CCM, Part II (3)

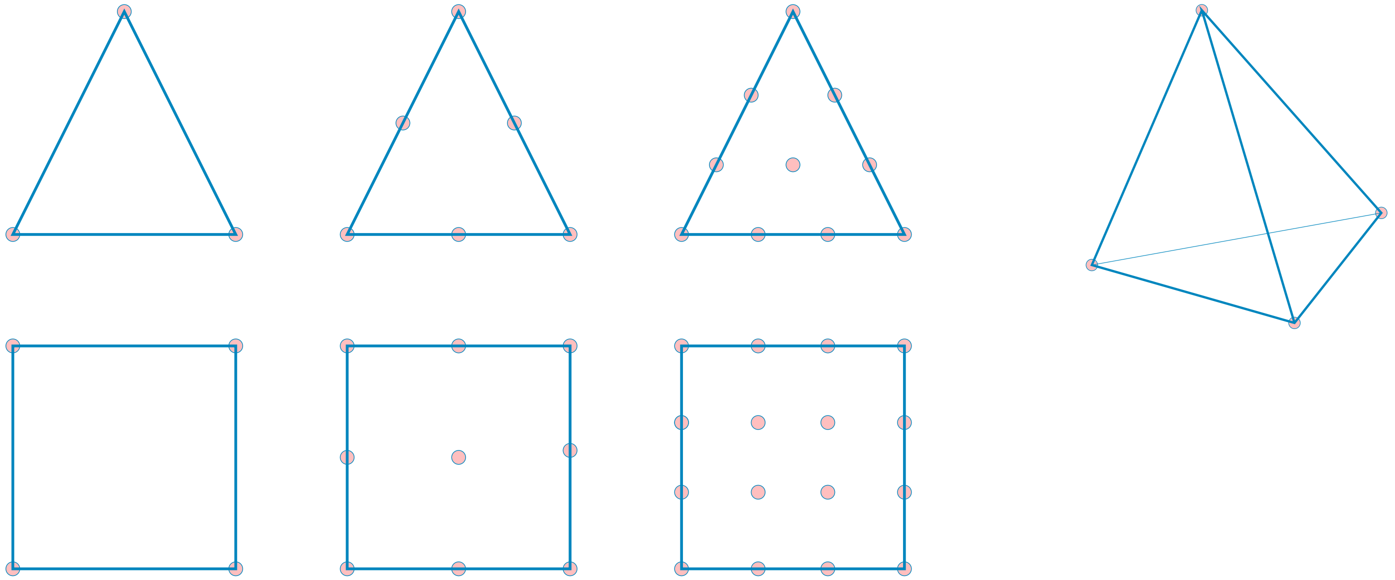
Daniele Boffi
Dipartimento di Matematica "F. Casorati"
Università di Pavia
daniele.boffi@unipv.it
http://www-dimat.unipv.it/boffi/

March 2005

Finite elements (cont'd)

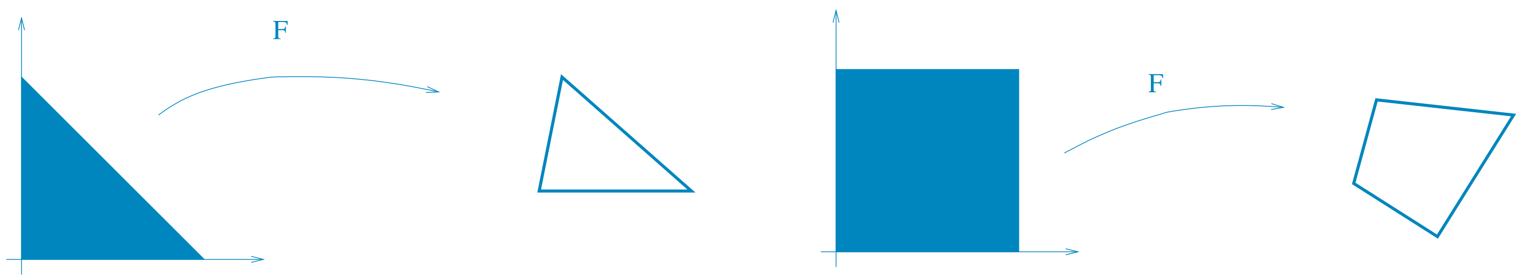
Generalization to more space dimensions

Example of unisolvent degrees of freedom



Finite elements (cont'ed)

How to construct stiffness matrix and load vector
In general one considers reference elements and mappings to actual elements



Notation: $\hat{K}$ reference element; K actual element
$\hat{\varphi}_1, \ldots \hat{\varphi}_N$ reference shape functions;
$\varphi_1, \ldots \varphi_N$ actual shape functions

Finite elements (cont'ed)

How to map the shape functions

$$F_K : \hat{K} \to K, \quad \vec{x} = F(\hat{\vec{x}})$$
$$\varphi(\vec{x}) = \hat{\varphi}(F^{-1}(\vec{x}))$$

Example of computation of *local* stiffness matrix (one dimensional)

$$A_{ji} = a(\varphi_i, \varphi_j) = \int_a^b \varphi_i'(x)\varphi_j'(x)\,dx = \sum_K \int_K \varphi_i'(x)\varphi_j'(x)\,dx$$
$$\int_K \varphi_i'(x)\varphi_j'(x)\,dx = \int_{\hat{K}} \frac{\hat{\varphi}_i'(\hat{x})}{F'(\hat{x})}\frac{\hat{\varphi}_j'(\hat{x})}{F'(\hat{x})}F'(\hat{x})\,d\hat{x} = \int_{\hat{K}} \frac{\hat{\varphi}_i'(\hat{x})\hat{\varphi}_j'(\hat{x})}{F'(\hat{x})}\,d\hat{x}$$

Finite elements (cont'ed)

$$\int_{\hat{K}} \frac{\hat{\varphi}_i'(\hat{x})\hat{\varphi}_j'(\hat{x})}{F'(\hat{x})}\, d\hat{x}$$

In general, $F = \alpha + \beta\hat{x}$ is affine so that $F' = \beta$ is constant (and equal to h)

$$\int_{\hat{K}} \frac{\hat{\varphi}_i'(\hat{x})\hat{\varphi}_j'(\hat{x})}{F'(\hat{\vec{x}})}\, dx = \frac{1}{h}\int_{\hat{K}} \hat{\varphi}_i'(\hat{x})\hat{\varphi}_j'(\hat{x})\, dx$$

In more space dimensions, F is affine for most popular elements.

$$\int_K \vec{\mathrm{grad}}\, \varphi_i(\vec{x}) \cdot \vec{\mathrm{grad}}\, \varphi_j(\vec{x})\, d\vec{x} = ?$$

Finite elements (cont'ed)

General strategy for assembling stiffness matrix and load vector

- Loop over elements $ie = 1, \ldots, ne$
- Compute local stiffness matrix $A^{loc}_{ji} = a(\varphi_i, \varphi_j)$, $i, j = 1, \ldots, ndof$ and local load vector $F^{loc}_i = F(\varphi_i)$, $i = 1, \ldots, ndof$
- Loop for $i, j = 1, \ldots, ndof$ and assembly of global matrix

$$A_{iglob,jglob} = A_{iglob,jglob} + A^{loc}_{ij}$$

- Account for boundary conditions

Finite elements (cont'ed)

Some remarks on the discrete linear system

- matrix is sparse (sparsity pattern, so called skyline, can be determined a priori)
- matrix is SPD (CG can be succesfully applied)
- conditioning of matrix grows as h goes to zero (need for preconditioning)

Convection diffusion equation

As usual. . . a one dimensional example

$$\begin{cases} -\varepsilon u''(x) + bu'(x) = 0 \qquad 0 < x < 1 \\ u(0) = 0, \; u(1) = 1 \end{cases}$$

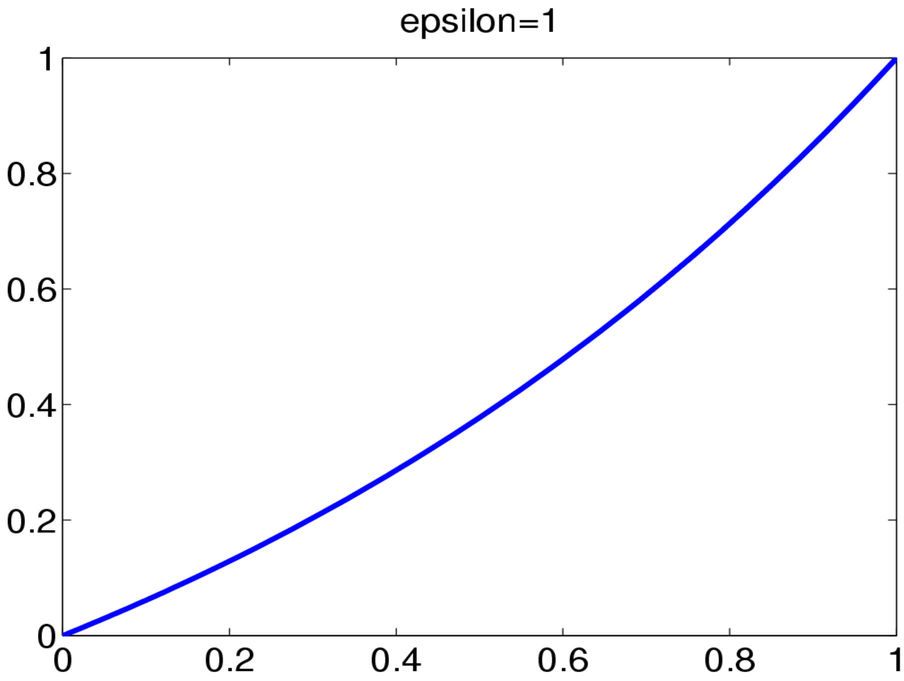
Non-homogeneous boundary conditions (!)
Péclet number $\mathbb{P} = |b|L/(2\varepsilon)$ ($L = 1$ in our case)

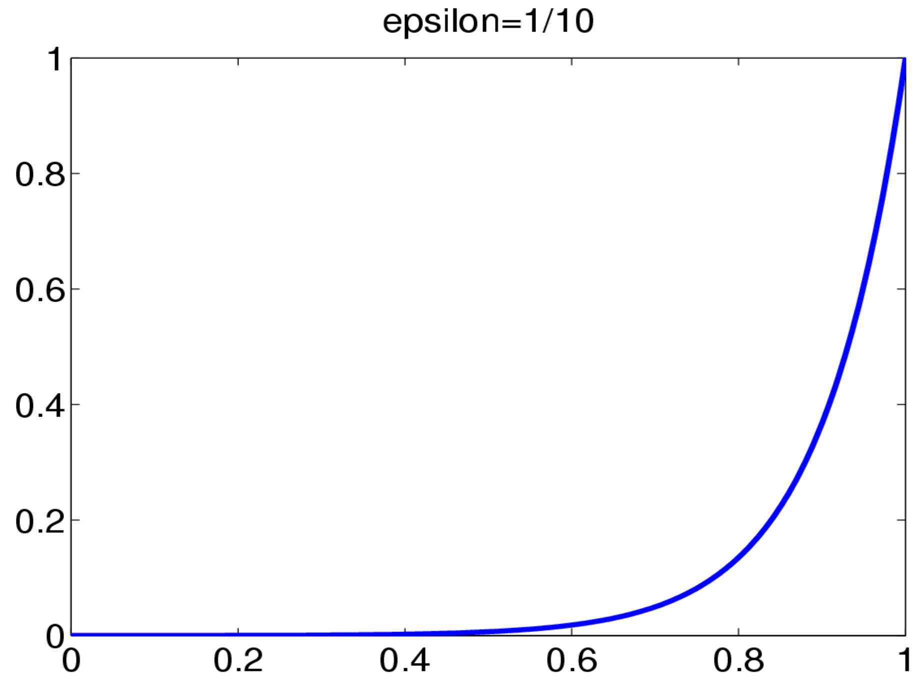
Closed form solution can be explicitly computed

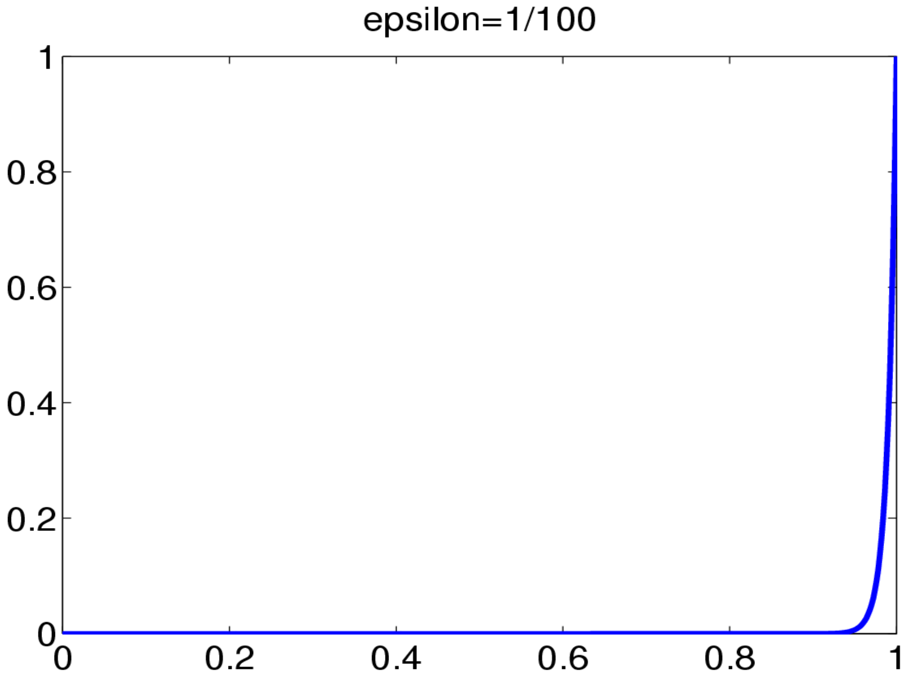
$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$

Convection diffusion equation (cont'ed)

$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$







If $b/\varepsilon \ll 1$ then $u(x) \simeq x$

If $b/\varepsilon \gg 1$ then $u(x) \simeq \exp(-b(1-x)/\varepsilon)$

In the second case, *boundary layer* of size $\mathcal{O}(\varepsilon/b)$

Convection diffusion equation (cont'ed)

Approximation by finite elements

$$a(u,v)=\int_0^1 \left(\varepsilon u'(x)v'(x)+bu'(x)v(x)\right)\,dx$$

After some computations. . . stiffness matrix is (uniform mesh):

$$\left(\frac{b}{2}-\frac{\varepsilon}{h}\right)u_{i+1}+\frac{2\varepsilon}{h}u_i+\left(-\frac{b}{2}-\frac{\varepsilon}{h}\right)u_{i-1}$$

Local (discrete) Péclet number is $\mathbb{P}(h)=|b|h/(2\varepsilon)$, so that our system has the structure

$$(\mathbb{P}(h)-1)u_{i+1}+2u_i-(\mathbb{P}(h)+1)u_{i-1}=0$$

Convection diffusion equation (cont'ed)

$$(\mathbb{P}(h) - 1)u_{i+1} + 2u_i - (\mathbb{P}(h) + 1)u_{i-1} = 0$$

General solution

$$u_i = \frac{1 - \left(\frac{1+\mathbb{P}(h)}{1-\mathbb{P}(h)}\right)^i}{1 - \left(\frac{1+\mathbb{P}(h)}{1-\mathbb{P}(h)}\right)^N} \quad i = 1, \ldots, N$$

If $\mathbb{P}(h) > 1$ solution oscillates!

Stabilization techiniques

- Upwind (finite differences)
- Artificial viscosity, streamline diffusion (loosing consistency)
- Petrov–Galerkin, SUPG (strongly consistent)

Hyperbolic equations

Let's consider the model problem (one dimensional convection equation)

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, & t > 0,\ x \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

Solution is a traveling wave $u(x,t) = u_0(x - at)$.

We consider a finite difference approximation.

Hyperbolic equations (cont'ed)

$u_j^n \simeq u(x_j, t_n)$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(h_{j+1/2}^n - h_{j-1/2}^n)$$

where $h_{j+1/2} = h(u_j, u_{j+1})$ is a *numerical flux*

Indeed,

$$\frac{\partial}{\partial t}U_j = -\left((au)(x_{j+1/2}) - (au)(x_{j-1/2})\right) \quad \text{with } U_j = \int_{x_{j-1/2}}^{x_{j+1/2}} u,\, dx$$

Hyperbolic equations (cont'ed)

Courant–Friedrichs–Lewy (CFL) condition

$$\left| a \frac{\Delta t}{\Delta x} \right| \leq 1$$

Very clear geometrical interpretation (see also multidimensional extension and generalization to systems)

Remark: implicit schemes (in time) don't have restrictions, but add artificial diffusion

End of part III