Quasi-conformal minimal Lagrangian diffeomorphisms of the hyperbolic plane

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(joint work with J.M. Schlenker)

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Quasi-symmetric homeomorphism of a circle

- A homeomorphism $\phi : S^1_\infty \to S^1_\infty$ is quasi-symmetric if there exists K such that

$$\frac{1}{K} \leq \frac{[\phi(a), \phi(b); \phi(c), \phi(d)]}{[a, b; c, d]} \leq K$$

for every $a, b, c, d \in S^1_\infty = \partial \mathbb{H}^2$.

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for every $a, b, c, d \in S^1_\infty = \partial\mathbb{H}^2$.

- A homeomorphism $g : S^1_\infty \to S^1_\infty$ is quasi-symmetric iff there exits a quasi-conformal diffeo ϕ of $\mathbb{H}^2$ such that $g = \phi|_{S^1_\infty}$.

The universal Teichmüller space

$\mathcal{T} = \{$quasi-conformal diffeomorphisms of $\mathbb{H}^2\}/\sim$
where $\phi \sim \psi$ is there is $A \in PSL_2(\mathbb{R})$ such that

$$\phi|_{S^1_\infty} = A \circ \psi|_{S^1_\infty}.$$

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Shoen conjecture

Conjecture (Shoen)

For any quasi-symmetric homeomorphism $g : S^1_\infty \to S^1_\infty$ there is a unique quasi-conformal harmonic *diffeo Φ of $\mathbb{H}^2$ such that $g = \Phi|_{S^1_\infty}$*

Main result

THM (B-Schlenker)

For any quasi-symmetric homeomorphism $g : S^1_\infty \to S^1_\infty$ there is a unique quasi-conformal minimal Lagrangian diffeomorphims $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ such that $g = \Phi|_{S^1_\infty}$

Minimal Lagrangian diffeomorphisms

A diffeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ is minimal Lagrangian if

- It is area-preserving;
- The graph of Φ is a minimal surface in $\mathbb{H}^2 \times \mathbb{H}^2$.

Minimal Lagrangian maps vs harmonic maps

Given a minimal Lagrangian diffemorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$, let $S \subset \mathbb{H}^2 \times \mathbb{H}^2$ be its graph, then the projections

$$\phi_1 : S \to \mathbb{H}^2 \qquad \phi_2 : S \to \mathbb{H}^2$$

are harmonic maps, and the sum of the corresponding Hopf differentials is 0.

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are harmonic maps, and the sum of the corresponding Hopf differentials is 0.

Conversely given two harmonic diffeomorphisms u, u^* such that the sum of the corresponding Hopf differentials is 0, then $u \circ (u^*)^{-1}$ is a minimal Lagrangian diffeomorphism.

Known results

- Labourie (1992): If S, S' are closed hyperbolic surfaces of the same genus, there is a unique $\Phi : S \to S'$ that is minimal Lagrangian.
- Aiyama-Akutagawa-Wan (2000): Every quasi-symmetric homeomorphism with small dilatation of S^1_∞ extends to a minimal Lagrangian diffeomorphism.
- Brendle (2008): If K, K' are two convex subsets of $\mathbb{H}^2$ of the same finite area, there is a unique minimal lagrangian diffeomorphism $g : K \to K'$.

The AdS geometry

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- Given a qs homeo g of the circle, we prove that minimal Lagrangian diffeomorphisms extending g correspond bijectively to maximal surfaces in AdS_3 satisfying some asymptotic conditions (determined by g).
- We prove that there exists a unique maximal surface satisfying these asymptotic conditions.

Remark

The correspondence

$$\{\textit{minimal Lagrangian maps of } \mathbb{H}^2\} \quad \leftrightarrow \{\textit{maximal surfaces in } AdS_3\}$$

is analogous to the classical correspondence

$$\{\textit{harmonic diffeomorphisms of } \mathbb{H}^2\} \quad \leftrightarrow \{\textit{surfaces of } H = 1 \textit{ in } \mathbb{M}^3\}$$

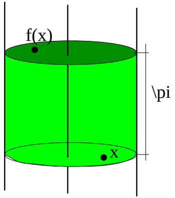
The Anti de Sitter space

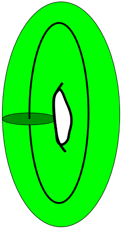
AdS_3= model manifolds of Lorentzian geometry of constant curvature -1.
$\tilde{AdS}_3 = (\mathbb{H}^2 \times \mathbb{R}, g)$ where

$$g_{(x,t)} = (g_{\mathbb{H}})_x - \phi(x) d\theta^2$$

$\phi(x) = \mathrm{ch}(d_{\mathbb{H}}(x, x_0))^2$ [Lapse function]

$AdS_3 = \tilde{AdS}_3/f$ where
$f(x,\theta) = (R_\pi(x), \theta+\pi)$ and R_π is the rotation of π around x_0





The boundary of AdS_3

$\partial_\infty AdS_3 \cong S^1 \times S^1$.

- The conformal structure of AdS_3 extends to the boundary.
- Isometries of AdS_3 extend to conformal diffeomorphisms of the boundary.

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- The conformal structure of AdS_3 extends to the boundary.
- Isometries of AdS_3 extend to conformal diffeomorphisms of the boundary.
- There are exactly two foliations of $\partial_\infty AdS_3$ by lightlike lines. They are called the left and right foliations.
- Leaves of the left foliation meet leaves of the right foliation exactly in one point.

The double foliation of the boundary of AdS_3

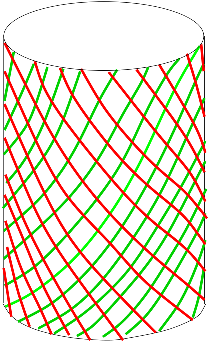

Figure: The l behaviour of the double foliation of $\partial_\infty \tilde{AdS}_3$.

The boundary of AdS_3

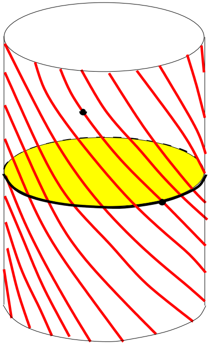

Figure: Every leaf of the left (right) foliation intersects $S^1_\infty \times \{0\}$ exactly once.

The product structure

The map

$$\pi : \partial_\infty AdS_3 \to S^1_\infty \times S^1_\infty$$

obtained by following the left and right leaves is a diffeomorphism.

Spacelike meridians

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- A a-causal curve in $\partial_\infty AdS_3$ is locally the graph of an orientation preserving homeomorphism between two intervals of S^1_∞.
- A-causal meridians are the graphs of orientation preserving homeomorphisms of S^1_∞ .

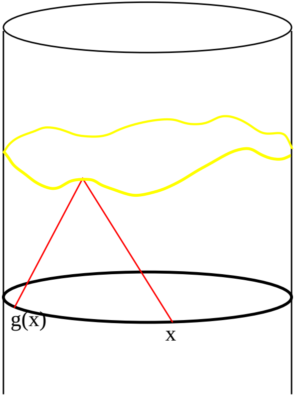

Figure: Every leaf of the left/right foliation intersects the meridian just in one point

Spacelike surfaces in AdS_3

- A smooth surface $S \subset AdS_3$ is spacelike if the restriction of the metric on TS is a Riemannian metric.
- Spacelike surfaces are locally graphs of some real function u defined on some open set of $\mathbb{H}^2$ verifying

$$\phi^2 ||\nabla u||^2 < 1 \, .$$

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- Spacelike compression disks lift in $\tilde{AdS}_3$ to graphs of entire spacelike functions $u : \mathbb{H}^2 \to \mathbb{R}$.

The asymptotic boundary of spacelike graphs

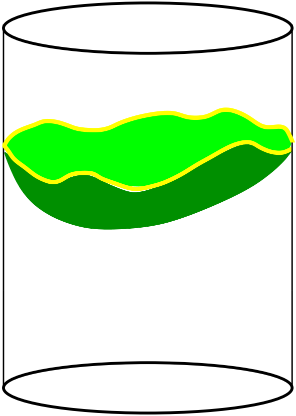

Figure: If $S = \Gamma_u$ is a spacelike graph in $\tilde{AdS}_3$, then u extends on the boundary and S projects to spacelike compression disk.

Notations

Let S be a spacelike surface in AdS_3. We consider:

1. I= the restriction of the Lorentzian metric on S;
2. J= the complex structure on S;
3. k= the intrinsic sectional curvature of S;
4. $B : TS \to TS$= the shape operator;
5. $E : TS \to TS$= the identity operator;
6. $H = \mathrm{tr} B$= the mean curvature of the surface S.

The Gauss-Codazzi equations are

$$d^{\nabla} B = 0 \qquad k = -1 - \det B\,.$$

Maximal surfaces

A surface $S \subset AdS_3$ is maximal if $H = 0$.

From maximal graphs to minimal diffeomorphisms of $\mathbb{H}^2$

Let S be any spacelike surface in AdS_3. We consider two bilinear forms on S

$$\mu_l(x, y) = I((E+JB)x, (E+JB)y) \qquad \mu_r = I((E-JB)x, (E-JB)y)$$

From maximal graphs to minimal diffeomorphisms of $\mathbb{H}^2$

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Prop (Krasnov-Schlenker)

Around points where μ_l (resp. μ_r) is not degenerate, it is a hyperbolic metric.

- When S is totally geodesic then $I = \mu_l = \mu_r$;
- $\det(E+JB) = \det(E-JB) = 1 + \det B = -k$

From maximal graphs to minimal diffeomorphisms

Let S be a spacelike graph:

- if $k < 0$ then μ_l and μ_r are hyperbolic metrics on S;
- if $k \leq -\epsilon < 0$ then μ_l and μ_r are complete hyperbolic metrics.

From maximal graphs to minimal diffeomorphisms

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Prop

Let S be a maximal graph with uniformly negative curvature and let

$$\phi_{S,l} : S \to \mathbb{H}^2 \qquad \phi_{S,r} : S \to \mathbb{H}^2 .$$

be the developing maps for μ_l and μ_r respectively.
The diffeomorphism $\Phi_S = \phi_{S,r} \circ \phi_{S,l}^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ is minimal Lagrangian. Moreover

- *It is C-quasi-conformal for some $C = C(\sup_S k)$*
- *The graph of $\Phi_S|_{S^1_\infty}$ is $\partial_\infty S$.*

From a minimal Lagrangian map to a maximal surface

Prop

Given any quasi-conformal minimal Lagrangian map $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ there is a unique maximal surface S with uniformly negative curvature producing Φ.

The proof relies on the fact that μ_l and μ_r determines I and B in some explicit way.

Given a quasi-symmetric homeo g of S^1_∞ the following facts are equivalent:

- There exists a unique quasi-conformal minimal Lagrangian diffeomorphism Φ of $\mathbb{H}^2$ such that $\Phi|_{S^1_\infty} = g$.
- There exists a maximal surface $S \subset AdS_3$ with uniformly negative curvature such that $\partial_\infty S = \Gamma_g$.

AdS results

THM (B-Schlenkler)

We fix a homeomorphism $g : S^1_\infty \to S^1_\infty$.

- *There is a maximal graph S such that $\partial_\infty S = \Gamma_g$.*
- *If g is quasi-symmetric then there is a unique S as above with uniformly negative curvature.*

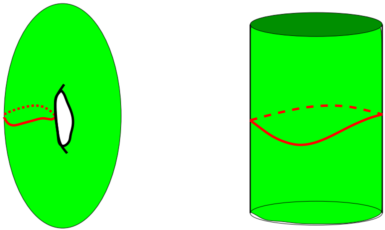
Higher dimension result

$$AdS_{n+1} = \mathbb{H}^n \times \mathbb{R}$$

THM (B-Schlenker)

Let Γ be any acausal subset of $\partial_\infty AdS_{n+1}$ that is a graph of a function $u : S^{n-1}_\infty \to \mathbb{R}$. Then there exists a maximal spacelike graph M in AdS_{n+1} such that $\partial_\infty M = \Gamma$.

We fix a homeomorphism $g : S^1_\infty \to S^1_\infty$.



We consider the lifting of the graph of g in $\tilde{AdS}_3$ that is a closed curve Γ_g.
We have to find a function u such that

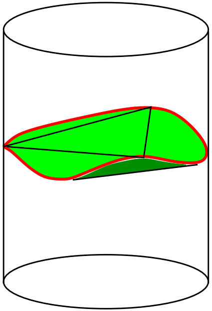
- its graph is spacelike $\Rightarrow \phi|\nabla u| < 1$;
- its graph is maximal $\Rightarrow Hu = 0$;
- the closure of its graph in ∂_∞ is Γ_g.

The convex hull of Γ_g

There is a minimal convex set K in $\tilde{AdS}_3$ containing Γ_g. Moreover:

- $\partial_\infty K = \Gamma_g$;
- The boundary of K is the union of two $\mathrm{C}^{0,1}$-spacelike graphs $\partial_- K, \partial_+ K$;

The convex hull of Γ_g

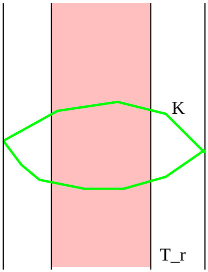


The approximations surfaces

Let $T_r = B_r(x_0) \times \mathbb{R}^1 \subset AdS_3$ and consider $U_r = T_r \cap \partial_- K$.

Prop (Bartnik)

There is a unique maximal surface S_r contained in T_r such that $\partial U_r = \partial S_r$. Moreover S_r is the graph of some function u_r defined on $B_r(x_0)$.



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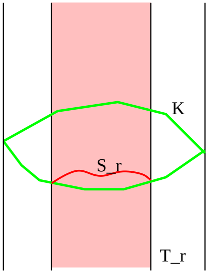


The existence of the maximal surface

Step 1 There is a sequence r_n such that $u_n := u_{r_n}$ converge to a function u_∞ uniformly on compact subset of $\mathbb{H}^2$. Moreover if S is the graph of u_∞ we have that $\partial_\infty S = \Gamma_g$.

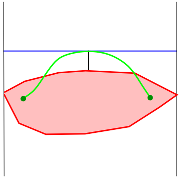
Step 2 The surface S is a maximal surface.

Surfaces S_r are contained in K

Lemma

If M is a cpt maximal surface such that ∂M is contained in K, then M is contained in K.

By contradiction suppose that M is not contained in K



$p \in M \setminus K$ = point that maximizes the distance from K.
$q \in \partial K$ = point such that $d(p, q) = d(p, K)$.
P= plane through p orthogonal to $[q, p]$.

P is tangent to M and does not disconnect $M \Rightarrow$ principal curvatures at p are negative.

The construction of the limit

- $S_r \subset K \Rightarrow u_r$ are uniformly bounded on $B_R(x_0)$.
- $\phi||\nabla u_r|| < 1 \Rightarrow$ The maps u_r are uniformly Lipschitz on any $B_R(x_0)$.

We conclude:

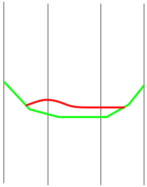
- There is a sequence r_n such that $u_n = u_{r_n}$ converge uniformly on compact sets of $\mathbb{H}^2$ to a function u_∞.
- The graph of the map u_∞ – say S – is a weakly spacelike surface: it is Lipschitz and satisfies $\phi|\nabla u_\infty| \leq 1$.

The asymptotic boundary of S

- S is contained in $K \Rightarrow \partial_\infty S \subset \Gamma_g$.
- $\partial_\infty S$ is a spacelike meridian of $\partial_\infty AdS_3$

$$\partial_\infty S = \Gamma_g \,.$$

A possible degeneration



The surface S could contain some lightlike ray.

Remark

We have to prove that the surfaces S_n are uniformly spacelike in T_ρ.

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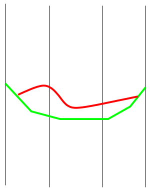

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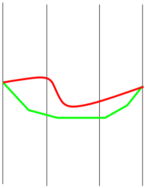

The surface S could contain some lightlike ray.

Remark

We have to prove that the surfaces S_n are uniformly spacelike in T_R.

Uniformly spacelike surfaces

Let U be a compact domain of $\mathbb{H}^2$.
The graph of a function $u : U \to \mathbb{R}$ is spacelike if

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A family of graphs over U – $\{\Gamma_{u_i}\}_{i \in I}$ is uniformly spacelike if there exists $\epsilon > 0$ such that

$$\phi^2 ||\nabla u_i||^2 < (1 - \epsilon)$$

holds for every $x \in U$ and $i \in I$.

The main estimate

Prop

For every $R > 0$ there is a constant $\epsilon = \epsilon(R, K)$ such that

$$\sup_{B_R(x_0)} \phi|\nabla u_n| < (1 - \epsilon)$$

for $n > n(R)$

The proof is based on the maximum principle using a localization argument due to Bartnik.

The conclusion of the proof of the existence

Let $\Omega_R = \{u : B_R(x_0) \to \mathbb{R} | \Gamma_u \text{ is spacelike}\}$
We consider the operator $H : \Omega_R \to \mathbb{C}^\infty(B_R(x_0))$

$$Hu(x) = \text{mean curvature at } (x, u(x)) \text{ of } \Gamma_u \, .$$

$Hu = \sum a_{ij}(x, u, \nabla u)\partial_{ij}u + \sum b_k(x, u, \nabla u)\partial_k u$.
H is an elliptic operator on Ω_R at point $u \in \Omega_R$.
H is uniformly elliptic on any family of uniformly spacelike functions.

How to conclude

- $u_n|_{B_R(x_0)}$ is a uniformly spacelike functions;
- they are solution of a uniformly elliptic equation $Hu_n = 0$;

By standard theory of regularity of elliptic equations $\rightarrow$ the limit u_∞ is smooth and $Hu_\infty = 0$.

The uniform estimate

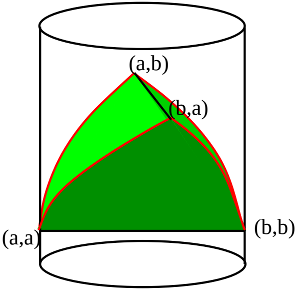
The width of the convex hull K

$$\delta = \inf\{d(x,y) | x \in \partial_- K, y \in \partial_+ K\} .$$

Lemma

In general $\delta \in [0, \pi/2]$*. It is* 0 *exactly when g is a symmetric map.*
If $\delta = \pi/2$ *and there are points at distance* $\pi/2$*, then* K *is a standard tetrahedron* K_0*.*

The standard tetrahedron



Characterization of quasi-symmetric maps

Prop

The following facts are equivalent:

1. *g is a quasi-symmetric homeomorphism.*
2. $\delta < \pi/2$.
3. *Any maximal surface S such that $\partial_\infty S = \Gamma_g$ has uniformly negative curvature.*

(3) $\Rightarrow$ (1)

S determines a quasi-conformal minimal Lagrangian map Φ such that $\Phi|_{S^1_\infty} = g$.
Thus g is quasi-symmetric.

$(1) \Rightarrow (2)$

Suppose there exists $x_n \in \partial_- K$ and $y_n \in \partial_+ K$ such that $d(x_n, y_n) \to \pi/2$
We find a sequence of isometries γ_n of AdS_3 such that

- $\gamma_n(x_n) = x_0$.
- the geodesic joining $\gamma_n(x_n)$ to $\gamma_n(y_n)$ is vertical.

$(1) \Rightarrow (2)$

Let $K_n = \gamma_n(K)$.

- $\partial_\infty K_n = \Gamma_{g_n}$ and $\{g_n\}$ are uniformly quasi-symmetric.
- $K_n \to K_0$ and $\Gamma_{g_n} \to \partial_\infty K_0$.

$(1) \Rightarrow (2)$

Let $K_n = \gamma_n(K)$.

- $\partial_\infty K_n = \Gamma_{g_n}$ and $\{g_n\}$ are uniformly quasi-symmetric.
- $K_n \to K_0$ and $\Gamma_{g_n} \to \partial_\infty K_0$.
- The boundary of K_0 cannot be approximated by a family of uniformly quasi-symmetric maps.

(2) ⇒ (3)

We consider $\chi = \log(-(\det B)/4)$. We have $k = -1 + e^{4\chi}$ and

$$\Delta\chi = k$$

[Schlenker-Krasnov].
If p is a local maximum for k then $k(p) \leq 0$. Moreover if $k(p) = 0$, then S is flat and $K = K_0$.

$(2) \Rightarrow (3)$

Lemma

If $\delta < \pi/2$ *then* $\sup_S ||B|| < C$.

$(2) \Rightarrow (3)$

Take any sequence x_n such that $k(x_n) \to \sup k$.
Let γ_n be a sequence such that $\gamma_n(x_n) = x_0$ and $\nu_n(x) = e$ (where ν_n is the normal field of $S_n = \gamma_n(S)$.
$S_n \to S_\infty$ and $x \in S_\infty$, $k_\infty = \sup k$ and x is a local maximum for k_∞.
$\sup k \leq 0$.

$(2) \Rightarrow (3)$

If $\sup k = 0$ then S_∞ is a flat maximal surface $\to$ its convex core is K_0. In particular $\delta(K_0) = \pi/2$
On the other hand $K_n = \gamma_n(S_n) \to K_0$.

- $\delta(K_n) = \delta < \pi/2$.
- $\delta(K_n) \to \delta(K_0) = \pi/2$.