the following assumptions on the involved bilinear forms $a : \Sigma \times \Sigma \to \mathbb{R}$ and $b : \Sigma \times V \to \mathbb{R}$:

$$a(\cdot, \cdot) \text{ and } b(\cdot, \cdot) \text{ are continuous, that is}$$
$$\exists M_a > 0 : \forall \sigma, \tau \in \Sigma \ a(\sigma, \tau) \leq M_a ||\sigma||_\Sigma ||\tau||_\Sigma,$$
$$\exists M_b > 0 : \forall \sigma \in \Sigma, \forall v \in V \ b(\sigma, v) \leq M_b ||\sigma||_\Sigma ||v||_V.$$

We also assume that $a(\cdot, \cdot)$ is symmetric and positive semidefinite and set

$$|\tau|_a := (a(\tau, \tau))^{1/2}.$$

It is immediate to check that $|\cdot|_a$ is a seminorm on Σ and that for all $\sigma, \tau \in \Sigma$ we have

$$a(\sigma, \tau) \leq |\sigma|_a |\tau|_a.$$

We suppose that problem (8) is well-posed and that the following a priori estimates hold true:

$$\max_{t \in [0,T]} ||u(t)||_H^2 + \int_0^T ||u(t)||_V^2 \, dt \leq ||u_0||_H^2 + C \int_0^T ||g(t)||_{V'}^2 \, dt, \tag{9}$$

$$\int_0^T \mu^2(t) ||\sigma(t)||_\Sigma^2 \, dt \leq C \left(||u_0||_H^2 + \int_0^T ||g(t)||_{V'}^2 \, dt \right), \tag{10}$$

where $\mu(t)$ is a suitably chosen weight function which might tend to zero as t goes to zero.

Let Σ_h and V_h be finite dimensional subspaces of Σ and V, respectively. The space semidiscretization of problem (8) reads: for almost every $t \in]0, T[$, find $\sigma_h(t) \in \Sigma_h$ and $u_h(t) \in V_h$ such that

$$a(\sigma_h(t), \tau) + b(\tau, u_h(t)) = 0 \qquad\qquad \forall \tau \in \Sigma_h,$$
$$b(\sigma_h(t), v) - \frac{d}{dt}(u_h(t), v) = -_{V'}< g(t), v >_V \qquad \forall v \in V_h, \tag{11}$$
$$u_h(0) = u_{0,h},$$

where $u_{0,h} \in V_h$ is a suitable approximation of u_0.

EXAMPLE 2 (The heat equation in mixed form). *Let Ω be an open Lipschitz polygon in $\mathbb{R}^2$ or polyhedron in $\mathbb{R}^3$. Setting $\Sigma = H(\mathrm{div}; \Omega)$, $V = H = V' = L^2(\Omega)$, the formulation given in equation (8) is a weak form of the heat equation ($\boldsymbol{\sigma}(t) = \mathrm{grad}\, u(t)$) with the choices*

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau}), \qquad b(\boldsymbol{\tau}, v) = (\mathrm{div}\, \boldsymbol{\tau}, v).$$

It can be directly checked that the a priori estimates (10) hold with $\mu(t) = t$. The estimates can be improved to get $\mu(t) = 1$ if more regularity is assumed on u_0, namely $u_0 \in H_0^1(\Omega)$ or, analogously, $\boldsymbol{\sigma}_0 = \mathrm{grad}\, u_0 \in L^2(\Omega)$.

For the mixed spatial semidiscretization of the heat equation we construct two sequences of finite element spaces $\Sigma_h \subset H(\mathrm{div}; \Omega)$ and $V_h \subset L^2(\Omega)$ and consider the discrete problem: for almost every $t \in]0, T[$, find $\boldsymbol{\sigma}_h(t) \in \Sigma_h$ and $u_h(t) \in V_h$ such that

$$(\boldsymbol{\sigma}_h(t), \boldsymbol{\tau}) + (\mathrm{div}\, \boldsymbol{\tau}, u_h(t)) = 0 \qquad\qquad \forall \boldsymbol{\tau} \in \Sigma_h$$
$$(\mathrm{div}\, \boldsymbol{\sigma}_h(t), v) - \frac{d}{dt}(u_h(t), v) = -(g(t), v) \qquad \forall v \in V_h \tag{12}$$
$$u_h(0) = u_{0,h}.$$

Several possible choices for the spaces Σ_h and V_h have been presented in the literature for the corresponding source problem. For instance, in the case of triangular or tetrahedral meshes, we can choose as Σ_h the spaces of the elements RT introduced in [21, 19] or the elements BDM and BDFM introduced in [12, 11] (see [13] for a unified presentation; see, also, [16] for the use of RT elements in the context of parabolic problems in mixed form). In all these cases V_h is equal to $\mathrm{div}\, \Sigma_h$. On quadrilaterals or hexahedrons the situation is more complicated; a two-dimensional theory has been developed recently in [1], showing that the standard families

just listed do not achieve optimal approximation properties in $H(\operatorname{div};\Omega)$ for general quadrilateral meshes. There, a new family ABF has been proposed by adding internal degrees of freedom to the RT elements in order to recover the optimal accuracy. In this case, the inclusion $\operatorname{div}\Sigma_h \subseteq V_h$ is no longer valid (see [1] for the details about the definition of V_h).

All these spaces satisfy the conditions for the well-posedness of the steady problem, namely there exist two positive constants α and β such that

$$(\boldsymbol{\tau},\boldsymbol{\tau}) \geq \alpha||\boldsymbol{\tau}||^2_{H(\operatorname{div};\Omega)} \quad \forall \boldsymbol{\tau} \in \Sigma_h \ s.t. \ (\operatorname{div}\boldsymbol{\tau},v)=0 \quad \forall v \in V_h, \tag{13}$$

$$\sup_{\boldsymbol{\tau}\in\Sigma_h} \frac{(\operatorname{div}\boldsymbol{\tau},v)}{||\boldsymbol{\tau}||_{H(\operatorname{div};\Omega)}} \geq \beta||v||_{L^2(\Omega)} \quad \forall v \in V_h. \tag{14}$$

In Section 4 we shall test the lowest order RT element for the approximation of the heat equation, together with another less standard element which also satisfies (13) and (14).

3.2. $\binom{\mathrm{f}}{0}$-type problems. We consider three Hilbert spaces V, H and Q such that $V \subseteq H \simeq H' \subseteq V'$ with dense and continuous inclusions. As in the previous section, we shall refer to the scalar product of H with $(\cdot,\cdot)$. A model $\binom{f}{0}$-type evolution problem reads: given $T > 0$, $f :]0,T[\to V'$, and $u_0 \in H$, for almost every $t \in]0,T[$ find $u(t) \in V$ and $p(t) \in Q$ such that

$$\begin{aligned}
\frac{d}{dt}(u(t),v) + a(u(t),v) + b(v,p(t)) &= {}_{V'}\!< f, v >_V & &\forall v \in V, \\
b(u(t),q) &= 0 & &\forall q \in Q, \\
u(0) &= u_0,
\end{aligned} \tag{15}$$

where the first two equations are defined in the sense of distributions in $]0,T[$. We recall the definitions of the bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ and make the following standard hypotheses:

$a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, that is
$\exists M_a > 0 : \forall u,v \in V \ a(u,v) \leq M_a||u||_V||v||_V,$
$\exists M_b > 0 : \forall v \in V, \forall q \in Q \ b(v,q) \leq M_b||v||_V||q||_Q.$

Moreover, we assume that the form a is coercive on the kernel of B

$$K = \{v \in V : b(v,q)=0 \ \forall q \in Q\},$$

i.e., there exists $\alpha > 0$ such that

$$a(v,v) \geq \alpha||v||^2_V \quad \forall v \in K. \tag{16}$$

A weaker ellipticity could be considered when dealing with parabolic problems. However, it can be reduced to (16) with a change of variables. We suppose that problem (15) is well-posed and that the following a priori estimates holds true:

$$\max_{t\in[0,T]} ||u(t)||^2_H + \alpha \int_0^T ||u(t)||^2_V \, dt \leq ||u_0||^2_H + C \int_0^T ||f(t)||^2_{V'} \, dt, \tag{17}$$

$$\int_0^T ||p(t)||^2_Q \, dt \leq C \left(||u_0||^2_H + \int_0^T ||f(t)||^2_{V'} \, dt \right). \tag{18}$$

Let V_h and Q_h be finite dimensional subspaces of V and Q; then the Galerkin space semidiscretization of problem (15) reads: for almost every $t \in]0,T[$, find $u_h(t) \in V_h$ and $p_h(t) \in Q_h$ such that

$$\begin{aligned}
\frac{d}{dt}(u_h(t),v) + a(u_h(t),v) + b(v,p_h(t)) &= {}_{V'}\!< f, v >_V & &\forall v \in V_h, \\
b(u_h(t),q) &= 0 & &\forall q \in Q_h, \\
u_h(0) &= u_{0,h},
\end{aligned} \tag{19}$$

where $u_{0,h} \in V_h$ is an approximation of u_0.

EXAMPLE 3 (The Stokes equations). *Let Ω be an open Lipschitz polyhedron in $\mathbb{R}^n$ (with $n = 2$ or 3); then the Stokes equations fit in a natural way within our setting with the following definitions:*

$$V = (H_0^1(\Omega))^n, \quad Q = L^2(\Omega)/\mathbb{R}, \quad H = (L^2(\Omega))^n,$$
$$a(\mathbf{u}, \mathbf{v}) = (\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})), \qquad b(\mathbf{v}, q) = (\operatorname{div}\mathbf{v}, q),$$

where ε is, as usual, the linearized strain tensor. It is known (see, for instance, [22]) that estimates (17) and (18) hold true in this case. Moreover, whenever Ω is convex (with the usual modifications for the nonconvex case), if $f \in L^2(]0, T[; H)$ and $u_0 \in K$ then $u \in L^2(]0, T[; (H^2(\Omega))^n)$, $\partial u/\partial t \in L^2(]0, T[; H)$, and $p \in L^2(]0, T[; H^1(\Omega))$.

For the discretization of the steady problem, we refer to [13]. We shall see at the end of Section 6 that a good approximation of the steady Stokes equations provides a convergent approximation to the time dependent problem also.

4. Numerical investigations. In this section we report on some numerical tests for various mixed discretization to the heat equation presented in Example 2. The discrete problem we are dealing with is the one presented in equation (12). We consider two possible choices for the discrete spaces Σ_h and V_h. Given $\Omega =]0, \pi[^2$, and a triangular mesh, the first method consists in choosing Σ_h as the lowest order RT element [21] and as V_h the space of piecewise constant functions. We shall refer to this choice as the RT method. In the second method, which has been analyzed in [8], the space Σ_h consists of continuous piecewise linear (in each component) vectorfields and V_h is simply defined as $V_h = \operatorname{div}\Sigma_h$ and turns out to be a subset of the space of piecewise constant functions. We shall refer to this example as P1 method. The RT method is well-known to be stable [21], when applied to the steady problem, see (13) and (14). The P1 method has been introduced in [8] in order to construct a counterexample for the approximation of eigenvalue problems. It has been shown to be stable on a special mesh sequence of the square which is built of uniform subsquares, each of them subdivided into four triangles (criss-cross mesh). In this section, we shall denote by N the number of subdivisions of each side of Ω, so that a criss-cross mesh contains $4N^2$ triangles. On general triangular meshes, the P1 element does not satisfy the inf-sup condition (14) in the sense that the inf-sup constant β tends to zero as h goes to zero.

If not otherwise indicated, we take $T = 10$ and advance in time using an implicit Euler scheme with step dt = 0.1.
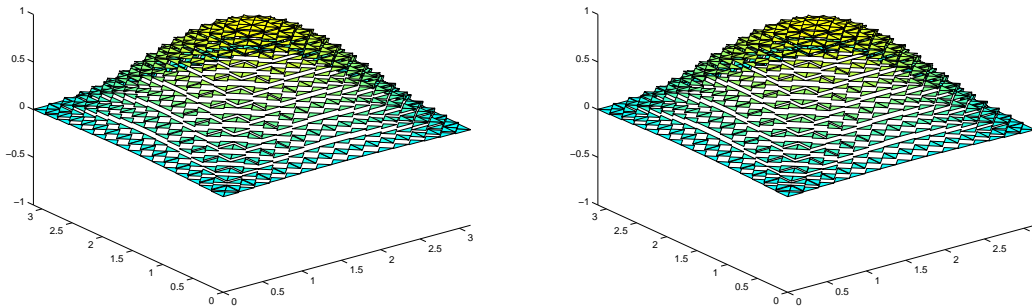

FIG. 1. *$u(10)$ on 16-by-16 criss-cross mesh computed with the RT (left) and P1 (right) methods with $g = 2\sin x \sin y$.*

In all our tests, g does not depend on t and $u_0 \equiv 0$, so that the solution $(\boldsymbol{\sigma}(t), u(t))$ asymptotically tends to the solution $(\boldsymbol{\sigma}_\infty, u_\infty) \in H(\operatorname{div};\Omega) \times L^2(\Omega)$ of the steady problem

$$(\boldsymbol{\sigma}_\infty, \boldsymbol{\tau}) + (\operatorname{div}\tau, u_\infty) = 0 \quad \forall \boldsymbol{\tau} \in H(\operatorname{div};\Omega)$$
$$(\operatorname{div}\boldsymbol{\sigma}_\infty, v) = -(g, v) \qquad \forall v \in L^2(\Omega).$$

In our first test, we choose $g(x, y) = 2\sin x \sin y$, so that $u_\infty(x, y) = \sin x \sin y$. In Figure 1 the component $u(t)$ of the solution at time $T = 10$ is plotted for both methods on a criss-cross mesh with $N = 16$. The

results look quite similar; the $L^2(\Omega)$ norms of the solution is 1.5725 and 1.5674 (the reference value is $\pi/2 = 1.57079\ldots$) for the RT and P1 methods, respectively, and the corresponding solution value at the center are 0.9957 and 0.9925 (the reference value is 1.0). In the second test, considering again criss-cross meshes, we take the function $g = c(h)$ depending on the meshsize h as a checkerboard function with values ± 1 on the underlying mesh of subsquares. For example, the case $N = 4$ is plotted in Figure 2. Let $u_\infty(h)$
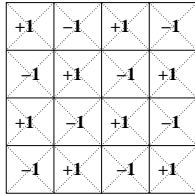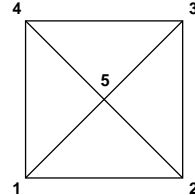

FIG. 2. *The checkerboard $c(h)$ for $N = 4$.*

FIG. 3. *The reference criss-cross macroelement*

be the asymptotic solution of our problem with datum $g = c(h)$. As h goes to zero, the function $c(h)$ tends weakly to zero in $L^2(\Omega)$, so that, for the compactness of the inverse Laplace operator, we have that $u_\infty(h)$ tends to zero strongly in $L^2(\Omega)$. We expect a good numerical method to provide a solution $u_h(t) \in V_h$ tending to zero as h goes to zero. We explicitly observe that our expectation is related to a sort of uniform convergence, since the solution is computed, for each h, with respect to a different right hand side. On the other hand, this kind of convergence is what we usually obtain from the error estimates (see (27) and (36)).

<table>
<tr><td colspan="5">TABLE 1
L^2 norms of the solution with $g = c(h)$.</td></tr>
<tr><td></td><td colspan="2">dt = .1</td><td colspan="2">dt = .01</td></tr>
<tr><td>N</td><td>RT</td><td>P1</td><td>RT</td><td>P1</td></tr>
<tr><td>4</td><td>0.080756</td><td>0.451091</td><td>0.080756</td><td>0.451091</td></tr>
<tr><td>8</td><td>0.020186</td><td>0.430821</td><td>0.020186</td><td>0.430821</td></tr>
<tr><td>16</td><td>0.005047</td><td>0.427416</td><td>0.005047</td><td>0.427416</td></tr>
<tr><td>32</td><td>0.001262</td><td>0.426687</td><td>0.001262</td><td>0.426687</td></tr>
</table>

<table>
<tr><td colspan="3">TABLE 2
L^2 norms of the solution when g is the 8-by-8 checkerboard.</td></tr>
<tr><td>N</td><td>RT</td><td>P1</td></tr>
<tr><td>4</td><td>0.053830</td><td>0.103860</td></tr>
<tr><td>8</td><td>0.020186</td><td>0.430821</td></tr>
<tr><td>16</td><td>0.022569</td><td>0.020642</td></tr>
<tr><td>32</td><td>0.020689</td><td>0.019728</td></tr>
</table>

In Table 1 we report the L^2 norm of $u_h(10)$ for various values of N computed with RT and P1 methods. It is evident that the solution computed with the P1 method does not go to zero. This fact clearly indicates a different behavior for the two methods. In particular, it is evident that the stability of the steady problem (see (13) and (14)) is not sufficient for the good approximation of the evolution problem in mixed form. Here, by *good approximation* we mean a convergence like it comes from estimates (27) or (36). In Table 1 we also show that the bad behavior of the P1 approximation is not related to a poor time discretization. Indeed, a refinement in t does not produce any improvement. At the end of Section 5, we shall come back to this example and analyze more deeply the differences between the two methods. In Figure 4 we show the solution $u_h(10)$ computed with the two methods. It can be observed the resonance induced by the checkerboard in the case of the P1 scheme. This will be related to the presence of spurious eigenvalues (see [8]) at the end of Section 5. On the other hand, if we keep g fixed and let h go to zero, then we observe that the solution computed with the P1 method is similar to the one obtained with the RT method (see Table 2), the only exception being when the mesh is the one underlying the structure of g.

We introduce now another example, in which the P1 method is not converging as h goes to zero in the $L^\infty(L^2)$ norm. The load term g is now a function of t; namely, at time t, $g(t)$ is the $N(t)$ by $N(t)$ checkerboard function with $N(t) = \lceil t \rceil$. The norm of the solution $u(t)$, computed with RT and P1 methods using three different meshes ($N = 8, 12, 16$), is plotted in Figure 5. We notice that, when the P1 method is used, there is a pick with constant height appearing at an increasing value of t as h decreases. This happens when the structure of g *resonates* with the mesh and cannot be avoided even if the mesh is refined.

5. Convergence analysis for the $\binom{0}{g}$-type evolution problems. In this section we shall develop two different error estimates. The first one, stated in Theorem 3, is new and is based on the compatibility assumption (23). This assumption is satisfied for most standard finite elements used for the approximation of
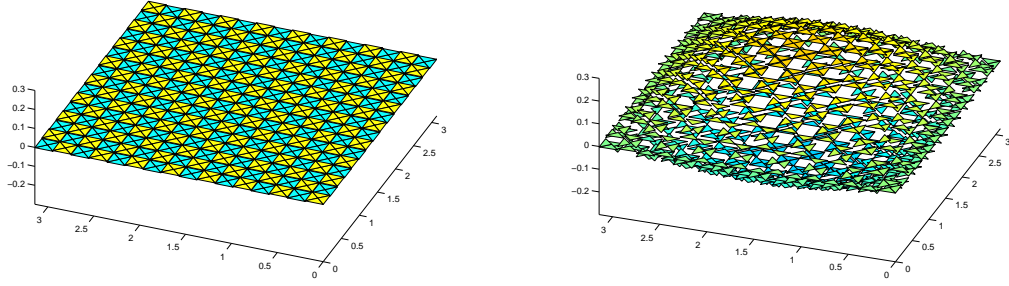
FIG. 4. *u(10) on 16-by-16 criss-cross mesh computed with the RT (left) and P1 (right) methods when g is a checkerboard.*
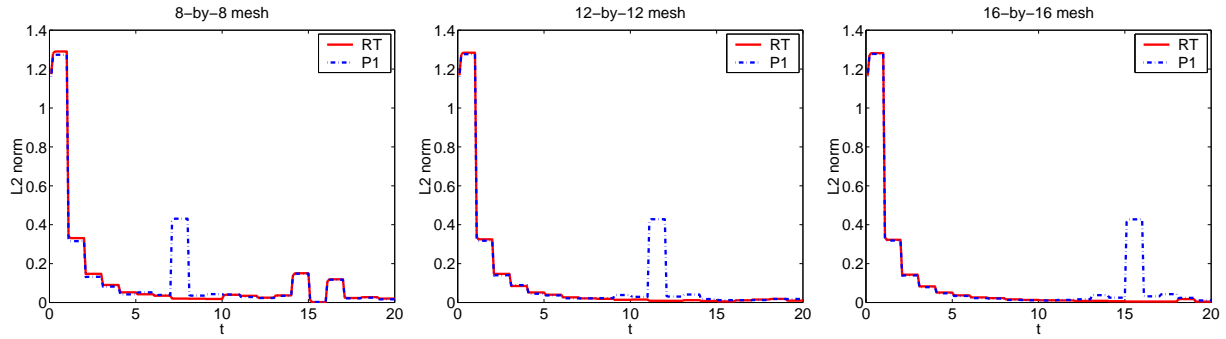

FIG. 5. *L^2-norm of $u_h(t)$ for $t \in [0, 20]$ with $g(t)$ equal to $N(t)$ by $N(t)$ checkerboard function and Dirichlet (natural) boundary conditions*

the heat equation. The second estimate, given in Theorem 5, is basically equivalent to the one given in Theorem 2.1 of [16]. Our proof is new and we explicitly observe that, in this case, the regularity hypothesis (34) has to be made.

We start this section by recalling the continuous problem (8) and its space semidiscretization (11). The hypotheses on the involved bilinear forms have been made explicit in Section 3.1. Given $T > 0$, $g :]0, T[\rightarrow V'$, and $u_0 \in H$, for almost every $t \in]0, T[$ we look for $\sigma(t) \in \Sigma$ and $u(t) \in V$ such that

$$a(\sigma(t), \tau) + b(\tau, u(t)) = 0 \qquad\qquad \forall \tau \in \Sigma,$$

$$b(\sigma(t), v) - \frac{d}{dt}(u(t), v) = -_{V'}< g(t), v >_V \qquad \forall v \in V,$$

$$u(0) = u_0.$$

The semidiscrete counterpart reads: for almost every $t \in]0, T[$, find $\sigma_h(t) \in \Sigma_h$ and $u_h(t) \in V_h$ such that

$$a(\sigma_h(t), \tau) + b(\tau, u_h(t)) = 0 \qquad\qquad \forall \tau \in \Sigma_h,$$

$$b(\sigma_h(t), v) - \frac{d}{dt}(u_h(t), v) = -_{V'}< g(t), v >_V \qquad \forall v \in V_h,$$

$$u_h(0) = u_{0,h}.$$

The error equations are

$$a(\sigma(t) - \sigma_h(t), \tau) + b(\tau, u(t) - u_h(t)) = 0 \quad \forall \tau \in \Sigma_h,$$

$$b(\sigma(t) - \sigma_h(t), v) - \frac{d}{dt}(u(t) - u_h(t), v) = 0 \quad \forall v \in V_h.$$

Given two linear operators $\Pi : \Sigma \to \Sigma_h$ and $P : H \to V_h$, and taking $\tau = \Pi\sigma(t) - \sigma_h(t)$, $v = Pu(t) - u_h(t)$ in the error equations we get, after summation,

$$
\left(\frac{\partial}{\partial t}\left(Pu(t) - u_h(t) \right), Pu(t) - u_h(t) \right) + a(\Pi\sigma(t) - \sigma_h(t), \Pi\sigma(t) - \sigma_h(t)) =
$$
$$
b(\sigma(t) - \Pi\sigma(t), Pu(t) - u_h(t)) - \left(\frac{\partial}{\partial t}\left(u(t) - Pu(t) \right), Pu(t) - u_h(t) \right) -
$$
$$
a(\sigma(t) - \Pi\sigma(t), \Pi\sigma(t) - \sigma_h(t)) - b(\Pi\sigma(t) - \sigma_h(t), u(t) - Pu(t)).
$$
(20)

Let us now take Π as a Fortin operator and P as the H projection onto V_h, namely

$$
\begin{aligned}
&\Pi : \Sigma \to \Sigma_h \\
&b(\tau - \Pi\tau, v) = 0 \quad \forall v \in V_h
\end{aligned}
$$
(21)

and

$$
\begin{aligned}
&P : H \to V_h \\
&(w - Pw, v) = 0 \quad \forall v \in V_h.
\end{aligned}
$$

Then, equation (20) reduces to

$$
\left(\frac{\partial}{\partial t}\left(Pu(t) - u_h(t) \right), Pu(t) - u_h(t) \right) + a(\Pi\sigma(t) - \sigma_h(t), \Pi\sigma(t) - \sigma_h(t)) =
$$
$$
a(\sigma(t) - \Pi\sigma(t), \Pi\sigma(t) - \sigma_h(t)) - b(\Pi\sigma(t) - \sigma_h(t), u(t) - Pu(t)),
$$
(22)

where we used the fact that Π is a Fortin operator (see (21)) and that the projection P commutes with the time derivative.

In several interesting applications, the last term in the right hand side of (22) vanishes. Let $B : \Sigma \to V'$ denote the canonical operator defined by $_{V'}\!<B\sigma, v>_V = b(\sigma, v)$ for all $\sigma \in \Sigma$ and $v \in V$; if

$$
B(\Sigma_h) \subseteq V_h
$$
(23)

then we can show that, for any $v \in V$,

$$
b(\tau, v - Pv) = 0 \quad \forall \tau \in \Sigma_h.
$$

Indeed, in this case, $b(\tau, v - Pv) = {}_{V'}\!<B\tau, v - Pv>_V = (B\tau, v - Pv) = 0$, due to the identification $H \simeq H'$.

Hence, we can easily obtain the following result.

LEMMA 2. *Let (σ, u) and (σ_h, u_h) be the solutions of problems (8) and (11), respectively. If the inclusion (23) holds, then the following error estimate is true:*

$$
\max_{t \in [0,T]} ||u(t) - u_h(t)||_H^2 + \int_0^T |\sigma(t) - \sigma_h(t)|_a^2 \, dt \le
$$
$$
||Pu_0 - u_{0,h}||_H^2 + \max_{t \in [0,T]} ||u(t) - Pu(t)||_H^2 + 2\int_0^T |\sigma(t) - \Pi\sigma(t)|_a^2 \, dt.
$$

In order to obtain a convergence result, we need some assumptions on the operators P and Π. Let us denote by V^+ a subspace of V such that the second component of the solution u belongs to $L^2(]0,T[;V^+)$ whenever $g \in L^2(]0,T[;H)$ and by Σ^+ a space such that the first component of the solution σ belongs to $L^2(]0,T[;\Sigma^+)$ whenever $g \in L^2(]0,T[;H)$. Moreover, let us assume that the following estimate holds

$$
\|\sigma\|_{L^2(\Sigma^+)} + \|u\|_{L^2(V^+)} \le C\|g\|_{L^2(H)}.
$$
(24)

Then we assume that there exists $\rho_1(h)$, tending to zero as h goes to zero, such that for every $u \in V^+$ it holds

$$
||u - Pu||_H \le \rho_1(h)||u||_{V^+}.
$$
(25)

In general, estimate (25) can be achieved if P is a suitable approximation operator. As far as the operator Π is concerned, in the spirit of [7], we make the following assumption: there exists $\rho_2(h)$, tending to zero as h goes to zero, such that

$$|\sigma - \Pi\sigma|_a \leq \rho_2(h)||\sigma||_{\Sigma^+}. \tag{26}$$

The following theorem summarizes the results obtained so far.

THEOREM 3. *Let (σ, u) and (σ_h, u_h) be the solutions of problems (8) and (11), respectively, and suppose that the inclusion (23) holds true. If there exist P and Π satisfying (25) and (26), then we have the following estimate*

$$||u(t) - u_h(t)||_{L^\infty(H)} + \left(\int_0^T |\sigma(t) - \sigma_h(t)|_a^2 \, dt\right)^{1/2} \leq$$
$$C||Pu_0 - u_{0,h}||_H + C(\rho_1(h) + \rho_2(h))||g||_{L^2(H)}. \tag{27}$$

A different estimate, which does not require the inclusion (23), but for which some more regularity assumption on the solution is required, can be obtained by using the approach of the *elliptic projection* (see Section 2). Let us consider the steady problem associated to (8): given $g \in V'$ find $(\sigma, u) \in \Sigma \times V$ such that

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= 0 && \forall \tau \in \Sigma, \\ b(\sigma, v) &= -_{V'}< g, v >_V && \forall v \in V, \end{aligned} \tag{28}$$

and the corresponding discrete problem: find $(\sigma_h, u_h) \in \Sigma_h \times V$ such that

$$\begin{aligned} a(\sigma_h, \tau) + b(\tau, u_h) &= 0 && \forall \tau \in \Sigma_h, \\ b(\sigma_h, v) &= -_{V'}< g, v >_V && \forall v \in V_h. \end{aligned} \tag{29}$$

We make the assumption that (28) and (29) are well-posed. Then, given $(\sigma, u) \in \Sigma \times V$ solution of (28), we can define $\Pi\sigma \in \Sigma_h$ and $P\sigma \in V_h$ as the solution of (29) with right hand side $_{V'}< g, v >_V = -b(\sigma, v)$, namely

$$\begin{aligned} a(\Pi\sigma, \tau) + b(\tau, P\sigma) &= 0 && \forall \tau \in \Sigma_h, \\ b(\Pi\sigma, v) &= b(\sigma, v) && \forall v \in V_h. \end{aligned} \tag{30}$$

We observe that, thanks to the well-posedness of problem (28), the continuous solution of (30) is (σ, u). Since basically Π and P depend explicitly only upon σ, we omit u from the notation for simplicity. It should be reminded, however, that $P\sigma$ is a discrete counterpart to u. Moreover, the operator $\Pi : \Sigma \to \Sigma_h$ defined in (30) is a Fortin operator in the sense of (21). Inserting now the *elliptic projections* Π and P defined in (30) into (20), we obtain

$$\left(\frac{\partial}{\partial t}\left(P\sigma(t) - u_h(t)\right), P\sigma(t) - u_h(t)\right) + a(\Pi\sigma(t) - \sigma_h(t), \Pi\sigma(t) - \sigma_h(t)) =$$
$$-\left(\frac{\partial}{\partial t}\left(u(t) - P\sigma(t)\right), P\sigma(t) - u_h(t)\right),$$

where we made use of the fact that Π is a Fortin operator and we took advantage of the following error equation

$$a(\sigma(t) - \Pi\sigma(t), \tau) + b(\tau, u(t) - P\sigma(t)) = 0 \quad \forall \tau \in \Sigma_h.$$

Hence, using standard arguments and Gronwall's lemma, we obtain the following result.

LEMMA 4. *Let (σ, u) and (σ_h, u_h) be the solutions of problems (8) and (11), respectively. Let $\Pi : \Sigma \to \Sigma_h$ and $P : V \to V_h$ be defined as in (30). Suppose, moreover, that u_0 is in V and is such that there exists $\sigma_0 \in \Sigma$ with (σ_0, u_0) solution to (28) for a suitable $g \in V'$. Then the following error estimate is true:*

$$\max_{t \in [0,T]} ||u(t) - u_h(t)||_H^2 + \int_0^T |\sigma(t) - \sigma_h(t)|_a^2 \, dt \leq ||P\sigma_0 - u_{0,h}||_H^2 +$$
$$\max_{t \in [0,T]} ||u(t) - P\sigma(t)||_H^2 + 2\int_0^T |\sigma(t) - \Pi\sigma(t)|_a^2 \, dt + C\int_0^T \left\|\frac{\partial}{\partial t}(u(t) - P\sigma(t))\right\|_H^2 \, dt.$$

In order to get the convergence estimate from the previous lemma, we make the following assumptions. First, we assume that there exists $\omega_1(h)$, going to zero as h goes to zero, such that

$$||u - P\sigma||_H \leq \omega_1(h)||u||_{V+}. \tag{31}$$

Then, we need an estimate for $||\sigma(t) - \Pi\sigma(t)||_a$. In analogy to (26), we suppose that there exists $\omega_2(h)$, going to zero as h goes to zero, such that

$$|\sigma - \Pi\sigma|_a \leq \omega_2(h)||\sigma||_{\Sigma+}. \tag{32}$$

Finally, we suppose the existence of $\omega_3(h)$, going to zero as h goes to zero, such that

$$\left\|\frac{\partial}{\partial t}(u - P\sigma)\right\|_{L^2(H)} \leq \omega_3(h)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(V+)}. \tag{33}$$

REMARK 1. *We explicitly notice that estimate (33) is not an immediate consequence of (31) unless the operator P commutes with the time derivative. Actually, this property is true provided some regularity assumption is made. Indeed, we need assume*

$$\frac{\partial\sigma}{\partial t} \in L^2(]0,T[\,;\Sigma) \tag{34}$$

in order to define $P(\partial\sigma(t)/\partial t) \in V_h$, according to (30):

$$a\left(\Pi\frac{\partial\sigma(t)}{\partial t}, \tau\right) + b\left(\tau, P\frac{\partial\sigma(t)}{\partial t}\right) = 0 \qquad \forall\tau \in \Sigma_h$$

$$b\left(\Pi\frac{\partial\sigma(t)}{\partial t}, v\right) = b\left(\frac{\partial\sigma(t)}{\partial t}, v\right) \qquad \forall v \in V_h.$$

Differentiating with respect to t equations (30), we get

$$a\left(\frac{\partial\Pi\sigma(t)}{\partial t}, \tau\right) + b\left(\tau, \frac{\partial P\sigma(t)}{\partial t}\right) = 0 \qquad \forall\tau \in \Sigma_h$$

$$b\left(\frac{\partial\Pi\sigma(t)}{\partial t}, v\right) = b\left(\frac{\partial\sigma(t)}{\partial t}, v\right) \qquad \forall v \in V_h.$$

Comparing the last two equations, and using the uniqueness hypothesis on problem (29), we finally obtain

$$P\frac{\partial\sigma(t)}{\partial t} = \frac{\partial P\sigma(t)}{\partial t} \quad \text{for a.e. } t \in]0,T[. \tag{35}$$

Relation (35), which relies on the regularity assumption (34), can be used to get the estimate

$$\left\|\frac{\partial}{\partial t}(u - P\sigma)\right\|_{L^2(H)} \leq \omega_1(h)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(V+)}.$$

Assumptions (31), (32), and (33) allow us to state the following theorem.

THEOREM 5. *Let (σ, u) and (σ_h, u_h) be the solutions of problems (8) and (11), respectively. Let Π and P be the two components of the elliptic projection (30) and let σ_0 be as in Lemma 4. If (31), (32), and (33) are satisfied, then we have the following estimate*

$$||u(t) - u_h(t)||_{L^\infty(H)} + \left(\int_0^T |\sigma(t) - \sigma_h(t)|_a^2\, dt\right)^{1/2} \leq$$
$$C||P\sigma_0 - u_{0,h}||_H + C(\omega_1(h) + \omega_2(h))||g||_{L^2(H)} + C\omega_3(h)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(V+)}. \tag{36}$$

EXAMPLE 4 (Convergence analysis for the mixed approximation to the heat equation). *We review here the numerical examples presented in Section 4. We start with the analysis of the RT, BDM and BDFM method for mesh of triangles, tetrahedrons, rectangles and parallelepipeds. In these cases, Theorem 3 gives the optimal k-th order rate of convergence (see [13, 8]).*

On general quadrilateral meshes, we can use the ABF family introduced in [1]. In this case we cannot use Theorem 3, since the inclusion (23) is not satisfied. However, we can invoke Theorem 5 and conclude with the optimal estimate, provided some extra regularity on the time derivative of u is assumed.

On the other hand, the P1 scheme does not fit within our results. Indeed, it has been proved in [8] that (32) does not hold. We shall make it clear, how things might go wrong (as shown numerically in Figure 4) with the following considerations. In [8] it has been shown that the P1 element, when applied to the mixed eigenvalue problem associated with Laplace operator, presents spurious eigenvalues. Let Ω be the square of side length π and define f^h as the eigenfunction associated to the first spurious eigenvalue $\bar{\lambda}_h$. We normalize f^h, so that $||f^h||_{L^2} = 1$. This function, in particular, has a clear checkerboard pattern and a numerical evidence shows that $\bar{\lambda}_h$ tends to a number close to six (see [8] and Section 7). Take $u_0 = 0$, then the continuous solution u^h to the heat equation is (see (3))

$$u^h(t) = \sum_{i=1}^{\infty} w_i \int_0^t (f^h, w_i) e^{\lambda_i(s-t)} \, ds.$$

Let us consider the discrete eigenmodes $\lambda_{i,h} \in \mathbb{R}$ and $w_{i,h} \in V_h$, $\sigma_{i,h} \in \Sigma_h$, $i = 1, \ldots, \mathcal{N}(h)$, (where $\mathcal{N}(h)$ is the dimension of V_h) satisfying

$$(\boldsymbol{\sigma}_{i,h}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, w_{i,h}) = 0 \quad \forall \boldsymbol{\tau} \in \Sigma_h$$
$$(\operatorname{div} \boldsymbol{\sigma}_{i,h}, v) = -\lambda_{i,h}(w_{i,h}, v) \quad \forall v \in V_h.$$

With this notation, a solution expansion holds also at the discrete level, namely

$$u_h^h(t) = \sum_{i=1}^{\mathcal{N}(h)} w_{i,h} \int_0^t (f^h, w_{i,h}) e^{\lambda_{i,h}(s-t)} \, ds = \sum_{i=1}^{\mathcal{N}(h)} w_{i,h}(f^h, w_{i,h}) \frac{1 - e^{-\lambda_{i,h}t}}{\lambda_{i,h}}. \tag{37}$$

Since f^h tends to zero weakly in $L^2(\Omega)$, the continuous asymptotic solution u_∞^h tends to zero strongly in $L^2(\Omega)$. On the other hand, from (37) we get

$$u_h^h(t) = f^h \frac{1 - e^{-\bar{\lambda}_h t}}{\bar{\lambda}_h} \quad and \quad ||u_h^h||_{L^\infty(L^2)} = \frac{1 - e^{-\bar{\lambda}_h T}}{\bar{\lambda}_h}.$$

This last relation implies that an estimate like (27) cannot hold for the P1 method if we can show that $||u^h||_{L^\infty(L^2)}$ tends to zero as h goes to zero. Indeed, we can split u^h as the sum of u_1 and u_2 defined as follows:

$$\frac{\partial u_1}{\partial t} - \Delta u_1 = f^h \quad in \ \Omega \times]0, T[\qquad \frac{\partial u_2}{\partial t} - \Delta u_2 = 0 \quad in \ \Omega \times]0, T[$$
$$u_1(0) = u_\infty^h \qquad in \ \Omega \qquad\qquad u_2(0) = -u_\infty^h \qquad in \ \Omega.$$

It is clear that $u_1(t) = u_\infty^h$ for all t, so that

$$||u^h||_{L^\infty(L^2)} \le ||u_1||_{L^\infty(L^2)} + ||u_2||_{L^\infty(L^2)} \le 2||u_\infty^h||_{L^2}.$$

6. Convergence analysis for the $\binom{f}{0}$-type evolution problems. Also in this section we present two different error estimates. In the first one, we shall make the hypothesis that the bilinear form a is coercive on the whole space V (see Theorem 9). This estimate applies, for instance, to the Stokes problem introduced in Example 3 and in this case provides optimal rate of convergence only when lowest order elements are used. In the second estimate, presented in Theorem 11, we only assume the ellipticity in the kernel (16), but, in order to get the result, we require an additional approximation property. If applied to the Stokes problem, this estimate turns out to be optimal also for higher order schemes.

We recall the continuous problem (15) and its space semidiscretization (19). The hypotheses on the forms, in particular the ellipticity in the kernel (16), have been made in Section 3.2. Given $T > 0$, $f :]0, T[\to V'$, and $u_0 \in H$, for almost every $t \in]0, T[$ find $u(t) \in V$ and $p(t) \in Q$ such that

$$\frac{d}{dt}(u(t), v) + a(u(t), v) + b(v, p(t)) = {}_{V'}< f, v >_V \qquad \forall v \in V,$$
$$b(u(t), q) = 0 \qquad\qquad\qquad\qquad \forall q \in Q,$$
$$u(0) = u_0.$$

The discrete counterpart reads: for almost every t, find $u_h(t) \in V_h$ and $p_h(t) \in Q_h$ such that

$$\frac{d}{dt}(u_h(t), v) + a(u_h(t), v) + b(v, p_h(t)) = {}_{V'}< f, v >_V \qquad \forall v \in V_h,$$
$$b(u_h(t), q) = 0 \qquad\qquad\qquad\qquad \forall q \in Q_h,$$
$$u_h(0) = u_{0,h},$$

where $u_{0,h} \in V_h$ is an approximation of u_0. The error equations are:

$$\frac{d}{dt}(u(t) - u_h(t), v) + a(u(t) - u_h(t), v) + b(v, p(t) - p_h(t)) = 0 \quad \forall v \in V_h,$$
$$b(u(t) - u_h(t), q) = 0 \quad \forall q \in Q_h.$$

In the next lemma, we shall use the kernel of the discrete operator associated to b, namely $K_h = \{v_h \in V_h : b(v_h, q_h) = 0 \ \forall q_h \in Q_h\}$.

LEMMA 6. *Let us suppose that the form a is elliptic in V, that is (16) holds for any $v \in V$. Let (u, p) and (u_h, p_h) be the solutions of problems (15) and (19), respectively. If the time derivative of u and u_h are bounded in $L^2(]0, T[; H)$ then the following estimate holds true:*

$$\max_{t \in [0,T]} ||u(t) - u_h(t)||_H^2 + \alpha \int_0^T ||u(t) - u_h(t)||_V^2 \le ||u_0 - u_{0,h}||_H^2 + $$

$$C \left(\int_0^T ||u(t) - \Pi u(t)||_H^2 \, dt \right)^{1/2} + C \int_0^T ||u(t) - \Pi u(t)||_V^2 \, dt + \int_0^T b(\Pi u(t) - u_h(t), p(t)) \, dt, \tag{38}$$

where Π is an operator from V^+ to the discrete kernel K_h.

Proof. From the ellipticity and the error equations we get

$$\frac{1}{2}\frac{d}{dt}||u(t) - u_h(t)||_H^2 + \alpha||u(t) - u_h(t)||_V^2 = $$

$$\left(\frac{\partial}{\partial t}\Big(u(t) - u_h(t)\Big), u(t) - u_h(t) \right) + a(u(t) - u_h(t), u(t) - u_h(t)) = $$

$$\left(\frac{\partial}{\partial t}\Big(u(t) - u_h(t)\Big), u(t) - \Pi u(t) \right) + a(u(t) - u_h(t), u(t) - \Pi u(t)) - $$

$$b(\Pi u(t) - u_h(t), p(t) - p_h(t)). \tag{39}$$

From our assumption on the time derivative of u and u_h and the fact that $\Pi u(t) \in K_h$ we easily get the result integrating from 0 to T. $\square$

REMARK 2. *From the previous proof it follows that the first constant C appearing on the right hand side of (38) is related to the bound of the time derivatives of u and u_h in $L^2(]0, T[; H)$. This is a regularity assumption which is in general not too strong if $f \in L^2(]0, T[; H)$. On the other hand, if the time derivatives of u and u_h are only in $L^2(]0, T[; V')$, then a similar result as (38) can be obtained but with the V norm instead of the H one in the second term in the right hand side.*

In order to obtain a rate of convergence from the previous lemma, we consider $V^+ \subseteq V$ and $Q^+ \subseteq Q$ such that the solution to (15) satisfies $u \in L^2(]0, T[; V^+)$ and $p \in L^2(]0, T[; Q^+)$ if $f \in L^2(]0, T[; H)$ with the a priori estimate $||u||_{L^2(V^+)} + ||p||_{L^2(Q^+)} \le C||f||_{L^2(H)}$. Then we introduce the following definitions (see [7]).

DEFINITION 7. *We say that the* weak approximability *property of the space Q^+ holds, if there exists $\rho_1(h)$, tending to zero as h goes to zero, such that*

$$\sup_{v_h \in K_h} \frac{b(v_h, p)}{||v_h||_V} \le \rho_1(h) ||p||_{Q^+} \quad \forall p \in Q^+.$$

Moreover, we need also an approximability property of the kernel, hence we define the following *strong approximability* of V^+.

DEFINITION 8. *The* strong approximability *of V^+ with respect to Π is satisfied if there exists $\rho_2(h)$ tending to zero as h goes to zero, such that for all $u \in V^+$ it holds:*

$$||u - \Pi u_h||_V \le \rho_2(h) ||u||_{V^+}.$$

If the above definitions are fulfilled, then the following theorem holds true:

THEOREM 9. *Let us assume that the form a is elliptic in V and that Definitions 7 and 8 hold true. Moreover, let us denote by $\rho_3(h)$ a function going to zero as h tends to zero such that $||u - \Pi u||_H \le \rho_3(h) ||u||_{V^+}$. Let (u,p) and (u_h, p_h) be the solutions of problems (15) and (19), respectively. If the time derivatives of u and u_h are bounded in $L^2(]0, T[; H)$ then we have the following error estimate:*

$$||u - u_h||_{L^\infty(H)} + ||u - u_h||_{L^2(V)} \le ||u_0 - u_{0,h}||_H + C \left(\sqrt{\rho_3(h)||f||_{L^2(H)}} + (\rho_1(h) + \rho_2(h)) ||f||_{L^2(H)} \right). \quad (40)$$

REMARK 3. *Since $V \subseteq H$ with continuous embedding, we could take $\rho_3(h) = \rho_2(h)$ in the previous theorem. However, we prefer to keep separated the two functions, since in general the approximation in H might be of higher order than in V.*

From estimate (40), it is clear that, in order to derive an optimal order of convergence, we need a good balance among the ρ_i, $i = 1, 2, 3$. We shall discuss this issue in more detail in Example 5.

Going back to the proof of the previous theorem, we notice that, taking in equation (39) Π as the H projection onto the discrete kernel K_h, we obtain the following different estimate, provided u belongs to $L^\infty(V)$ (see [14] for a similar approach to the analysis of the Navier–Stokes equations)

$$\begin{aligned}
||u - u_h||_{L^\infty(H)} + ||u - u_h||_{L^2(V)} \le \\
||u_0 - u_{0,h}||_H + C \left(\rho_4(h)||u||_{L^\infty(V)} + \rho_2(h)||u||_{L^2(V^+)} + \rho_1(h)||p||_{L^2(Q^+)} \right),
\end{aligned} \quad (41)$$

where $\rho_4(h)$, going to zero as h tends to zero, is such that

$$||u - \Pi u||_H \le \rho_4(h) ||u||_V. \quad (42)$$

We notice, that this estimate is not fully satisfactory either. Indeed, in some cases, in order to get a good bound for $\rho_4(h)$, one should use an inverse inequality, leading to stronger assumptions on the mesh sequence.

We are now ready to present the second estimate of this section. Let $\Pi : V^+ \times Q^+ \to K_h$ and $P : V^+ \times Q^+ \to Q_h$ denote the elliptic projections, that is, for $u \in V^+$ and $p \in Q^+$,

$$\begin{aligned}
a(\Pi(u,p), v) + b(v, P(u,p)) &= a(u,v) + b(v,p) \quad &\forall v \in V_h \\
b(\Pi(u,p), q) &= 0 \quad &\forall q \in Q_h.
\end{aligned} \quad (43)$$

In order to give sense to equation (43), we make the assumption that the approximation to the steady equation associated with problem (15) is stable in the sense of [13].

LEMMA 10. *Let (u,p) and (u_h, p_h) be the solutions of problems (15) and (19), respectively. Assume that the form a is uniformly elliptic in the discrete kernel, that is (16) is satisfied for any $v \in K_h$ with α independent of h. Then we have*

$$\max_{t \in [0,T]} ||u(t) - u_h(t)||_H^2 + \alpha \int_0^T ||u(t) - u_h(t)||_V^2 \, dt \le ||\Pi(u_0, 0) - u_{0,h}||_H^2 +$$

$$\max_{t \in [0,T]} ||u(t) - \Pi(u(t), p(t))||_H^2 + \int_0^T \left(\left\| \frac{\partial}{\partial t}(u(t) - \Pi(u(t), p(t))) \right\|_H^2 + ||u(t) - \Pi(u(t), p(t))||_V^2 \right) dt, \quad (44)$$

where Π is the elliptic projector mapping $V^+ \times Q^+$ into K_h (see equation (43)).

Proof. Using the ellipticity in the discrete kernel we have

$$\frac{1}{2}\frac{d}{dt}||\Pi(u(t),p(t)) - u_h(t)||_H^2 + \alpha||\Pi(u(t),p(t)) - u_h(t)||_V^2 \leq$$

$$- \left(\frac{\partial}{\partial t}\Big(u(t) - \Pi(u(t),p(t))\Big), \Pi(u(t),p(t)) - u_h(t)\right) -$$

$$a(u(t) - \Pi(u(t),p(t)), \Pi(u(t),p(t)) - u_h(t)) - b(\Pi(u(t),p(t)) - u_h(t), p(t) - p_h(t))$$

In the last term, p_h can be replaced by $P(u,p) \in Q_h$, since $\Pi(u(t),p(t)) - u_h(t) \in K_h$. Then, using the definition of the elliptic projections (see (43)), we get the desired estimate. $\square$

The following theorem gives the convergence result.

THEOREM 11. *Let the hypotheses of Lemma 10 be satisfied and suppose that the* strong approximability *property (see Definition 8) is fulfilled. Then, we have the estimate*

$$||u - u_h||_{L^\infty(H)} + \alpha||u - u_h||_{L^2(V)} \leq$$

$$||\Pi(u_0,0) - u_{0,h}||_H + \rho_4(h)\left(||u||_{L^\infty(V)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(V)}\right) + \rho_2(h)||u||_{L^2(V^+)}, \tag{45}$$

where $\rho_4(h)$ has been introduced in (42).

Proof. The proof easily follows from the previous lemma, by noticing that the elliptic projection operator Π commutes with the time derivative. $\square$

We now present an estimate for the pressure.

THEOREM 12. *Let (u,p) and (u_h,p_h) be the solutions of problems (15) and (19), respectively. Let the hypotheses of Lemma 10 be satisfied and suppose that the following discrete inf-sup condition holds with $\beta > 0$ independent of h*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{||v_h||_V ||q_h||_Q} \geq \beta.$$

If there exists $\rho_5(h)$ going to zero as h tends to zero such that

$$||p - P(u,p)||_Q \leq \rho_5(h)||p||_{Q^+} \quad \forall q \in Q^+,$$

then we have the following estimate

$$||p - p_h||_{L^2(Q)} \leq C\left(\rho_5(h)||p||_{L^2(Q^+)} + \rho_4(h)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(V)} + ||\Pi(u_0,0) - u_{0,h}||_V\right). \tag{46}$$

Proof. For almost any t we have

$$\beta||P(u(t),p(t)) - p_h(t)||_Q \leq \sup_{v_h \in V_h} \frac{b(v_h, P(u(t),p(t)) - p_h(t))}{||v_h||_V} =$$

$$\sup_{v_h \in V_h} \frac{b(v_h, P(u(t),p(t)) - p(t)) - \left(\frac{\partial}{\partial t}\Big(u(t) - u_h(t)\Big), v_h\right) - a(u(t) - u_h(t), v_h)}{||v_h||_V} \leq$$

$$||p(t) - P(u(t),p(t))||_Q + \left\|\frac{\partial}{\partial t}\Big(u(t) - u_h(t)\Big)\right\|_{V'} + ||u(t) - u_h(t)||_V.$$

In order to conclude the proof we need to estimate the second term. We subtract from the first equation in (43) the first equation in (19) and, taking $v_h \in K_h$, we get

$$\left(\frac{\partial}{\partial t}\Big(\Pi(u(t),p(t)) - u_h(t)\Big), v_h\right) + a(\Pi(u(t),p(t)) - u_h(t), v_h) = -\left(\frac{\partial}{\partial t}\Big(u(t) - \Pi(u(t),p(t))\Big), v_h\right).$$

We take $v_h = \partial(\Pi(u(t), p(t)) - u_h(t))/\partial t \in K_h$ and we have

$$\left\| \frac{\partial}{\partial t} \Big(\Pi(u(t), p(t)) - u_h(t) \Big) \right\|_H^2 + \frac{1}{2} \frac{d}{dt} a(\Pi(u(t), p(t)) - u_h(t), \Pi(u(t), p(t)) - u_h(t)) \leq$$
$$\left\| \frac{\partial}{\partial t} \Big(u(t) - \Pi(u(t), p(t)) \Big) \right\|_H \left\| \frac{\partial}{\partial t} \Big(\Pi(u(t), p(t)) - u_h(t) \Big) \right\|_H .$$

Integrating over $[0, T]$, we obtain the result in a standard way. □

EXAMPLE 5 (Convergence analysis for the approximation to the evolution Stokes problem). *Estimate (40) can be used when lowest order are used. For instance, if we consider the MINI element (see [2]), then we have $\rho_2(h) = Ch$, $\rho_3(h) = Ch^2$, and $\rho_1(h)$ can be bounded by Ch in a standard way as follows:*

$$\sup_{\mathbf{v}_h \in K_h} \frac{b(\mathbf{v}_h, p)}{\|\mathbf{v}_h\|_{H^1}} = \sup_{\mathbf{v}_h \in K_h} \frac{b(\mathbf{v}_h, p - p^I)}{\|\mathbf{v}_h\|_{H^1}} \leq Ch\|p\|_{H^1},$$

where p^I is an approximation of p satisfying $\|p - p^I\|_{L^2} \leq Ch\|p\|_{H^1}$. Then, Theorem 9 gives a first order convergence estimate.

Estimate (45) can be used to analyze higher order methods. For instance, when using generalized k-th order Hood–Taylor schemes (see [5] and [6]), we have $\|\mathbf{u} - \Pi(\mathbf{u}, p)\|_{L^2} + h\|\mathbf{u} - \Pi(\mathbf{u}, p)\|_{H^1} \leq Ch^{k+1}|\mathbf{u}|_{H^{k+1}}$ and Theorem 11 gives a k-th order estimate, provided suitable regularity on the solution is assumed, in particular on the time derivative of $\mathbf{u}$. An alternative estimate can be obtained from (41) with no regularity assumptions on $\partial u/\partial t$ but with the need for an inverse inequality. As far as the approximation of the pressure is concerned, the conclusions of Theorem 12 are that $\|p - p_h\|_{L^2(L^2)}$ is $O(h^k)$, as expected, provided the solution is smooth enough (see (46)).

7. Further considerations on the heat equation in mixed form. In this section we study in more detail the numerical results reported at the end of Section 4. We introduce a modified P1 element on criss-cross meshes, which we call P1*, following the notation of [8], and which has a similar behavior as the P1 element with respect to the convergence of eigenmodes. We recall that the criss-cross mesh is constructed by dividing Ω into N by N subsquares (macroelements) which are then partitioned into four subtriangles by their diagonals. The elements P1 and P1* present different definitions of both spaces Σ_h and V_h. For the P1* approximation, the space of scalars V_h is made of piecewise constants on the square macroelements and the number of degrees of freedom in Σ_h has been reduced by eliminating those ones corresponding to the centers of the macroelements. The elimination is performed in such a way that the divergences of the elements in Σ_h are constant on each macroelement. We refer the reader to [8] for more details on how to perform the elimination of such degrees of freedom.

To get started, we recall the mixed formulation of Laplace eigenproblem: find $\lambda \in \mathbb{R}$ such that there exist $w \in V = L_0^2(\Omega)$ and $\boldsymbol{\sigma} \in \Sigma = H_0(\mathrm{div}; \Omega)$ with $w \not\equiv 0$ satisfying

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, w) &= 0 \quad \forall \boldsymbol{\tau} \in \Sigma \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) &= -\lambda(w, v) \quad \forall v \in V \end{aligned} \tag{47}$$

and its numerical approximation with the P1* method: find $\lambda_h \in \mathbb{R}$ such that there exist $w_h \in V_h$ and $\boldsymbol{\sigma}_h \in \Sigma_h$ with $w_h \not\equiv 0$ satisfying

$$\begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, w_h) &= 0 \quad \forall \boldsymbol{\tau} \in \Sigma_h \\ (\mathrm{div}\,\boldsymbol{\sigma}_h, v) &= -\lambda_h(w_h, v) \quad \forall v \in V_h. \end{aligned} \tag{48}$$

It can be easily shown (in (48), set $w_h = -1/\lambda_h \, \mathrm{div}\,\boldsymbol{\sigma}_h$ which comes from the second equation, into the first equation) that the eigenvalues of (48) correspond to the nonvanishing ones of the following problem: find $\lambda_h \in \mathbb{R}$ such that there exists $\boldsymbol{\sigma}_h \in \Sigma$ with $\boldsymbol{\sigma}_h \not\equiv 0$ satisfying

$$(\mathrm{div}\,\boldsymbol{\sigma}_h, \mathrm{div}\,\boldsymbol{\tau}) = \lambda_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \Sigma_h. \tag{49}$$

In [8] it has been proved and numerically demonstrated that this method does not work. Namely, some *spurious* solutions appear which pollute the numerical spectrum. Here we present the following new result
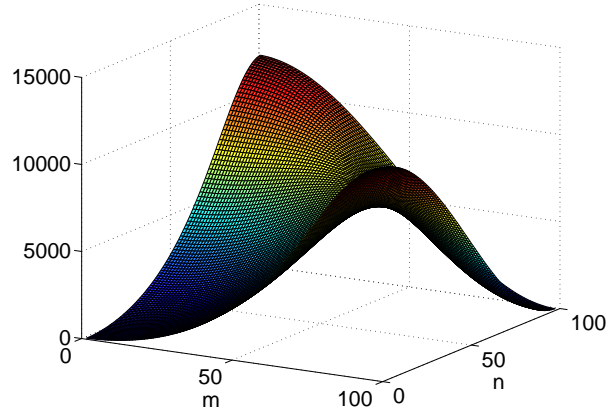
FIG. 6. *Eigenvalues corresponding to formula (50) as functions of m and n when $N = 100$*

which shows that the only pathology of the method under consideration is the presence of spurious modes; namely all continuous eigenmodes are correctly approximated.

Let $\Omega =]0, \pi[\times]0, \pi[$, then by separation of variable it is easy to obtain the exact solution to (47)

$$\lambda^{mn} = m^2 + n^2, \quad m, n \in \mathbb{N}, \ n + m \neq 0$$
$$\boldsymbol{\sigma}^{mn}(x, y) = (m \sin(mx) \cos(ny), n \cos(mx) \sin(nx))$$
$$w^{mn} = -\cos(mx) \cos(ny).$$

The following proposition and the next corollary give the expressions of the discrete solutions to (49) and (48).

PROPOSITION 13. *Given $N \in \mathbb{N}$, and defined*

$$h = \pi/N, \quad \text{cm} = \cos(mh), \quad \text{cn} = \cos(nh),$$
$$A = (1 + 1/3\cos(mh) + 1/3\cos(nh) + 1/3\cos(mh)\cos(nh))(\cos(nh) - \cos(mh))$$
$$B_1 = \sin(mh)\sin(nh)(4/3 + 2/3\cos(mh))$$
$$B_2 = \sin(mh)\sin(nh)(4/3 + 2/3\cos(nh)),$$

for $m, n = 0, \ldots, N - 1$, the eigenvalues of scheme (49) are given by $\lambda_h^{00} = 0$,

$$\lambda_h^{m0} = \frac{6}{h^2} \frac{1 - \cos(mh)}{2 + \cos(mh)}, \qquad \lambda_h^{0n} = \frac{6}{h^2} \frac{1 - \cos(nh)}{2 + \cos(nh)},$$
$$\lambda_h^{mn} = \frac{2}{h^2} \frac{4 + \text{cm} + \text{cn} - (\text{cm} + \text{cn})^2 - \text{cm}\,\text{cn}(\text{cm} + \text{cn}) + 3\sqrt{A^2 + B_1 B_2}}{(1 + \text{cm}/3 + \text{cn}/3 + \text{cm}\,\text{cn}/3)(4 + \text{cm} + \text{cn}) - \sqrt{A^2 + B_1 B_2}}, \qquad mn > 0.$$
(50)

Setting for $mn > 0$

$$\overline{m} = \sqrt{\frac{A + \sqrt{A^2 + B_1 B_2}}{B_2}} \sqrt{nm}, \qquad \overline{n} = \sqrt{\frac{-A + \sqrt{A^2 + B_1 B_2}}{B_1}} \sqrt{nm},$$
(51)

the eigenfunctions corresponding to (50) are

$$\boldsymbol{\sigma}_h^{mn}(x_i, y_j) = \begin{cases} (m \sin(mx_i), 0) & \text{if } n = 0, \\ (0, n \sin(ny_j)) & \text{if } m = 0, \\ (\overline{m} \sin(mx_i) \cos(ny_j), \overline{n} \cos(mx_i) \sin(ny_j)) & \text{if } mn > 0. \end{cases}$$
(52)

Proof. We provide a sketch of the proof which is mainly a tedious calculation. Considering the reference criss-cross macroelement (based on the square $[0, 1] \times [0, 1]$), we denote by $\varphi_i(\hat{x}, \hat{y})$, $i = 1, \ldots, 5$ the standard

continuous piecewise linear functions associated to the nodes represented in Figure 3. It turns out that the eight basis functions for the space Σ_h are

$$\psi_1(\hat{x}, \hat{y}) = (\varphi_1(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4, \varphi_5(\hat{x}, \hat{y})/4)$$
$$\psi_2(\hat{x}, \hat{y}) = (\varphi_5(\hat{x}, \hat{y})/4, \varphi_1(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4)$$
$$\psi_3(\hat{x}, \hat{y}) = (\varphi_2(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4, -\varphi_5(\hat{x}, \hat{y})/4)$$
$$\psi_4(\hat{x}, \hat{y}) = (-\varphi_5(\hat{x}, \hat{y})/4, \varphi_2(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4)$$
$$\psi_5(\hat{x}, \hat{y}) = (\varphi_3(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4, \varphi_5(\hat{x}, \hat{y})/4)$$
$$\psi_6(\hat{x}, \hat{y}) = (\varphi_5(\hat{x}, \hat{y})/4, \varphi_3(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4)$$
$$\psi_7(\hat{x}, \hat{y}) = (\varphi_4(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4, -\varphi_5(\hat{x}, \hat{y})/4)$$
$$\psi_8(\hat{x}, \hat{y}) = (-\varphi_5(\hat{x}, \hat{y})/4, \varphi_4(\hat{x}, \hat{y}) + \varphi_5(\hat{x}, \hat{y})/4),$$

where the numbering is taken such that $\psi_{2i-1}(\hat{x}, \hat{y})$ and $\psi_{2i}(\hat{x}, \hat{y})$ are the two shape functions associated with node i ($i = 1, \ldots, 4$). Taking $\boldsymbol{\sigma}_h$ as given by (52), it is easy (even if long) to check that the expression for λ_h is the one reported in (50). We acknowledge that, in order to guess the correct expression for (52), we have used Mathematica [24]. □

COROLLARY 14. *With the same notation as in Proposition 13, the solutions $(\lambda_h, \boldsymbol{\sigma}_h, w_h)$ of (48) are given by (50), (52), and by the following expressions:*

$$w_h^{m0}|_{K_{ij}} = -\frac{2}{h\lambda_h^{m0}} m \sin\left(m\frac{h}{2}\right) \cos\left(mx_{i+\frac{1}{2}}\right)$$

$$w_h^{0n}|_{K_{ij}} = -\frac{2}{h\lambda_h^{0n}} n \sin\left(n\frac{h}{2}\right) \cos\left(ny_{j+\frac{1}{2}}\right) \tag{53}$$

$$w_h^{mn}|_{K_{ij}} = -\frac{2}{h\lambda_h^{mn}} \left(\overline{m} \sin\left(m\frac{h}{2}\right) \cos\left(n\frac{h}{2}\right) + \overline{n} \sin\left(n\frac{h}{2}\right) \cos\left(m\frac{h}{2}\right)\right) \cos\left(mx_{i+\frac{1}{2}}\right) \cos\left(ny_{j+\frac{1}{2}}\right),$$

where the last term in (53) holds for $mn > 0$ and $x_{i+1/2} = x_i + h/2$, $y_{j+1/2} = y_i + h/2$, and K_{ij} is the square of vertices (x_i, y_j), (x_{i+1}, y_j), (x_{i+1}, y_{j+1}), and (x_i, y_{j+1}).

Proof. The proof follows from the expressions for λ_h and $\boldsymbol{\sigma}_h$ given in Proposition 13 and from the formula $w_h = -\operatorname{div}\boldsymbol{\sigma}_h/\lambda_h$. □

Let m and n be fixed, then from (50) and (51) we have

$$\lim_{h\to 0} \lambda_h^{mn} = m^2 + n^2 \quad \text{and} \quad \lim_{h\to 0} \overline{m} = m, \quad \lim_{h\to 0} \overline{n} = n,$$

so that it is evident that all continuous eigensolutions are approximated by a suitably chosen discrete one. On the other hand, if we take $m = n = N - 1 = \pi/h - 1$, then

$$\lim_{h\to 0} \lambda_h^{N-1, N-1} = 6,$$

which does not correspond to any continuous eigenvalue of (47). This result is in agreement with the numerical experiments presented in [8], where, in particular, it was apparent the presence of a spurious eigenvalues converging to 6. For a better understanding of the behavior of the discrete eigenvalues, in Figure 6 we present the graph of λ_h^{mn} as a function of m, n when $N = 100$. The obtained surface is similar to the one obtained in [9] in an analogue situation. We now present the main result of this section, which is closely related to the numerical tests reported in Section 4. In that section, we showed two examples in order to demonstrate that the P1 method provides results which are acceptable in one case (*regular* right hand side) and awful in the other (oscillatory right hand side). Here we theoretically substantiate those tests, proving that, in general, the P1* methods works if the right hand side is regular enough. Our theorem is proved under the general hypothesis that any continuous eigensolution to (47) is well approximated by the numerical scheme, no matter whether other spurious solutions are present. For this reason we analyze the P1* method, since to our best knowledge estimates like (50), (52), and (53) are not available for the P1 method. On the other hand, we chose to perform the numerical tests using the P1 method (which seems more natural), even though the P1* method would behave similarly.

THEOREM 15. *Let us consider the P1* element for the approximation of the heat equation as in scheme (12) and let $g \in L^\infty(]0, T[; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$.*

Then, if $(\boldsymbol{\sigma}, u)$ is the continuous solution and $(\boldsymbol{\sigma}_h, u_h)$ the discrete one, we have that u_h converges to u in $L^\infty([0,T]; L^2(\Omega))$.

Proof. The regularity assumptions on g and u_0 can be written in a more convenient way as follows: for any $\varepsilon > 0$ there exists N such that

$$\sum_{i>N}(g(t), w_i)^2 < \varepsilon, \quad \forall t \in [0, T], \qquad \sum_{i>N}(u_0, w_i)^2 < \varepsilon, \tag{54}$$

where w_i denote the eigenfunctions of problem (47).

The following structure for the continuous and discrete solutions of the mixed heat equation holds

$$
\begin{aligned}
u(t) &= \sum_{i=1}^{\infty}\left((u_0, w_i)e^{-\lambda_i t} + \int_0^t (g(s), w_i)e^{-\lambda_i(t-s)}\, ds\right) w_i \\
u_h(t) &= \sum_{i=1}^{\infty}\left((u_0, w_{i_h})e^{-\lambda_{i,h} t} + \int_0^t (g(s), w_{i,h})e^{-\lambda_{i,h}(t-s)}\, ds\right) w_{i,h},
\end{aligned}
\tag{55}
$$

where λ_i and w_i (resp. $\lambda_{i,h}$ and $w_{i,h}$ for $i = 1, \ldots, \mathcal{N}(h)$) denote continuous (resp. discrete) eigensolutions of problem (47) (resp. (48)). According to the analysis presented at the beginning of this section (see in particular (50) and (53)) they can be ordered in such a way that for any i

$$\lambda_{i,h} \to \lambda_i \quad \text{and} \quad w_{i,h} \to w_i \quad \text{pointwise in } \Omega \tag{56}$$

as h goes to zero. We explicitly note that in the previous notation we have associated with any eigenvalue λ_i (resp. $\lambda_{i,h}$) a one dimensional eigenspace spanned by the eigenfunction w_i (resp. $w_{i,h}$); this means in particular that it might be $\lambda_i = \lambda_j$ (and/or resp. $\lambda_{i,h} = \lambda_{j,h}$) for some $i \neq j$. Moreover, we shall use the orthogonalities $(w_i, w_j) = 0$ (resp. $(w_{i,h}, w_{j,h})$) for $i \neq j$.

The aim of our proof is to show that for any $\varepsilon > 0$ we have $\|u(t) - u_h(t)\|_0 < \varepsilon$ for any $t \in [0, T]$ when h is small enough. Using (55), we have

$$\|u(t) - u_h(t)\|_0^2 \leq T_1 + T_2 + T_3 + T_4,$$

with

$$T_1 = \sum_{i=N+1}^{\infty}\left((u_0, w_i)^2 e^{-2\lambda_i t} + \int_0^t (g(s), w_i)^2 e^{-2\lambda_i(t-s)}\, ds\right)$$

$$T_2 = \sum_{i=N+1}^{\mathcal{N}(h)}\left((u_0, w_{i,h})^2 e^{-2\lambda_{i,h} t} + \int_0^t (g(s), w_{i,h})^2 e^{-2\lambda_{i,h}(t-s)}\, ds\right)$$

$$T_3 = \left\|\sum_{i=1}^{N}\left((u_0, w_i)w_i e^{-\lambda_i t} - (u_0, w_{i,h})w_{i,h}e^{-\lambda_{i,h} t}\right)\right\|_0^2$$

$$T_4 = \int_0^t \left\|\sum_{i=1}^{N}\left((g(s), w_i)w_i e^{-\lambda_i(t-s)} - (g(s), w_{i,h})w_{i,h}e^{-\lambda_{i,h}(t-s)}\right)\right\|_0^2 ds.$$

Given $\varepsilon > 0$, thanks to the regularity hypotheses (54), we can choose N such that $T_1 < \varepsilon$. The convergence of the eigenvalues and eigenvectors (56) gives that, for h small enough, we also have $T_3 < \varepsilon$ and $T_4 < \varepsilon$. It remains to estimate T_2 which we do now. We shall show that $\sum_{i>N}(u_0, w_{i,h})^2$ can be bounded by 2ε if h is small enough. The term involving $g(s)$ can be handled in the same way and, putting things together, this is what we need in order to get $T_2 < \varepsilon$. The term $\sum_{i>N}(u_0, w_{i,h})^2$ can indeed be estimated in the following way. We have

$$\sum_{i=1}^{N}(u_0, w_{i,h})^2 + \sum_{i=N+1}^{\mathcal{N}(h)}(u_0, w_{i,h})^2 = \sum_{i=1}^{\mathcal{N}(h)}(u_0, w_{i,h})^2 \leq$$

$$\sum_{i=1}^{\infty}(u_0, w_i)^2 = \sum_{i=1}^{N}(u_0, w_i)^2 + \sum_{i=N+1}^{\infty}(u_0, w_i)^2 \leq \sum_{i=1}^{N}(u_0, w_i)^2 + \varepsilon.$$

From the convergence of the eigenvectors (56), we have, for h small enough,

$$\sum_{i=1}^{N}(u_0, w_{i,h})^2 - \sum_{i=1}^{N}(u_0, w_i)^2 \leq \varepsilon, \qquad \text{and the bound} \qquad \sum_{i=N+1}^{\mathcal{N}(h)}(u_0, w_{i,h})^2 \leq 2\varepsilon.$$

$\square$

REMARK 4. *The proof of the previous theorem strongly relies on (54) and (56). In particular, it shows that any scheme which provides convergent eigenmodes for problem (47) (no matter whether spurious solutions are present) can be successfully applied to the approximation of the heat equation in the case of smooth data (in the sense of (54)).*

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