Mean curvature flow in Anti de Sitter space

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(joint work with J.M. Schlenker)

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The space

$\mathbb{H}^n = (B^n, g_{\mathbb{H}})$ where

$$B^n = \{(x_1, \ldots, x_n) | \sum x_i^2 < 1\}$$

$$g_{\mathbb{H}} = \frac{4}{(1-r^2)^2}(dx_1^2 + \ldots + dx_n^2) \qquad r = \sqrt{x_1^2 + \ldots + x_n^2}$$

The spacetime

AdS_{n+1}=$(\mathbb{H}^n \times \mathbb{R}, g)$ where

$$g = g_{\mathbb{H}} - \left(\frac{1+r^2}{1-r^2}\right)^2 dt^2$$

It is the space-form of constant curvature -1 in Lorentzian geometry.

The asymptotic boundary

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On $\partial_\infty AdS_{n+1}$ a conformal Lorentzian structure is defined.

Spacelike graphs

Let M_0 be a spacelike hypersurface in AdS_{n+1} that is a graph of a function

$$u_0 : \mathbb{H}^n \longrightarrow \mathbb{R}$$

Remark (Mess)

Every complete spacelike surface in AdS_{n+1} is a graph.

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Remark

The function u_0 extends to a function $\bar{u}_0 : \overline{\mathbb{H}}^n \to \mathbb{R}$. The graph of $\bar{u}_0$ is the closure of M_0 into $\overline{AdS}^{n+1}$.

$\partial\infty M =$ closure of M in $\partial_\infty AdS_{n+1}$

The mean curvature flow

Let M_{u_0} be a spacelike graph in AdS_{n+1}.
The mean curvature flow starting from M_{u_0} is a map

$$\sigma : M_{u_0} \times [0, T] \to AdS_{n+1}$$

such that

- For every s the map $\sigma(\cdot, s)$ is an embedding of M_{u_0} onto a spacelike graph M_{u_s};
- $\frac{\partial \sigma}{\partial s} = H\nu$;
- $\sigma(x, 0) = x$;
- $\partial_\infty M_{u_s} = \partial_\infty M_{u_0}$.

Longtime existence of MCF

Let M be a spacelike graph. In general $\partial_\infty M$ is achronal.

If $\partial_\infty M$ is acausal we prove that there exists a family of spacelike hypersurfaces $(M_s)_{s\in[0,+\infty)}$ such that

- M_s are moving by Mean Curvature Flow;
- $M_0 = M$;
- $\partial_\infty M_s = \partial_\infty M$;
- M_s is the graph of a function $u_s : \mathbb{H}^n \to \mathbb{R}$;

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- $\partial_\infty M_s = \partial_\infty M$;
- M_s is the graph of a function $u_s : \mathbb{H}^n \to \mathbb{R}$;.
- For any divergent sequence s_n, there is a subsequence s_{n_k} such that $M_{s_{n_k}}$ converge to a maximal surface N with $\partial_\infty N = \partial_\infty M$.

Let Γ be a spacelike curve in ∂AdS_3 that is the graph of some function

$$\psi : \partial\mathbb{H}^2 \to \mathbb{R}\,.$$

Let K be the convex hull of Γ. The width of Γ is

$$w(\Gamma) = \sup\{\ell(c) | c \text{ is a timelike path contained in } K\}\,.$$

We have $w(\Gamma) \in [0, \pi/2]$, and $w(\Gamma) = 0$ iff K is a totally geodesic plane.

Maximal surfaces in AdS_3

THM

If $w(\Gamma) < \pi/2$, then there is a unique maximal surface $M \subset AdS_3$ such that

- $\partial_\infty M = \Gamma$.
- *There is $\epsilon > 0$ such that $k_M < -\epsilon^2$.*

Moreover, the second fundamental form is bounded.

Quasi-conformal diffeomorphism of the hyperbolic plane

- A diffeomorphism $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ is quasi-conformal if there exists $C < 1$ such that

$$|\partial_{\bar{z}}\phi| < C|\partial_z \phi| \, .$$

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- Any quasi-conformal diffeomorphism of $\mathbb{H}^2$ extends to a homeomorphism of $S^1_\infty = \partial\mathbb{H}^2$ that is quasi-symmetric.

Quasi-symmetric homeomorphism of a circle

- A homeomorphism $g : S^1_\infty \to S^1_\infty$ is quasi-symmetric if there exists K such that

$$\left| \frac{g(\theta + h) - g(\theta)}{g(\theta - h) - g(\theta)} \right| < K$$

for every $\theta, h \in \mathbb{R}$.
- A homeomorphism $g : S^1_\infty \to S^1_\infty$ is quasi-symmetric iff there exists a quasi-conformal diffeo ϕ of $\mathbb{H}^2$ such that $g = \phi|_{S^1_\infty}$.

Schoen conjecture

Conjecture (Shoen)

For any quasi-symmetric homeomorphism $g : S^1_\infty \to S^1_\infty$ there is a unique quasi-conformal harmonic diffeo ϕ of $\mathbb{H}^2$ such that $g = \phi|_{S^1_\infty}$

Main result

THM (B-Schlenker)

For any quasi-symmetric homeomorphism $g : S^1_\infty \to S^1_\infty$ there is a unique quasi-conformal minimal Lagrangian diffeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ such that $g = \Phi|_{S^1_\infty}$

Minimal Lagrangian diffeomorphisms

A diffeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ is minimal Lagrangian if

- It is area-preserving;
- The graph of Φ is a minimal surface in $\mathbb{H}^2 \times \mathbb{H}^2$.

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- Given a qs homeo g of the circle, we construct a curve Γ_g in $\partial_\infty AdS_3$ such that
  1. $w(\Gamma_g) < \pi/2$.
  2. qc minimal Lagrangian diffeomorphisms extending g correspond bijectively to maximal surfaces in AdS_3 with uniformly negative curvature spanning Γ_g.

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- The result eventually follows from THM 1.

Spacelike graphs in AdS^3

Let M be a spacelike graph in AdS^3 and let us consider the following tensors on M

- γ: the first fundamental form;
- b: the shape operator;
- J: the positive rotation of $\pi/2$.

Consider the symmetric two forms

$$\gamma_\pm(v, w) = \gamma((Id \pm Jb)(v), (Id \pm Jb)(w))$$

THM (Krasnov-Schlenker)

If the sectional curvature of M is negative, then $\gamma_\pm$ are non-degenerated hyperbolic metrics. The metrics $\gamma_\pm$ are complete provided that M has bounded second fundamental form and uniformly negative curvature.

Spacelike graphs in AdS^3

In the hypotheses of the theorem there are two isometries

$$\phi_\pm : \mathbb{H}^2 \to (M, \gamma_\pm)$$

Consider the diffeo $f_M = \phi_+^{-1} \circ \phi_- : \mathbb{H}^2 \to \mathbb{H}^2$:

- f_M is area-preserving.

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- f_M is quasi-conformal: there is $K < 1$ $K = K(\sup ||A||, -1/k)$ such that

$$|\partial_{\bar{z}} f_M| \leq K |\partial_z f_M| \, .$$

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- If M is a maximal surface then the graph of f_M is a minimal surface in $\mathbb{H}^2 \times \mathbb{H}^2$.

Spacelike functions

A function

$$u : \mathbb{H}^n \longrightarrow \mathbb{R}$$

is spacelike if its graph

$$M_u = \{(x, u(x)) \in AdS_{n+1} | x \in \mathbb{H}^n\}$$

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is spacelike.
The function u is spacelike iff

$$\sum \left(\frac{\partial u}{\partial x_i}\right)^2 < \left(\frac{2}{1+r^2}\right)^2 .$$

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The function u is 4 Lipschitz w.r.t. the Euclidean metric of the ball.
The function u extends to the boundary and $u|_{\partial \mathbb{H}^n}$ is 1-Lipschitz.

Let $\psi : \partial\mathbb{H}^n \to \mathbb{R}$ be a 1-Lipschitz function. Then there is a spacelike function $u : \mathbb{H}^n \to \mathbb{R}$ such that

$$u|_{\partial\mathbb{H}^2} = \psi\,.$$

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There are two functions

$$u_\pm : \mathbb{H}^n \to \mathbb{R}$$

such that

- they extend ψ;
- if u is any spacelike function extending ψ then

$$u_-(x) \leq u(x) \leq u_+(x)\,.$$

The domain D_ψ

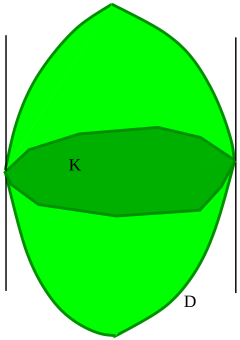
$D_\psi = \{(x,t) | u_-(x) \leq t \leq u_+(x)\}$

- If u is a spacelike function extending ψ, then M_u is contained in D_ψ.
- D_ψ is a convex region of AdS_{n+1}.

The convex hull of M_u

Let u be a spacelike function and $\psi = u|_{\partial\mathbb{H}^n}$.

- The convex hull of M_u say K_u is contained in D_ψ.
- If the graph of ψ is a-causal, then K_u is contained in the interior of D_ψ.



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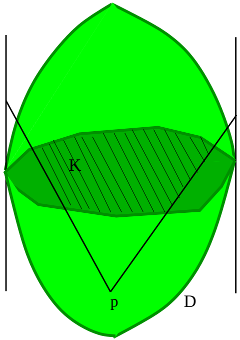
- The convex hull of M_u say K_u is contained in D_ψ.
- If the graph of ψ is a-causal, then K_u is contained in the interior of D_ψ.
- For any p in the interior of D_ψ, $I^+(p) \cap K_u$ is a compact region.



The modified version of the flow

The modified mean curvature flow is a map

$$\hat{\sigma} : \mathbb{H}^n \times [0, T] \to AdS_{n+1}$$

such that

- $\hat{\sigma}(x, s) = (x, u_s(x))$;
- u_s is a spacelike function;
- $g(\frac{\partial \hat{\sigma}}{\partial s}, \nu) = -H$;
- $u_s(x) = u_0(x)$ for every $x \in \partial\mathbb{H}^n$ and $s \leq T$

The statement

THM

Let M_0 be a spacelike graph in AdS_{n+1} such that $\partial_\infty M_0$ is acausal. Then there is a modified MCF

$$\hat{\sigma} : \mathbb{H}^n \times [0, +\infty) \to AdS_{n+1}$$

starting from M_0.

Scheme of the proof

Step 1: Let M^k be the intersection of M_0 with $C_k = B(0,k) \times \mathbb{R}$. Prove that there is a MCF M^k_s for $s \in (0,+\infty)$ such that

1. $M^k_0 = M^k$;
2. $\partial M^k_s = \partial M^k$;
3. M^k_s is the graph of a function $u^{(k)}_s : B(0,k) \to \mathbb{R}$.

Step 2: Prove that for every $r > 0$ and $T > 0$ the family

$$\{u^{(k)}|_{B(0,r)\times[1/T,+\infty)}\}$$

is compact in the space of spacelike functions with respect to the topology of $C^\infty(B(0,r) \times [1/T,+\infty))$.

Scheme of the proof

By a diagonal process we find a sequence k_n such that $u^{(k_n)}$ converges to some map

$$u : \mathbb{H}^n \times (0, +\infty) \to \mathbb{R}$$

in the topology of $C^\infty(\mathbb{H}^n \times (0, +\infty))$. We have that $\hat{\sigma}(x, s) = (x, u(x, s))$ is a modified MCF.

Step3: Prove that $u(x, s)$ converges to u_0 as $s \to 0$ and that $u(x, s) = \psi(x)$ for $x \in \partial\mathbb{H}^2$ and $s \in [0, +\infty)$.

Step 1

Let M^k be the intersection of M_{u_0} with the cylinder $C_k = B(0,k) \times \mathbb{R}$.
Ecker $\Rightarrow$ There exists a smooth family of surfaces M^k_s moving by MCF such that

- M^k_s is the graph of some spacelike function $u^{(k)}_s : B(0,k) \to \mathbb{R}$.
- $\partial M^k_s = \partial M^k$.

The gradient function

Given a spacelike hypersurface M in AdS_{n+1}, let ν be the normal field. The gradient function on M is defined by

$$v_M(x) = -\langle \nu(x), \frac{1}{|\partial_t|}\partial_t \rangle .$$

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Suppose that we have a family of modified MCF $\sigma_k : \Omega \times [0, T] \to AdS_{n+1}$ ans set $N_{k,s} = \sigma_k(\Omega, s)$. Suppose that there are constants $h, c, c_0, c_1, c_2, \ldots$ such that for every $s \in [0, T]$ and every $k \in \mathbb{N}$

$$\begin{aligned} &\sup_{N_{k,s}} |t| \leq h \\ &\sup_{N_{k,s}} v < c \\ &\sup_{N_{k,s}} |\nabla^m A|^2 < c_m \qquad \text{for } m = 0, 1, \ldots \end{aligned}$$

then, up to some subsequence, σ_k converges to some modified MCF $\sigma_\infty : \Omega \times [0, T] \to AdS_{n+1}$ in C^∞-topology.

MCF and convex hull

Let K be the convex hull of M_0. By maximum principle, M_s^k is contained in K for every $k > 0$ and $s > 0$.

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Let K be the convex hull of M_0. By maximum principle, M_s^k is contained in K for every $k > 0$ and $s > 0$.
An a-priori height estimate for M_s^k:

$$\sup_{M_s^k \cap C_r} |t| \leq \sup_{K \cap C_r} |t| .$$

The main estimate

Prop

For every $r > 0$ there are constants $c, c_0, c_1, c_2, \ldots$ such that

$$\sup_{M_s^k \cap C_r} v < c$$
$$\sup_{M_s^k \cap C_r} |\nabla^m A| < c_m \, .$$

for $k > \bar{k}(r)$.

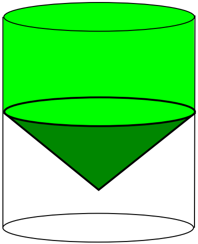
The proof is based on the maximum principle using a localization argument due to Ecker.

The future of a point in AdS_{n+1}

Given $p \in AdS_{n+1}$ we consider the function $\tau : I^+(p) \to \mathbb{R}$ that is the Lorentzian distance from p

$$\tau(q) = \sup\{\ell(c) | c \text{ causal path joining } p \text{ to } q\}$$

For $\epsilon \geq 0$ we put $I^+_\epsilon(p) = \tau^{-1}((\epsilon, +\infty))$.



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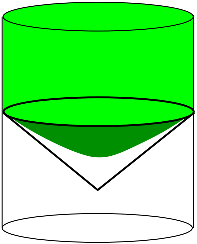


Ecker estimate

Let $p \in AdS_{n+1}$ be such that $I^+(p) \cap K$ is compact.
We consider any family of surfaces N_s moving by MCF such that

- $N_s \subset K$;
- $\partial N_s \cap (I^+(p) \cap K) = \emptyset$

We find a uniform estimate for v_{N_s}, A and $|\nabla^m A|$ on the domain $I^+_\epsilon(p) \cap K$

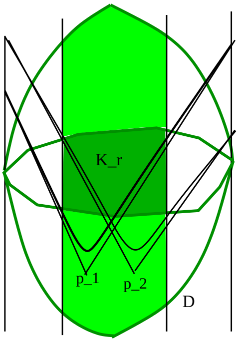
Lemma

For every $T > 0$, there are constants $c, c_0, c_1, \ldots$ only depending on p, K, T, ϵ such that for every N_s as above with $s > 1/T$

$$\sup_{I^+_\epsilon(p)\cap N_s} v \leq c$$
$$\sup_{I^+_\epsilon(p)\cap N_s} |\nabla^m A| < c_m .$$

The main estimate

Let $K_r = K \cap C_r$.
There is a finite family of points $p_1 \dots, p_n \in D$ and numbers $\epsilon_1, \dots \epsilon_n$ such that $K_r \subset \bigcup I^+_{\epsilon_j}(p_j)$.



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There is a finite family of points $p_1 \dots, p_n \in D$ and numbers $\epsilon_1, \dots \epsilon_n$ such that $K_r \subset \bigcup I^+_{\epsilon_j}(p_j)$.
For $k >> 0$, $\partial M^k_s \cap I^+(p_j) = \emptyset$ for $j = 1, \dots, n$. So there exists c such that for every $j = 1, \dots, k$

$$\sup_{M^k_s \cap I^+_{\epsilon_j}(p_j)} v < c\,.$$

for $s > 1/T$. In particular

$$\sup_{M^k_s \cap K_r} v \leq c\,.$$

Step 3: The condition at infinity

We prove that for every k for every $x \in B(0,k)$ and for every s we have that

$$u_-(x) \leq u^{(k)}(x,s) \leq u_+(x)$$

As a consequence we have that

$$u_-(x) \leq u(x,s) \leq u_+(x)$$

for every $x \in \mathbb{H}^n$ and for every $s \in [0,+\infty)$.
Since u_- and u_+ coincides with u_0 on the boundary, then the same holds for every $u(\cdot,s)$.

An a priori estimate for H

$$\left(\frac{\partial}{\partial s} - \Delta\right)(sH^2) \leq H^2 - \frac{n}{2}sH^4$$

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If Ω is a compact domain in $\mathbb{H}^n$ and

$$\sigma : \Omega \times [0, T] \to AdS_{n+1}$$

is a MCF such that $\sigma(x, s) = \sigma(x, 0)$ for every $s > 0$, then $H(x, s) = 0$ for every $x \in \partial\Omega$ and every $s > 0$.
By maximum principle we get

$$H^2_{\max}(s) \leq \frac{2}{ns}$$

Estimate for H on M_{u_s}

By the previous argument for every $u_s^{(k)}$ we get

$$H^2_{u_s^{(k)}}(x) < \frac{2}{ns}$$

for every $x \in B(0,k)$.

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for every $x \in B(0, k)$. Passing to the limit

$$H^2_{u_s} < \frac{2}{ns}$$

for every $x \in \mathbb{H}^n$.

Step 3: convergence of $u(x, s)$ as $s \to 0$

Since $\partial_s u_s^{(k)} v = H$ we derive that

$$\partial_s u_s^{(k)} \leq \sqrt{\frac{2}{ns}}$$

In particular we get

$$|u_s^{(k)}(x) - u_0(x)| \leq \sqrt{\frac{2s}{n}} .$$

Passing to the limit on k we have

$$|u(x, s) - u_0(x)| \leq \sqrt{\frac{2s}{n}} .$$