Summary We consider the Maxwell's equation with periodic coefficients as it is usually done for the modeling of photonic crystals. Using Bloch/Floquet theory, the problem reduces in a standard way to a modification of the Maxwell's cavity eigenproblem with periodic boundary conditions. Following [8], a modification of edge finite elements is considered for the approximation of the band gap. The method can be used with meshes of tetrahedrons or parallelepipeds. A rigorous analysis of convergence is presented, together with some preliminary numerical results in 2D, which fully confirm the robustness of the method. The analysis uses well established results on the discrete compactness for edge elements, together with new sharper interpolation estimates.

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Modified edge finite elements for photonic crystals

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1 Introduction

Photonic crystals are periodic structures composed of dielectric materials. The reason for the increase of the interest in this subject is that the spectrum of the Maxwell operator for such media is expected to have gaps. The presence of gaps means that there are prohibited frequencies of propagation of electromagnetic waves going through such crystals. This fact has many potential applications, for example, in optical communications, filters, lasers and microwaves. See [13, 20], for an introduction to photonic crystals, photonic band gap structures and some of their applications. The mathematical model can be written as a modified Maxwell's system with periodic boundary conditions. In recent papers [7,8] a finite element method to approximate such problem based on a modification of Nedéléc edge element spaces was proposed. The convergence of the finite element scheme was proved under severe regularity restrictions and in the case of uniform mesh sequences. Here we present a proof which holds under minimal assumptions on the regularity of the eigensolutions and on the mesh sequences. In order to do so, we derive new sharper interpolation estimates for edge elements.

The outline of the paper is the following. The next section is devoted to the presentation of the problem together with some properties of the analytical framework. Section 3 contains the discretization of the problem and recalls the abstract setting under which the convergence of the eigensolutions can be proved. In Section 4, the finite element spaces are described together the properties which yield the convergence. In particular Lemma 9 implies the discrete compactness property and Lemma 6 provides new interpolation estimates. In the last section some numerical examples are presented, which confirm the good behavior of the modified edge element scheme.

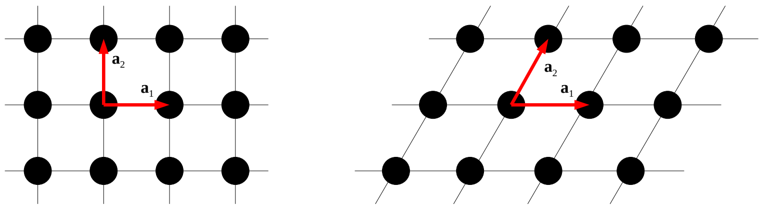

Fig. 1. Two dimensional crystal lattices and corresponding lattice vectors

2 Setting of the problem

In this section we recall some known results concerning the mathematical formulation of a model problem involving a photonic crystal. We consider the Maxwell's equations in $\mathbb{R}^3$

$$\begin{aligned} \nabla\times \mathbf{E} - i\omega\mu\mathbf{H} = 0 \\ \nabla\times \mathbf{H} + i\omega\varepsilon\mathbf{E} = 0. \end{aligned} \tag{1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and the magnetic fields. We assume that the magnetic permeability is constant with $\mu = 1$. The dielectric permittivity ε is assumed to be piecewise constant and uniformly bounded away from zero. In the case of photonic crystals, the medium has a certain periodicity. This means that the function ε is invariant under any translation equal to an integral multiple of suitably chosen lattice vectors. The general situation is presented (in 2D for simplicity) in Fig. 1, where the two lattice vectors are denoted by $\mathbf{a}_1$ and $\mathbf{a}_2$. The periodicity can be stated mathematically as

$$\varepsilon(\mathbf{x} + A\mathbf{k}) = \varepsilon(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^3 \tag{2}$$

where A is the 3-by-3 matrix whose columns are given by $\mathbf{a}_i$, $i = 1, 2, 3$ and $\mathbf{k} \in \mathbb{Z}^3$ denotes a generic vector of relative integers. Moreover, we denote by $\mathbf{R}$ the equivalence relation

$$x\mathbf{R}y \iff y = x + A\mathbf{k} \quad \text{for some } \mathbf{k} \in \mathbb{Z}^3.$$

In Fig. 1 the black regions denote the presence of a material with higher permittivity. One of the main feature of a photonic crystal is the presence of the so called band gap, which is guaranteed by suitable choice of the materials. For more information, we refer the interested reader, for instance, to [11, 10].

Eliminating $\mathbf{E}$ from equation (1) we obtain

$$\begin{aligned} &\boldsymbol{\nabla}\times \varepsilon^{-1}\,\boldsymbol{\nabla}\times \mathbf{H} = \omega^2\mathbf{H} \text{ in } \mathbb{R}^3 \\ &\boldsymbol{\nabla}\cdot \mathbf{H} = 0 \qquad\qquad\qquad \text{in } \mathbb{R}^3. \end{aligned} \tag{3}$$

Let Ω be the periodic domain $\mathbb{R}^3/\mathbf{R}$ which is isomorphic to the cell

$$\mathbf{C} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3,\ (x, y, z) \in [0, 1]^3\} \tag{4}$$

with the identifications of its opposite faces.

Following the Bloch theory (see, for instance, [15]), to a given lattice as in Fig. 1, we can always associate its first irreducible Brillouin zone (see Fig. 2). We denote this zone by K. Then the Bloch waves satisfy $\mathbf{H}(\mathbf{x}) = e^{i\boldsymbol{\alpha}\cdot\mathbf{x}}\mathbf{u}(\mathbf{x})$, where $\mathbf{u}$ is periodic in $\mathbf{x}$ and $\boldsymbol{\alpha} \in K$. Hence,

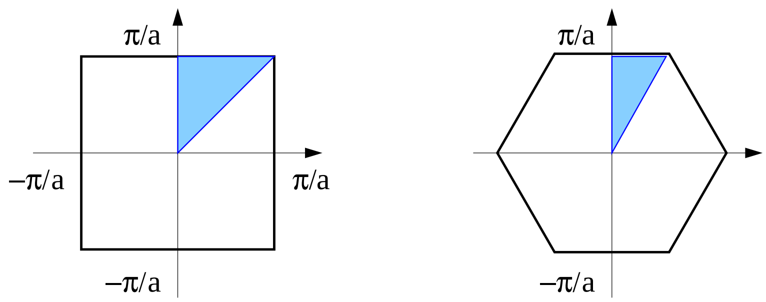

Fig. 2. First Brillouin zones corresponding to the configurations in Fig. 1. The shaded regions are the irreducible zones

for each $\boldsymbol{\alpha} \in K$, our equation reads

$$\begin{aligned} &\boldsymbol{\nabla}_\alpha \times \varepsilon^{-1} \boldsymbol{\nabla}_\alpha \times \mathbf{u} = \omega^2 \mathbf{u} \quad \text{in } \Omega \\ &\boldsymbol{\nabla}_\alpha \cdot \mathbf{u} = 0 \qquad\qquad\qquad\quad \text{in } \Omega. \end{aligned} \tag{5}$$

Here $\boldsymbol{\nabla}_\alpha = \boldsymbol{\nabla} + i\boldsymbol{\alpha} I$ (I being the identity operator). In practical applications, equation (5) allows us to compute the band gap of the device. Indeed, for suitable choices of $\boldsymbol{\alpha}$, one computes the eigenmodes $(\omega, \mathbf{u})$ (with $\mathbf{u} \not\equiv 0$) corresponding to (5). This provides a plot in which different values of ω are drawn as functions of $\boldsymbol{\alpha}$. The regions which are not touched by the plotted curves are band gaps.

In order to introduce the variational formulation of (5) we define the periodic versions of usual Hilbert spaces. Let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$.

$$
\begin{aligned}
&H^1_{\mathrm{p}}(\Omega) = \{v \in L^2(\Omega) : \ \boldsymbol{\nabla}\, v \in \mathbf{L}^2(\Omega)\} \\
&\mathbf{H}_{\mathrm{p}}(\mathrm{curl}; \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \boldsymbol{\nabla}\times \mathbf{v} \in \mathbf{L}^2(\Omega)\} \\
&\mathbf{H}_{\mathrm{p}}(\mathrm{div}; \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \boldsymbol{\nabla}\cdot \mathbf{v} \in L^2(\Omega)\} \\
&\mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha; \Omega) = \{\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div}; \Omega) : \ \boldsymbol{\nabla}_\alpha\cdot \mathbf{v} = 0\}.
\end{aligned}
\tag{6}
$$

Note that the domain Ω has no boundary. For the above definitions, functions defined on Ω can be viewed as functions defined on the cell C with suitable periodic boundary conditions on the identified faces. Hence, the spaces in (6) contain functions which are implicitly periodic; moreover, the derivative operators respect the periodicity of the domain.

For all $\mathbf{u}, \mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl}; \Omega)$ and $q \in H^1_{\mathrm{p}}(\Omega)$, we define the sesquilinear forms

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \varepsilon^{-1}\, \boldsymbol{\nabla}_\alpha\times \mathbf{u} \cdot \overline{\boldsymbol{\nabla}_\alpha\times \mathbf{v}}\, dx, \\
b(q, \mathbf{u}) &= \int_\Omega \boldsymbol{\nabla}_\alpha\, q \cdot \overline{\mathbf{u}}\, dx, \\
(\mathbf{u}, \mathbf{v}) &= \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\, dx
\end{aligned}
\tag{7}
$$

Problem (5) can be rewritten in the following mixed form: find $\omega^2 \in \mathbb{R}$, $(\mathbf{u}, p) \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl}; \Omega) \times H^1_{\mathrm{p}}(\Omega)$, with $(\mathbf{u}, p) \neq (0, 0)$, such that

$$
\begin{cases}
a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl}; \Omega) \\
\overline{b(q, \mathbf{u})} = 0 & \forall q \in H^1_{\mathrm{p}}(\Omega),
\end{cases}
\tag{8}
$$

where the new variable p plays the role of a Lagrange multiplier in order to enforce the constraint $\boldsymbol{\nabla}_\alpha\cdot \mathbf{u} = 0$.

For the analysis of (8), it is convenient to introduce the kernel of $\boldsymbol{\nabla}_\alpha\cdot$, that is

$$\mathbb{K} = \{\mathbf{v} \in \mathbf{H}_\mathrm{p}(\mathrm{curl}; \Omega) : b(q, \mathbf{v}) = 0 \ \forall q \in H^1_\mathrm{p}(\Omega)\}. \tag{9}$$

Let $T \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{L}^2(\Omega))$ be the resolvent operator associated with (8) and defined as follows. For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $T\mathbf{f} = \mathbf{u} \in \mathbf{L}^2(\Omega)$, where $\mathbf{u}$ is the first component of the solution of the following problem:

$$\text{find } (\mathbf{u}, p) \in \mathbf{H}_\mathrm{p}(\mathrm{curl}; \Omega) \times H^1_\mathrm{p}(\Omega), \text{ such that}$$
$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \ \forall \mathbf{v} \in \mathbf{H}_\mathrm{p}(\mathrm{curl}; \Omega) \\ \overline{b(q, \mathbf{u})} = 0 \qquad\qquad\qquad \forall q \in H^1_\mathrm{p}(\Omega). \end{cases} \tag{10}$$

If $\boldsymbol{\alpha} = (0, 0, 0)$, it is well-known that T is compact. Before stating the compactness of T for all $\boldsymbol{\alpha} \in K$, we recall some results of [8].

Theorem 1 *Let $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq (0, 0, 0)$. Given $\mathbf{u} \in \mathbf{L}^2(\Omega)$, there exists unique functions $\mathbf{w} \in (H^1_\mathrm{p}(\Omega))^3$ and $\varphi \in H^1_\mathrm{p}(\Omega)$ satisfying*

$$\mathbf{u} = \boldsymbol{\nabla}_\alpha\times \mathbf{w} + \boldsymbol{\nabla}_\alpha \varphi \quad \text{with} \quad \boldsymbol{\nabla}_\alpha\cdot \mathbf{w} = 0,$$

$$\|\mathbf{w}\|_1 + \|\varphi\|_1 \leq C\|\mathbf{u}\|_0.$$

Lemma 1 *The sequence*

$$0 \to H^1_\mathrm{p}(\Omega) \xrightarrow{\boldsymbol{\nabla}_\alpha} \mathbf{H}_\mathrm{p}(\mathrm{curl}; \Omega) \xrightarrow{\boldsymbol{\nabla}_\alpha\times} \mathbf{H}_\mathrm{p}(\mathrm{div}; \Omega) \xrightarrow{\boldsymbol{\nabla}_\alpha\cdot} L^2(\Omega) \to 0 \tag{11}$$

is exact.

The following lemma states the compactness of T.

Lemma 2 *The operator T is compact and self-adjoint from $\mathbf{L}^2(\Omega)$ into itself.*

Proof. Thanks to Lemma 1, the second equation in (10) implies that $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$. The sesquilinear form $a(\mathbf{u},\mathbf{v})$ is hermitian, continuous and coercive on $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$. Hence there exists a unique $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$ solution of (10). Since $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$ is compactly embedded in $\mathbf{L}^2(\Omega)$, the operator T is compact and self-adjoint. □

As a consequence of that, T admits an increasing sequence of real, positive eigenvalues

$$0 < \omega_1^2 < \omega_2^2 < \cdots < \omega_n^2 < \cdots ,$$

each associated with a finite dimensional eigenspace.

Moreover, the following regularity result holds for solutions of problem (10), see [6].

Lemma 3 *There exists $s > 1/2$, such that $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega)$ is continuously embedded in $(H^s(\Omega))^3$. Moreover for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ the solution $\mathbf{u}$ of problem (10) satisfies*

$$\mathbf{u} \in (H^{1+r}(\Omega))^3, \qquad \textit{for some } r \textit{ satisfying } 0 < r < 1/2. \tag{12}$$

In the above lemma the value of r depends on the ratio between the different values of ε.

3 Discretization of the problem

Let $E_h \subseteq \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)$ and $Q_h \subseteq H^1_{\mathrm{p}}(\Omega)$ be finite dimensional spaces. Then the discretization of (8) reads:

find $\omega_h^2 \in \mathbb{R}$, $(\mathbf{u}_h, p_h) \in E_h \times Q_h$, with $(\mathbf{u}_h, p_h) \neq (0,0)$, such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) = \omega_h^2 (\mathbf{u}_h, \mathbf{v}_h) & \forall \mathbf{v}_h \in E_h \\ \overline{b(q_h, \mathbf{u}_h)} = 0 & \forall q_h \in Q_h. \end{cases} \tag{13}$$

Problem (13) can be reduced to an algebraic generalized eigenvalue problem of the form

$$\begin{pmatrix} A & B^H \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} = \omega_h^2 \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} \tag{14}$$

with A the hermitian matrix associated to the sesquilinear form a, B the rectangular matrix associated to b and M the hermitian matrix associated to the scalar product in $\mathbf{L}^2(\Omega)$. To have an idea of the practical computation of the eigenvalues of this generalized eigensystem see [19].

If the matrix B has full rank, then system (14) has exactly $N(h) = \dim(E_h)$ real and positive eigenvalues:

$$0 < \omega_{1,h}^2 \leq \omega_{2,h}^2 \leq \cdots \leq \omega_{N(h),h}^2.$$

In order to analyze the convergence of the discrete eigensolutions to the continuous ones we apply the abstract theory developed in [5]. Let us first introduce the discretization of the resolvent operator T_h : $\mathbf{L}^2(\Omega) \to E_h \subseteq \mathbf{L}^2(\Omega)$: for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $T_h\mathbf{f} = \mathbf{u}_h \in E_h$, where $\mathbf{u}_h$ is the first component of the solution of the problem:

$$\text{find } (\mathbf{u}_h, p_h) \in E_h \times Q_h, \text{ such that}$$

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \; \forall \mathbf{v}_h \in E_h \\ \overline{b(q_h, \mathbf{u}_h)} = 0 \qquad\qquad \forall q_h \in Q_h. \end{cases} \tag{15}$$

We recall that for compact and self-adjoint operator like T, a sufficient and necessary condition in order to have the convergence of the spectrum is the uniform convergence in the operator norm, that is:

$$\lim_{h \to 0} \|T_h - T\|_{\mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{L}^2(\Omega))} = 0. \tag{16}$$

Let us introduce the following assumptions on the finite element space, see [5] for the abstract setting.

H1 *Ellipticity on the discrete kernel* - There exists $\alpha > 0$, independent of h, such that

$$a(\mathbf{u}_h, \mathbf{u}_h) \geq \alpha \|\mathbf{u}_h\|^2_{\text{curl}} \quad \forall \mathbf{u}_h \in \mathbb{K}_h \tag{H1}$$

where the discrete kernel $\mathbb{K}_h$ is defined as

$$\mathbb{K}_h = \{\mathbf{u}_h \in E_h \text{ such that } b(q_h, \mathbf{u}_h) = 0 \; \forall q_h \in Q_h\}.$$

H2 *Weak approximability* - There exists $\rho_1(h)$, tending to zero as h goes to zero, such that for any $p \in H^1_{\mathrm{p}}(\Omega)$ it holds

$$\sup_{\mathbf{v}_h\in\mathbb{K}_h} \frac{b(p,\mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathrm{curl}}} \le \rho_1(h)\|p\|_1. \tag{H2}$$

H3 *Strong approximability* - There exists $\rho_2(h)$, tending to zero as h goes to zero, such that, for all $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)\cap(H^{1+r}(\Omega))^3$ with $\mathbf{u}\in\mathbb{K}$, there exists $\mathbf{u}^I \in \mathbb{K}_h$ satisfying

$$\|\mathbf{u}-\mathbf{u}^I\|_{\mathrm{curl}} \le \rho_2(h)\|\mathbf{u}\|_{1+r}. \tag{H3}$$

Then the following theorem has been proved in [5]:

Theorem 2 *Let us assume that assumptions* H1-H3 *are verified. Then the sequence T_h converges uniformly to T in $\mathcal{L}(\mathbf{L}^2(\Omega),\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega))$, that is there exists $\rho_3(h)$, tending to zero as h goes to zero, such that*

$$\|T\mathbf{f}-T_h\mathbf{f}\|_{\mathrm{curl}} \le \rho_3(h)\|\mathbf{f}\|_0, \quad \textit{for all } \mathbf{f}\in\mathbf{L}^2(\Omega). \tag{17}$$

4 Finite element spaces and convergence

Before introducing the finite element spaces we are using for the approximation of problem (8), we recall the approximation properties of standard edge finite elements. Let $\mathcal{T}_h$ be a triangulation of Ω. To fix ideas, we shall consider a mesh of tetrahedrons and the first family of Nédélec finite elements [17]. We denote by $\mathcal{E}_k$ the k-th order element of such family and by $\Pi^{\mathcal{E}}_h$ the corresponding interpolant. Other families can be handled similarly; difficulties arising when general hexahedral meshes are used are out of the topic of the present paper (see [2], for a two dimensional discussion). Standard interpolation estimates (see [12,16] for a review) read

$$\|\mathbf{v} - \Pi_h^{\mathcal{E}}\mathbf{v}\|_0 \le Ch^s \left(|\mathbf{v}|_s + \|\,\boldsymbol{\nabla}\times\mathbf{v}\|_s\right) \quad 1/2 < s \le k+1,$$

$$\|\,\boldsymbol{\nabla}\times\mathbf{v} - \boldsymbol{\nabla}\times\Pi_h^{\mathcal{E}}\mathbf{v}\|_0 \le Ch^s|\,\boldsymbol{\nabla}\times\mathbf{v}|_s \quad 0 < s \le k+1.$$

According to the regularity result of Lemma 3, the first interpolation estimate cannot be applied to the solution of (10). We now derive a sharper L^2-estimate which requires less regularity on $\boldsymbol{\nabla}\times\mathbf{v}$ (in particular, no additional requirements when $s > 1$). The arguments used for the proof are not new, nevertheless we believe such an estimate is not present in the literature.

Proposition 1 *The following estimates hold true*

$$\begin{aligned}
&\|\mathbf{v} - \Pi_h^{\mathcal{E}}\mathbf{v}\|_0 \le Ch^s \left(|\mathbf{v}|_s + \|\,\boldsymbol{\nabla}\times\mathbf{v}\|_{L^p(\Omega)}\right) && \frac{1}{2} < s \le 1,\ p > 2,\ k \ge 0\\
&\|\mathbf{v} - \Pi_h^{\mathcal{E}}\mathbf{v}\|_0 \le Ch^s|\mathbf{v}|_s && 1 < s \le k+1,\ k > 0
\end{aligned} \qquad (18)$$

for all functions $\mathbf{v}$ *for which the right hand sides of (18) are bounded.*

Proof As usual, we work element by element. Let $\hat{K}$ denote the reference tetrahedron, K the actual element, and $x = F_K(\hat{x}) = B_k\hat{x} + d_k$

the corresponding affine mapping. We recall the covariant transformation $\mathbf{v} \circ F_K = B_K^{-T}\hat{\mathbf{v}}$ and some useful scalings

$$
\begin{aligned}
&\|\mathbf{v}\|_{L^2(K)} \le |\det(B_K)|^{1/2}\|B_K^{-1}\|\|\hat{\mathbf{v}}\|_{L^2(\hat{K})} \le Ch^{1/2}\|\hat{\mathbf{v}}\|_{L^2(\hat{K})}\\
&|\hat{\mathbf{v}}|_H^s(\hat{K}) \le \|B_K\|^{s+5/2}|\det(B_K)|^{-1}|\mathbf{v}|_{H^s(K)} \le Ch^{s-1/2}|\mathbf{v}|_{H^s(K)}\\
&\|\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{v}}\|_{L^p(\hat{K})} \le |\det(B_K)|^{1-1/p}\|B_K^{-1}\|\|\boldsymbol{\nabla}\times\mathbf{v}\|_{L^p(K)}\\
&\qquad\qquad \le Ch^{2-3/p}\|\boldsymbol{\nabla}\times\mathbf{v}\|_{L^p(K)}.
\end{aligned}
\tag{19}
$$

In [1] the following continuity for the interpolation operator $\hat{\Pi}_h^{\mathcal{E}}$ on the reference element has been proved ($q > 2$)

$$
\|\hat{\Pi}_h^{\mathcal{E}}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \le C\left(\|\hat{\mathbf{v}}\|_{L^q(\hat{K})} + \|\hat{\mathbf{v}}\times\mathbf{n}\|_{L^q(\partial\hat{K})} + \|\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{v}}\|_{L^q(\hat{K})}\right). \tag{20}
$$

Let us start with the first estimate in (18) and suppose that $\mathbf{v}$ belongs to $H^s(\Omega)$ ($1/2 < s \le 1$) and $\boldsymbol{\nabla}\times\mathbf{v}$ is in $L^p(\Omega)$ for some $p > 2$. In this case we consider $k = 0$. Using trace theorem and Sobolev embedding, from (20) we get

$$
\|\hat{\Pi}_h^{\mathcal{E}}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \le C\left(\|\hat{\mathbf{v}}\|_{H^s(\hat{K})} + \|\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{v}}\|_{L^p(\hat{K})}\right). \tag{21}
$$

Bramble–Hilbert theorem gives

$$
\begin{aligned}
\|\hat{\mathbf{v}}-\hat{\Pi}_h^{\mathcal{E}}\hat{\mathbf{v}}\|_{L^2(\hat{K})} &\le C\inf_{\hat{\mathbf{p}}\in(\mathcal{P}_0)^3}\left(\|\hat{\mathbf{v}}-\hat{\mathbf{p}}\|_{H^s(\hat{K})} + \|\hat{\boldsymbol{\nabla}}\times(\hat{\mathbf{v}}-\hat{\mathbf{p}})\|_{L^p(\hat{K})}\right)\\
&\le C\inf_{\hat{\mathbf{p}}\in(\mathcal{P}_0)^3}\|\hat{\mathbf{v}}-\hat{\mathbf{p}}\|_{L^2(\hat{K})} + C|\hat{\mathbf{v}}|_{H^s(\hat{K})} + C\|\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{v}}\|_{L^p(\hat{K})}
\end{aligned}
$$

where $\mathcal{P}_0$ is the space of constants on $\hat{K}$ and the last inequality takes into account the fact that $|\hat{p}|_{H^s(\hat{K})}$ and $\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{p}}$ vanish. Then, from Proposition 6.1 of [9] (see also [12], Theorem 3.14) we obtain

$$\|\hat{\mathbf{v}} - \hat{\Pi}_h^{\mathcal{E}}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \leq C|\hat{\mathbf{v}}|_{H^s(\hat{K})} + C\|\,\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{v}}\|_{L^p(\hat{K})}.$$

Finally, from the scalings (19) we get

$$\begin{aligned}\|\mathbf{v} - \Pi_h^{\mathcal{E}}\mathbf{v}\|_{L^2(K)} &\leq Ch^{1/2}\|\hat{\mathbf{v}} - \hat{\Pi}_h^{\mathcal{E}}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \\ &\leq Ch^{1/2}\left(|\hat{\mathbf{v}}|_{H^s(\hat{K})} + \|\,\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{v}}\|_{L^p(\hat{K})}\right) \\ &\leq Ch^{1/2}\left(h^{s-1/2}|\mathbf{v}|_{H^s(K)} + h^{2-3/p}\|\,\boldsymbol{\nabla}\times\mathbf{v}\|_{L^p(K)}\right) \\ &\leq C\left(h^s|\mathbf{v}|_{H^s(K)} + h^{5/2-3/p}\|\,\boldsymbol{\nabla}\times\mathbf{v}\|_{L^p(K)}\right)\end{aligned}$$

which gives the required approximation order ($p > 2$ implies $s \leq 1 < 5/2 - 3/p$).

We now consider $s > 1$ and a general $k < s$. If $\mathbf{v}$ belongs to $H^s(\Omega)$, then there exists $p > 2$ such that $\boldsymbol{\nabla}\times\mathbf{v}$ belongs to $L^p(\Omega)$. From (21) we get

$$\|\hat{\Pi}_h^{\mathcal{E}}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \leq C\|\hat{\mathbf{v}}\|_{H^s(\hat{K})}.$$

Then by classical arguments we obtain

$$\|\hat{\mathbf{v}} - \hat{\Pi}_h^{\mathcal{E}}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \leq C\inf_{\hat{\mathbf{p}}\in(\mathcal{P}_k)^3}\|\hat{\mathbf{v}} - \hat{\mathbf{p}}\|_{H^s(\hat{K})} \leq C|\hat{\mathbf{v}}|_{H^s(\hat{K})}, \qquad (22)$$

where $\mathcal{P}_k$ denotes the space of the restrictions to $\hat{K}$ of polynomials of degree less than or equal to k. We explicitly remark that, in (22), the first inequality is a consequence of Bramble–Hilbert theorem (since

$(\mathcal{P}_k)^3 \subset \mathcal{E}$) and the second inequality follows from classical Deny–Lions theorem in the case when s is integer. The case of fractional exponents is covered by Theorem 6.1 of [9].

Finally, from the scalings (19) we get the second estimate of (18).

Following the ideas of [8], we introduce a modification of standard edge elements satisfying the commuting diagram property with respect to the differential operators $\boldsymbol{\nabla}_\alpha$, $\boldsymbol{\nabla}_\alpha\times$ and $\boldsymbol{\nabla}_\alpha\cdot$.

Let $\mathcal{T}_h$ be a triangulation of Ω. We consider a mesh of tetrahedrons (affine meshes of parallelepipeds can be handled similarly, general hexahedral elements are beyond the aims of this paper) and any order Nédélec elements of the first type [18] (the second family can be studied with similar arguments).

We define the following finite element spaces:

$$
\begin{aligned}
Q_h &= \{q \in H^1_{\mathrm{p}}(\Omega) :\ q|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{q},\ \text{with } \tilde{q} \in \mathcal{P}_{k+1}(K)\ \forall K \in \mathcal{T}_h\} \\
E_h &= \{\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) :\ \mathbf{v}|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{\mathbf{v}},\ \text{with } \tilde{\mathbf{v}} \in \mathcal{E}_k(K)\ \forall K \in \mathcal{T}_h\} \\
F_h &= \{\mathbf{w} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega) :\ \mathbf{w}|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{\mathbf{w}},\ \text{with } \tilde{\mathbf{w}} \in \mathcal{F}_k(K)\ \forall K \in \mathcal{T}_h\} \\
S_h &= \{v \in \mathbf{L}^2(\Omega) :\ v|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{v},\ \text{with } \tilde{v} \in \mathcal{P}_k(K)\ \forall K \in \mathcal{T}_h\}
\end{aligned}
\tag{23}
$$

where $\mathcal{P}_k(K)$ is the set of the restrictions to K of polynomials of degree less than or equal to k; the elements of $\mathcal{E}_k(K)$ have the form

$\mathbf{a}(\mathbf{x}) + \mathbf{b}(x) \times \mathbf{x}$ with $\mathbf{a}, \mathbf{b} \in \mathcal{P}_k^3$; the space $\mathcal{F}_k(K)$ contains the vector fields of the form $\mathbf{a}(\mathbf{x}) + b(\mathbf{x})\mathbf{x}$, with $\mathbf{a} \in \mathcal{P}_k^3$ and $b \in \mathcal{P}_k$.

Remark 1 Explicit computations show that the following relations hold between the original operators $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\times$, $\boldsymbol{\nabla}\cdot$, and the modified operators $\boldsymbol{\nabla}_\alpha$, $\boldsymbol{\nabla}_\alpha\times$, and $\boldsymbol{\nabla}_\alpha\cdot$, when applied to discrete functions

$$\begin{aligned} \boldsymbol{\nabla}_\alpha q &= e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}} \boldsymbol{\nabla} \tilde{q} \\ \boldsymbol{\nabla}_\alpha \times \mathbf{v} &= e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}} \boldsymbol{\nabla} \times \tilde{\mathbf{v}} \\ \boldsymbol{\nabla}_\alpha \cdot \mathbf{w} &= e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}} \boldsymbol{\nabla} \cdot \tilde{\mathbf{w}}. \end{aligned} \tag{24}$$

The interpolation operators onto the finite element spaces defined in (23) are the following ones. The degrees of freedom for the space Q_h are the nodal values, hence the interpolation operator Π_h^Q is the usual nodal interpolation operator.

The edge interpolation operator Π_h^E associates to each function $\mathbf{v}$ of $H^s(\Omega)^3$ the element $\Pi_h^E \mathbf{v} \in E_h$ using the following degrees of freedom on the tetrahedron $K \in \mathcal{T}_h$:

$$\begin{aligned} &\int_e p_e \left[e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_e)} (\mathbf{v} - \Pi_h^E \mathbf{v}) \cdot \mathbf{t} \right] ds = 0 && \forall p_e \in \mathcal{P}_k(e), \\ & && \forall e \text{ edge of } K, \\ &\int_f \mathbf{p}_f \cdot \left[e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_f)} (\mathbf{v} - \Pi_h^E \mathbf{v}) \times \mathbf{n} \right] d\sigma = 0 && \forall \mathbf{p}_f \in (\mathcal{P}_{k-1}(f))^2, \\ & && \forall f \text{ face of } K, \\ &\int_K \mathbf{p}_K \cdot \left[e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_K)} (\mathbf{v} - \Pi_h^E \mathbf{v}) \right] d\mathbf{x} = 0 && \forall \mathbf{p}_K \in (\mathcal{P}_{k-2}(K))^3, \end{aligned} \tag{25}$$

where $\mathbf{x}_e$, $\mathbf{x}_f$, and $\mathbf{x}_K$ are the barycenter of e, f, and K, respectively, $\mathbf{t}$ is the tangential unit vector of e and $\mathbf{n}$ is the outward normal unit vector of f.

Analogously, the face interpolation operator Π_h^F associates to any smooth enough vector field $\mathbf{w}$ a discrete element $\Pi_h^F \mathbf{w} \in F_h$ by using the following degrees of freedom on the tetrahedron $K \in \mathcal{T}_h$:

$$\begin{aligned}
&\int_f \mathbf{p}_f \cdot \left[e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_f)} (\mathbf{w} - \Pi_h^F \mathbf{w}) \cdot \mathbf{n} \right] d\sigma = 0 \quad && \forall \mathbf{p}_f \in (\mathcal{P}_k(f))^2, \\
& && \forall f \text{ face of } K, \\
&\int_K \mathbf{p}_K \cdot \left[e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_K)} (\mathbf{w} - \Pi_h^F \mathbf{w}) \right] d\mathbf{x} = 0 \quad && \forall \mathbf{p}_K \in (\mathcal{P}_{k-1}(K))^3.
\end{aligned} \tag{26}$$

At the end, the degrees of freedom used in order to define the interpolation operator Π_h^S are:

$$\int_K p_K \left[e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_K)} (v - \Pi_h^S v) \right] d\mathbf{x} = 0 \qquad \forall p_K \in \mathcal{P}_k(K),\ \forall K \in \mathcal{T}_h. \tag{27}$$

Adapting the idea of the analogous result in [8], we can prove that the finite element spaces defined in (23) enjoy the following property:

Lemma 4 *The spaces* Q_h*,* E_h*,* F_h *and* S_h *satisfy the commuting diagram property:*

$$\begin{array}{ccccccccc} 0 \to Q & \xrightarrow{\boldsymbol{\nabla}_\alpha} & E & \xrightarrow{\boldsymbol{\nabla}_\alpha \times} & F & \xrightarrow{\boldsymbol{\nabla}_\alpha \cdot} & S/\mathbb{R} & \to 0 \\ \downarrow \Pi_h^Q & & \downarrow \Pi_h^E & & \downarrow \Pi_h^F & & \downarrow \Pi_h^S & \\ 0 \to Q_h & \xrightarrow{\boldsymbol{\nabla}_\alpha} & E_h & \xrightarrow{\boldsymbol{\nabla}_\alpha \times} & F_h & \xrightarrow{\boldsymbol{\nabla}_\alpha \cdot} & S_h/\mathbb{R} & \to 0 \end{array} \qquad (28)$$

In the above diagram, the spaces Q*,* E*,* F*,* S *are suitable smooth dense subspaces of* $H^1_{\mathrm{p}}(\Omega)$*,* $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)$*,* $\mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega)$ *and* $L^2(\Omega)$*, respectively.*

Arguing again as in [8], the approximation properties for the interpolation operators in the modified spaces can be easily deduced from the corresponding ones in the standard spaces.

Lemma 5 *There exists* C*, independent of* h*, such that the following interpolation error estimates hold true for sufficiently regular func-*

tions:

$$\|q - \Pi_h^Q q\|_1 \leq C h^{s-1} \|q\|_s \qquad 1 \leq s \leq k+2, \tag{29}$$

$$\| \boldsymbol{\nabla}_\alpha \times \mathbf{v} - \boldsymbol{\nabla}_\alpha \times \Pi_h^E \mathbf{v}\|_0 \leq C h^s | \boldsymbol{\nabla}_\alpha \times \mathbf{v}|_s \quad 0 < s \leq k+1, \tag{30}$$

$$\|\mathbf{v} - \Pi_h^F \mathbf{v}\|_0 \leq C h^s |\mathbf{v}|_s \qquad 1/2 < s \leq k+1, \tag{31}$$

$$\| \boldsymbol{\nabla}_\alpha \cdot \mathbf{v} - \boldsymbol{\nabla}_\alpha \cdot \Pi_h^F \mathbf{v}\|_0 \leq C h^s | \boldsymbol{\nabla}_\alpha \cdot \mathbf{v}|_s \qquad 0 < s \leq k+1, \tag{32}$$

$$\|v - \Pi_h^S v\|_0 \leq C h^s \|v\|_s \qquad 0 < s < k+1. \tag{33}$$

Moreover, from Proposition 1, we have the following estimate.

Lemma 6 *There exists C, independent of h, such that the following interpolation error estimates hold true for sufficiently regular functions:*

$$\begin{aligned} &\|\mathbf{v} - \Pi_h^E \mathbf{v}\|_0 \leq C h^s \left(|\mathbf{v}|_s + \| \boldsymbol{\nabla} \times \mathbf{v}\|_{L^p(\Omega)} \right) && \frac{1}{2} < s \leq 1,\ p > 2,\ k \geq 0 \\ &\|\mathbf{v} - \Pi_h^E \mathbf{v}\|_0 \leq C h^s |\mathbf{v}|_s && 1 < s \leq k+1,\ k > 0. \end{aligned} \tag{34}$$

A consequence of Lemma 4 is the following discrete version of Theorem 1.

Lemma 7 *Let $\mathbf{u}_h \in E_h$, then there exist $\mathbf{z}_h \in E_h$ and $q_h \in Q_h$ such that*

$$\mathbf{u}_h = \mathbf{z}_h + \boldsymbol{\nabla}_\alpha\, q_h, \qquad \textit{and } b(q_h, \mathbf{z}_h) = 0, \tag{35}$$

where $\mathbf{z}_h \in E_h$ *can be characterized by means of the following mixed problem:*

find $\mathbf{z}_h \in E_h$ *and* $\boldsymbol{\sigma}_h \in \boldsymbol{\nabla}_\alpha\times E_h \subset F_h$ *such that*

$$\begin{cases} (\mathbf{z}_h, \mathbf{w}_h) + (\boldsymbol{\sigma}_h, \boldsymbol{\nabla}_\alpha\times \mathbf{w}_h) = 0 \;\; \forall \mathbf{w}_h \in E_h \\ \overline{(\boldsymbol{\tau}_h, \boldsymbol{\nabla}_\alpha\times \mathbf{z}_h)} = \overline{(\boldsymbol{\tau}_h, \boldsymbol{\nabla}_\alpha\times \mathbf{u}_h)} \; \forall \boldsymbol{\tau}_h \in \boldsymbol{\nabla}_\alpha\times E_h \subset F_h. \end{cases} \tag{36}$$

We have now all the elements which are needed for the proof of the assumptions H1-H3. This is done in the next three lemmas.

Lemma 8 *There exists a constant* C *not depending on* h *such that for all* $\mathbf{u}_h \in \mathbb{K}_h$ *it holds*

$$\|\mathbf{u}_h\|_0 \le C\| \boldsymbol{\nabla}_\alpha\times \mathbf{u}_h\|_0. \tag{37}$$

Proof. The proof follows from the analogous results for the $\boldsymbol{\nabla}\times$ operator given in [1] (see Proposition 4.6) and from Remark 1. Given $\mathbf{u}_h \in E_h$, we find $\tilde{\mathbf{u}}_h \in \mathcal{E}_k$ such that $\tilde{\mathbf{u}}_h = e^{i\boldsymbol{\alpha}\cdot\mathbf{x}}\mathbf{u}_h$. Then

$$\begin{aligned} \|\mathbf{u}_h\|_0 &= \|e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{\mathbf{u}}_h\|_0 = \|\tilde{\mathbf{u}}_h\|_0 \le C\| \boldsymbol{\nabla}\times \tilde{\mathbf{u}}_h\|_0 \\ &= C\|e^{i\boldsymbol{\alpha}\cdot\mathbf{x}} \boldsymbol{\nabla}_\alpha\times \mathbf{u}_h\|_0 = C\| \boldsymbol{\nabla}_\alpha\times \mathbf{u}_h\|_0. \end{aligned}$$

□

Let us now verify that assumption H2 holds true.

Lemma 9 *For all* $\mathbf{v}_h \in \mathbb{K}_h$ *there exists* $\mathbf{v} \in \mathbb{K}$ *such that*

$$\|\mathbf{v}_h - \mathbf{v}\|_0 \le Ch^s\|\mathbf{v}_h\|_{\mathrm{curl}}, \quad \textit{with } s > 1/2.$$

Proof. Since $\mathbf{v}_h \in \mathbb{K}_h$, it follows that $\mathbf{v}_h$ is the solution of a problem like (36) with datum $\boldsymbol{\nabla}_\alpha \times \mathbf{v}_h$. Let us define $\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)$ as the solution of the corresponding continuous problem, that is $\mathbf{v}$ is such that

$$(\mathbf{v}, \mathbf{w}) + (\boldsymbol{\sigma}, \boldsymbol{\nabla}_\alpha \times \mathbf{w}) = 0 \qquad \forall \mathbf{w} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)$$

$$\overline{(\boldsymbol{\tau}, \boldsymbol{\nabla}_\alpha \times \mathbf{v})} = \overline{(\boldsymbol{\tau}, \boldsymbol{\nabla}_\alpha \times \mathbf{v}_h)} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\nabla}_\alpha \times \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega).$$

From the a priori estimate $\|\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)} \leq C\| \boldsymbol{\nabla}_\alpha \times \mathbf{v}_h\|_0$, and from $\boldsymbol{\nabla}_\alpha \cdot \mathbf{v} = 0$, Lemma 3 gives the regularity $\mathbf{v} \in (H^s(\Omega))^3$ for some $s > 1/2$ with the uniform bound $\|\mathbf{v}\|_s \leq C\| \boldsymbol{\nabla}_\alpha \times \mathbf{v}_h\|_0$. Hence we can apply Theorem 1 of [4] in order to get the required estimate

$$\|\mathbf{v} - \mathbf{v}_h\|_0 \leq Ch^s\|\mathbf{v}\|_s \leq Ch^s\| \boldsymbol{\nabla}_\alpha \times \mathbf{v}_h\|_0.$$

□

In order to prove H2, we write:

$$\sup_{\mathbf{v}_h \in \mathbb{K}_h} \frac{b(p, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathrm{curl}}} = \sup_{\mathbf{v}_h \in \mathbb{K}_h} \frac{b(p, \mathbf{v}_h - \mathbf{v})}{\|\mathbf{v}_h\|_{\mathrm{curl}}} \leq Ch^s\|p\|_1,$$

where $\mathbf{v}$ is given by Lemma 9.

It remains to prove H3.

Lemma 10 *For all $r > 0$ there exists a constant C such that, for all $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap (H^{1+r}(\Omega))^3$ with $\mathbf{u} \in \mathbb{K}$, there is an element $u^I \in \mathbb{K}_h$ satisfying*

$$\|\mathbf{u} - \mathbf{u}^I\|_{\mathrm{curl}} \leq Ch^r\|\mathbf{u}\|_{1+r}. \tag{38}$$

Proof. Let us consider $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl}; \Omega) \cap (H^{1+r}(\Omega))^3$ with $\mathbf{u} \in \mathbb{K}$. Then there exists $\mathbf{g} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha; \Omega)$ such that the couple $(\mathbf{u}, p = 0)$ is solution of problem (10) with datum $\mathbf{g}$. Let us take as $\mathbf{u}^I$ the first component of the solution of (15) with the same datum. We can adapt to this problem the known error estimates for the standard Maxwell equations, see e.g. [14] and we obtain

$$\|\mathbf{u} - \mathbf{u}^I\|_{\mathrm{curl}} \leq C \inf_{\mathbf{v}_h \in E_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathrm{curl}}, \tag{39}$$

then the interpolation error estimates (34) and (30) give (38). □

As a consequence of the results of [5,3], we have proved the following theorem:

Theorem 3 *There exists a constant C such that for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ it holds*

$$\|T\mathbf{f} - T_h\mathbf{f}\|_{\mathrm{curl}} \leq Ch^t \|\mathbf{f}\|_0, \tag{40}$$

where $t = \inf(s, r) = r$, and s and r are given in Lemma 3.

Let ω_i^2 be an eigenvalue of problem (8), with multiplicity m_i and denote by E_i the corresponding eigenspace. Then, due to (40), exactly m_i discrete eigenvalues $\omega^2_{i_1,h}, \ldots, \omega^2_{i_{m_i},h}$ converge to ω_i^2. Moreover, setting $\hat{\omega}^2_{i,h} = (1/m_i) \sum_{j=1}^{m_i} \omega^2_{i_j,h}$ and denoting by $\hat{E}_{h,i}$ the direct sum of the eigenspaces corresponding to $\omega^2_{i_1,h}, \ldots, \omega^2_{i_{m_i},h}$, we have that *there exists h_0 such that for $0 < h < h_0$ the following inequalities hold:*

$$
\begin{aligned}
|\omega_i^2 - \hat{\omega}_{i,h}^2| &\le C e_h^{2t} \\
\delta(E_i, \hat{E}_{h,i}) &\le C e_h^{t},
\end{aligned}
\tag{41}
$$

where $\delta(E_i, \hat{E}_{h,i})$ denotes the gap between E_i and $E_{h,i}$.

5 Numerical results

In this section we show some numerical results in 2D. We consider several periodic structures in $x - y$ plane which extend indefinitely in z direction. Under this hypothesis we can always consider the electromagnetic field as a sum of two distinct fields, denoted by TE (Transverse Electric) and TM (Transverse Magnetic). TE modes have vanishing longitudinal (z direction) electric field component, whereas TM modes have vanishing longitudinal magnetic field component. We thus can compute all the eigenmodes $(\omega, \mathbf{u})$ of the problem (5) as the union of TE and TM eigenmodes.[1]

We solved problems (5) for a set of values of $\boldsymbol{\alpha}$ belonging to the boundary of the irreducible Brillouin zone. One can show that in order to compute the band structure of a crystal, this values are enough (see, for instance, [15]).

[1] for TE cases, first equation of problem (5) becomes $\boldsymbol{\nabla}_\alpha \times \boldsymbol{\nabla}_\alpha \times \mathbf{u} = \omega^2 \varepsilon \mathbf{u}$, where $\mathbf{u}$ is the electric field.

For all the experiment we used first order modified finite edge elements defined on a unstructured uniform mesh of triangles.

Let us start with one of most common structure treated in literature, i.e. a set of high refraction index circular rod immersed in air. Fig. 3 shows a sketch of the structure, the computational cell and the corresponding first Brillouin zone.

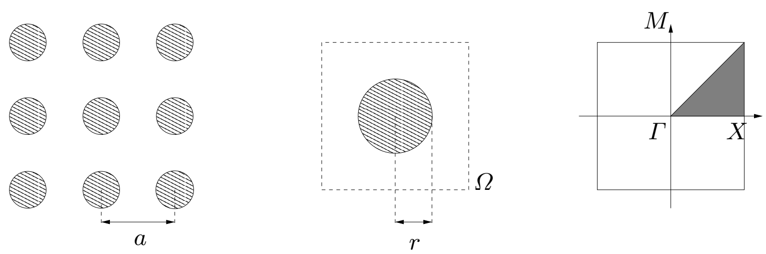

Fig. 3. Two dimensional photonic crystal made of a square lattice of circular dielectric rods ($\varepsilon_r = 8.9$, $r/a = 0.2$) in air.

In order to compute the band diagram, we discretized the domain $\Omega =]0,1[\times]0,1[$ with a mesh composed of 902 elements and 1393 degrees of freedom. Band structure for both TE and TM polarizations is plotted in Fig. 4. The graph shows the presence of a TM bandgap between first and second band, while TE polarization does not have any. Thanks to the scaling properties of Maxwell equations, the eigenvalues λ_i computed on this domain can be related to the ones of a structure whose elementary cell has side a with the equation $\lambda_i = \frac{\omega_i a}{2\pi c}$.

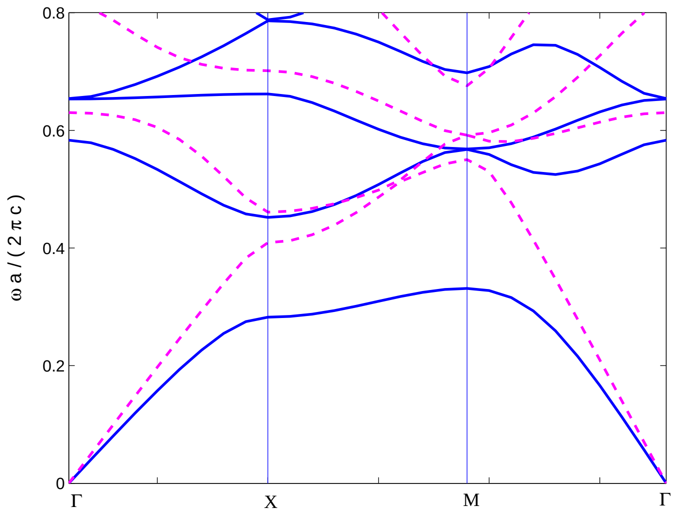

Fig. 4. Band structure of the square lattice of dielectric rods. Solid lines denote TM modes, dashed lines denotes the TE ones.

Another very common structure is a square lattice of thin dielectric veins (in this case alumina, $\varepsilon_r = 8.9$) immersed in air. Fig. 5 shows a sketch of the structure, the computational cell and the corresponding first Brillouin zone. We note that the Brillouin zone of this crystal is the the same of the former crystal, because the shape of the cell is different, but not the kind of periodicity.

The domain $\Omega =]0,1[\times]0,1[$ was discretized with a uniform mesh of 902 triangles (1393 degrees of freedom). In Fig. 6 TE and TM bands are plotted: in this case we note a TE band-gap between first and second band.

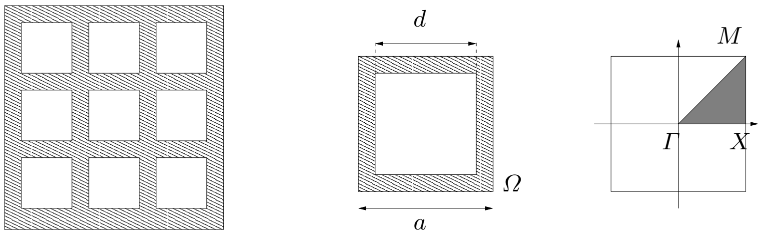

Fig. 5. Two dimensional photonic crystal made of a square lattice of dielectric veins ($\varepsilon_r = 8.9$, $d/a = 0.4$) in air.

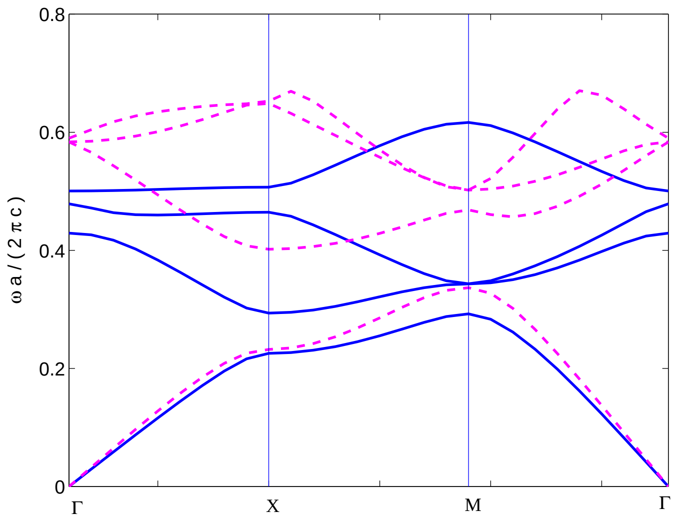

Fig. 6. Band structure of the square lattice of dielectric veins in air. Solid lines denote TM modes, dashed lines denotes the TE ones.

The last photonic crystal we study is obtained by a triangular lattice of circular holes in a dielectric substrate (in our case $\varepsilon_r = 13$). The Brillouin zone in this case is no longer a square, but a hexagon of apothem π/a. The irreducible zone is the triangle of vertex $\Gamma \equiv (0,0)$, $M \equiv (0,\pi/a)$, $K \equiv (\pi/(a\sqrt{3}),\pi/a)$. Fig. 7 shows a sketch of the crystal, the computational domain and the Brillouin zone.

The computational cell is the rhombus of unitary side, whose diago-

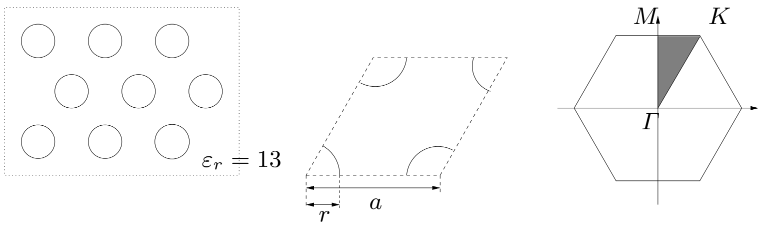

Fig. 7. Two dimensional photonic crystal made of a triangular lattice of circular rods in a dielectric substrate ($\varepsilon_r = 13$, $r/a = 0.48$).

nals measure 1 and $\sqrt{3}$, respectively. We discretized this domain with a mesh composed of 800 triangles and 1240 degrees of freedom. The band structure for both the polarizations is plotted in Fig. 8. The

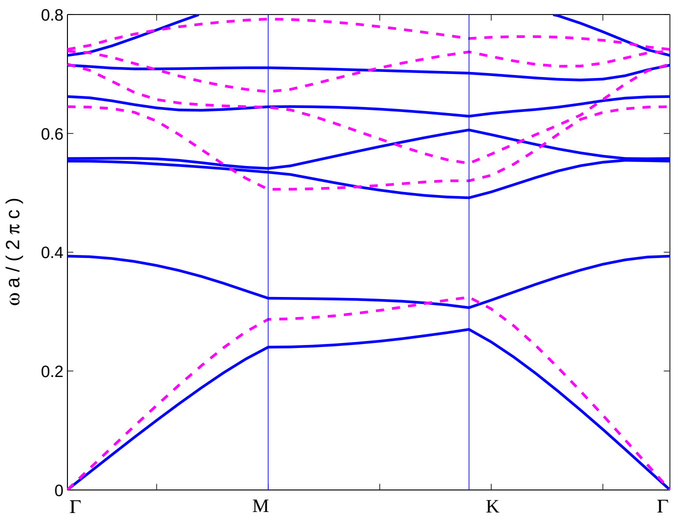

Fig. 8. Band structure of the triangular lattice of holes. Solid lines denote TM modes, dashed lines denote the TE ones.

figure shows a *complete band-gap*, i.e. a band gap for TE and TM polarizations.

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