On the Locus of Curves with Automorphisms (*).

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Sunto. – *Si descrivono le componenti del luogo delle curve con automorfismi non banali all'interno dello spazio dei moduli delle curve algebriche di genere maggiore o uguale a tre. Si descrivono poi le componenti del luogo singolare dello spazio dei moduli delle curve di genere maggiore o uguale a due.*

We shall describe the components of the locus of curves with non-trivial automorphisms in M_g, the moduli space of smooth genus g curves over the complex numbers; we shall denote this locus by S_g. As a byproduct, we shall obtain a description of the components of the singular locus of M_g. We thank Dan Madden for drawing our attention to this entertaining little problem.

We begin by discussing cyclic coverings of prime order p. Let X be a smooth curve, and $D = \sum a_i q_i$ an effective divisor on X. Suppose that $a_i < p$ for every i and that p divides $\sum a_i$. Then there is a line bundle L on X such that

$$L^p = \mathcal{O}(D) .$$

If $D = 0$ we also want L to be non-trivial; thus we have to exclude the case when X is rational. Let Γ be the inverse image of the section 1 of $\mathcal{O}(D)$ under the p-th power map $L \to L^p$, C the normalization of Γ, and $f\colon C \to X$ the natural projection. Then C is a connected p-fold cyclic covering of X, branched at the q_i. The covering transformations of C over X correspond to multiplication by p-th roots of unity in L. We shall write $C(p, X, D, L)$ to denote C when we will want to keep track of the way C was constructed.

Pick a primitive p-th root of unity ζ and let γ be the corresponding element of Aut (C/X). Denote by Q_i the inverse image of q_i in C. If we view the sections of L^{-1} as functions on L and hence on C, such a function φ obeys the transformation rule

$$\varphi(\gamma(P)) = \zeta\varphi(P) .$$

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Conversely, any function satisfying this rule extends, by linearity, to a section of L^{-1}. To see this it suffices to check that the section so obtained, a priori only meromorphic, is regular near each q_i. Let w and t be a coordinate on X centered at q_i and a fiber coordinate on L, respectively. The equation of Γ near Q_i is

$$t^p = w^{a_i}$$

and the normalization map from C to Γ is

$$t = z^{a_i}; \quad w = z^p ,$$

where z is a coordinate centered at Q_i. Then

$$t(\gamma(P)) = \zeta t(P) ,$$

hence

$$z(\gamma(P)) = \zeta^{b_i} z(P) ,$$

where b_i is the inverse of a_i modulo p. Writing

$$\varphi(z) = \sum_{j \geq 0} \alpha_j z^j ,$$

we find that α_j must be zero except when jb_i is congruent to 1 modulo p, i.e., except when j is congruent to a_i modulo p. Thus φ equals t times a holomorphic function of w. In conclusion, we have shown that, if we decompose $f_*(\mathcal{O}_C)$ according to the various irreducible representations of $\mathbf{Z}/(p)$, i.e., if we write

$$f_*(\mathcal{O}_C) = \bigoplus_{\xi} f_*(\mathcal{O}_C)^{\xi} ,$$

where ξ runs through the p-th roots of unity and a section of $f_*(\mathcal{O}_C)^{\xi}$ obeys the rule

$$\varphi(\gamma(P)) = \xi\varphi(P) ,$$

then $L^{-1} = f_*(\mathcal{O}_C)^{\zeta}$.

Any p-sheeted cyclic covering of X can be obtained by the construction we have outlined. To see this, let $\psi\colon E \to X$ be such a covering, and δ a generator of the group of its covering transformations. Set $L^{-1} = \psi_*(\mathcal{O}_E)^{\zeta}$. Let $q_1, \ldots, q_n$ be the branch points of ψ and $Q_1, \ldots, Q_n$ their inverse images in E. Choose local coordinates z_i centered at the Q_i in such a way that z_i^p is a local coordinate at q_i. Write

$$z_i(\delta(P)) = \zeta^{b_i} z_i(P) ,$$

and let a_i be the inverse of b_i modulo p. It follows from the same calculations that we used to identify $f_*(\mathcal{O}_C)^\zeta$ that $L^p = \mathcal{O}(D)$, where

$$D = \sum a_i q_i .$$

Then E is isomorphic to $C(p, X, D, L)$. The map j from E to L inducing this isomorphism is given by the following prescription. Let $\langle\, ,\rangle$ be the duality pairing between L and $\psi_*(\mathcal{O}_E)^\zeta$. Then

$$\langle j(P), \varphi\rangle = \varphi(P) .$$

LEMMA 1. – *Let* $D = \sum a_i q_i$, $D' = \sum a_i' q_i$ *be effective divisors and* L, L' *line bundles on* X *such that* $L^p = \mathcal{O}(D)$, $L'^p = \mathcal{O}(D')$. *Then there is an isomorphism of coverings of* X

$$\alpha\colon C(p, X, D, L) \to C(p, X, D', L')$$

if and only if there is an integer b, $1 \leqslant b < p$, *such that*

i) $ba_i \equiv a_i' \pmod p$, $\quad i = 1, \ldots, n$,

ii) $L^b \cong L'\left(\sum c_i q_i\right)$, $\quad$ *where* $ba_i = a_i' + c_i p$.

PROOF. – Set $C = C(p, X, D, L)$, $C' = C(p, X, D', L')$, and let $f\colon C \to X$, $f'\colon C' \to X$ be the projections. We know that $L = f_*(\mathcal{O}_C)^\zeta$, $L' = f'_*(\mathcal{O}_{C'})^\zeta$, with respect to generators γ, γ' of Aut (C/X) and Aut (C'/X). Suppose α exists; write

$$\alpha^{-1}\gamma'\alpha = \gamma^b .$$

Thus

$$L'^{-1} = f_*(\mathcal{O}_C)^{\zeta^b} .$$

Clearly L^{-b} is a subsheaf of $f_*(\mathcal{O}_C)^{\zeta^b}$ and agrees with it away from $q_1, \ldots, q_n$. Thus

$$f_*(\mathcal{O}_C)^{\zeta^b} = L^{-b}(\Delta) ,$$

where Δ is supported at $\sum q_i$. Let z, w be local coordinates on C, X such that w is centered at q_i and $w = z^p$. Let t be a fiber coordinate on L. As we have already observed, we can choose t in such a way that $t = z^{a_i}$. The function z^{a_i} is a local generator for L^{-1}, so

$$z^{ba_i} = z^{a_i'} w^{c_i}$$

is a local generator for L^{-b}. On the other hand, $z^{a_i'}$ is clearly a local generator for $f_*(\mathcal{O}_C)^{\zeta^b}$, therefore

$$\Delta = \sum c_i q_i$$

and, as a consequence,

$$L' = L^b(- \sum c_i q_i) .$$

Taking p-th powers, we find that

$$ba_i = a_i' + c_i p .$$

This proves the « only if » part of the lemma. To prove the converse, simply observe that the b-th power morphism from L to $L'(\sum c_i q_i)$ induces an isomorphism between C and C'. q.e.d.

REMARK 1. – Let $C_t = C(p, X, D_t, L_t)$, $0 \leqslant t \leqslant 1$, be a family of branched p-fold coverings of X, where $D_t = a_1 q_1(t) + a_2 q_2 + \ldots + a_n q_n$ and $q_1(t)$ moves in a closed loop. Let ξ be the homology class of the loop. Then L_1 equals $L_0 \otimes M$, where M is the p-torsion point in the Jacobian of X corresponding to ξ/p.

Now we can address our original problem of describing the components of the locus of curves with non-trivial automorphisms in M_g. Of course, this is a problem only if $g \geqslant 3$.

Let then C be a smooth curve of genus $g \geqslant 3$ with non-trivial automorphisms. Obviously C has an automorphism γ of prime order p and hence is a p-fold covering of $X = C/\langle\gamma\rangle$. Thus the locus S_g of curves with automorphisms is just the locus of curves which are p-fold cyclic coverings, for some prime p. If $g' \geqslant 0$ is an integer, p is *a* prime, and $a_1, \ldots, a_n$ are integers between 1 and $p-1$, we let

$$S(p, g'; a_1, \ldots, a_n)$$

be the locus of curves which are p-fold coverings of a smooth curve X of genus g' of the form $C(p, X, \sum a_i q_i, L)$ for some choice of the q_i and of L. We also allow n to be zero, meaning that we consider unbranched coverings. By the Riemann-Hurwitz formula, $S(p, g'; a_1, \ldots, a_n)$ is a subvariety of M_g if

$$2g - 2 = p(2g' - 2) + n(p-1) .$$

An easy parameter count shows that, when $g \geqslant 2$, $S(p, g'; a_1, \ldots, a_n)$ always has dimension $3g' - 3 + n$. Lemma 1 implies that

$$S(p, g'; a_1, \ldots, a_n) = S(p, g'; a_1', \ldots, a_n')$$

if there are an integer b and a permutation j such that $a_{j(i)}'$ is congruent to ba_i modulo p for every i. In particular we can always take a_1 to be equal to one. It follows from Remark 1 and the irreducibility of $M_{g'}$ that $S(p, g'; a_1, \ldots, a_n)$ is irreducible if $n > 0$. If $n = 0$, the same conclusion follows from the fact that the moduli space parametrizing couples

$$(\text{genus } g' \text{ curve } X,\ p\text{-torsion point in the Jacobian of } X)$$

is irreducible [2]. Thus

$$S_g = \bigcup \{S(p, g'; a_1, \ldots, a_n) \colon 2g - 2 = p(2g' - 2) + n(p-1)\}\ ,$$

and the problem of finding the components of S_g is simply the problem of determining all the inclusions among the $S(p, g'; a_1, \ldots, a_n)$'s. The following observation will be useful on several occasions.

REMARK 2. – Let C be a smooth connected curve and let Q be a point of C. If K is a finite subgroup of the isotropy group of Q in the automorphism group of C, then K is abelian. In fact, in a suitable local coordinate centered at C, the action of K is linear; in other words, K acts by multiplication by roots of unity in a neighbourhood of C. The conclusion follows by analytic continuation.

We begin our analysis of the inclusions among the $S(p, g'; a_1, \ldots, a_n)$ by studying $S(p, 0; a_1, a_2, a_3)$, where

$$1 = a_1 \leqslant a_2 \leqslant a_3 < p\,; \qquad \sum a_i = p\ .$$

Notice that this locus consists of a single point, corresponding to a curve $C = C(p, \boldsymbol{P}^1, \sum a_i q_i, \mathcal{O}(1))$. We let g be the genus of C, and γ a generator of $\operatorname{Aut}(C/\boldsymbol{P}^1)$. The Riemann-Hurwitz formula yields $g = (p-1)/2$. In particular $p \geqslant 3$. In the sequel, if D is a curve and A a subset of D, we shall denote by $\operatorname{Aut}(D, A)$ the group of those automorphisms φ of D such that $\varphi(A) = A$. If y is a point of D, we shall write $\operatorname{Aut}(D, y)$ instead of $\operatorname{Aut}(D, \{y\})$.

LEMMA 2. – *If $g \geqslant 2$, then $\operatorname{Aut}(C) = \operatorname{Aut}(C/\boldsymbol{P}^1) = \boldsymbol{Z}/(p)$ unless there is an automorphism τ of C that covers an automorphism σ of $\boldsymbol{P}^1$. This happens only in the following cases:*

i) *$a_2 = 1$ (or $a_2 = a_3$); σ has order two, leaves q_3 (resp. q_1) fixed, and interchanges q_1 with q_2 (resp. q_2 with q_3).*

ii) *a_2 is a non-trivial cubic root of 1 modulo p; σ has order three and permutes q_1, q_2, q_3 cyclically.*

It is always possible to choose τ to have the same order as σ.

Let y be the point of C that lies above q_1: Then, for any $g \geqslant 1$, $\operatorname{Aut}(C, y) = \operatorname{Aut}(C/\boldsymbol{P}^1)$ unless we are in case i) *and $a_2 = a_3$. If this is the case, then $\operatorname{Aut}(C, y)$ is cyclic of order $2p$ and generated by $\tau\gamma$.*

PROOF. – We set

$$G = \mathrm{Aut}\,(C)\,; \qquad P = \mathrm{Aut}\,(C/\boldsymbol{P}^1)\,.$$

Suppose $G \neq P$. To prove the first statement in the lemma we must show that P is strictly contained in its normalizer. This is clear if P is strictly contained in the p-Sylow subgroup of G, since this group has non-trivial center. It remains to examine the case when the order of G equals pk, with k prime to p. Suppose P equals its normalizer; it follows, in particular, that k is congruent to 1 modulo p. Since P is abelian, a theorem of Burnside (Theorem 2.10 in chapter 5 of [4]) shows that G has a normal subgroup H such that $G/H \simeq P$. Set $\Gamma = C/H$, $\Gamma' = C/G$, and let $\pi\colon C \to \Gamma$ be the projection. Since Γ' is covered by $\boldsymbol{P}^1$, it is a smooth rational curve; Γ is a cyclic p-fold covering of Γ'. Let $\tilde{\gamma}$ be the generator of $\mathrm{Aut}\,(\Gamma/\Gamma')$ corresponding to γ. The fixed points of γ map to fixed points of $\tilde{\gamma}$. If these are distinct, the Riemann-Hurwitz formula shows that the genus of Γ is not less than g, a contradiction since Γ is a quotient of C. Suppose then that two or all three of the fixed points of γ map to the same point x of Γ. If $\pi^{-1}(x)$ did contain fewer than k points, all of its points, in particular at least one of the fixed points of γ, would be fixed points for some non-trivial element of H. However, in view of Remark 2, this would contradict our assumption that P coincides with its normalizer. Since the k points of $\pi^{-1}(x)$ are partitioned into orbits of P, we find that k is congruent to 2 or 3 modulo p. This is impossible, since k is congruent to 1 modulo p and $p \geqslant 5$. This proves the first part of the lemma.

Now let τ be an element of the normalizer of P, not belonging to P; it induces an automorphism σ of $\boldsymbol{P}^1$ which permutes q_1, q_2, q_3. Thus the order of σ is 2 or 3, hence prime to p, and we may arrange things so that τ has the same order as σ. Suppose σ has order 2; thus it interchanges two of the q_i's (q_1 and q_2, say) and fixes the other. In this case, Lemma 1 says that there must be an integer b such that

$$a_2 \equiv b\,, \qquad 1 \equiv ba_2\,, \qquad a_3 \equiv ba_3 \qquad (\mathrm{mod}\ p)\,.$$

The only possibility is that $b = a_2 = 1$. If σ has order 3, it permutes the q_i's cyclically, sending q_1 to q_2 (say) and hence q_2 to q_3. Thus, by Lemma 1,

$$a_3 \equiv a_2^2\,, \qquad 1 \equiv a_2 a_3 \qquad (\mathrm{mod}\ p)\,.$$

In particular, a_2 is a cubic root of 1 modulo p; it is non-trivial since otherwise we would have $p = 3$. Conversely, if $a_2^3 \equiv 1$, $a_2 \not\equiv 1\ (\mathrm{mod}\ p)$, then $a_2^2 + a_2 + 1 \equiv 0\ (\mathrm{mod}\ p)$, hence $a_3 \equiv a_2^2\ (\mathrm{mod}\ p)$.

It remains to prove the last statement of the lemma. Suppose there is an element δ of $\mathrm{Aut}\,(C, y)$ not belonging to P. By Remark 2, δ centralizes P, hence descends to an automorphism σ of $\boldsymbol{P}^1$ that fixes q_1 and interchanges q_2 and q_3. It follows that we are in case i), $a_2 = a_3$, and δ is congruent to τ modulo P. q.e.d.

The description of the inclusions between the $S(p, g'; a_1, \ldots, a_n)$ is contained in the following result.

THEOREM 1. - *Let X be a general curve of genus g', $q_1, \ldots, q_n$ general points of X, $a_1, \ldots, a_n$ integers such that*

$$1 = a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n < p ,$$

$$\sum a_i \equiv 0 \pmod{p} ,$$

and let L be a non-trivial p-th root of $\mathcal{O}(\sum a_i q_i)$. Set $C = C(p, X, \sum a_i q_i, L)$, and suppose that C has genus $g \geqslant 2$. Then $\mathrm{Aut}(C) = \mathrm{Aut}(C/X) = \mathbf{Z}/(p)$, *except when there is an automorphism τ of C that covers an automorphism σ of X. This happens only in the following cases:*

i) *$g' = 0$, $n = 3$, $a_2 = 1$ (or $a_2 = a_3$); σ has order two, leaves q_3 (resp. q_1) fixed, and interchanges q_1 with q_2 (resp. q_2 with q_3).*

ii) *$g' = 0$, $n = 3$, a_2 is a non-trivial cubic root of 1 modulo p; σ has order three and permutes q_1, q_2, q_3 cyclically.*

iii) *$g' = 0$, $n = 4$, $a_2 = 1$, $a_3 = a_4 = p - 1$; σ acts on $\{q_1, q_2, q_3, q_4\}$ as the product of two disjoint transpositions.*

iv) *$g' = 1$, $n = 2$; σ is multiplication by -1 with respect to a suitable group law on X and interchanges q_1 and q_2.*

v) *$g' = 2$, $n = 0$; σ is the hyperelliptic involution.*

We can always choose τ to have the same order as σ. Cases i), ii), iii), iv), v) *are mutually exclusive.*

The proof is based on the following auxiliary result.

LEMMA 3. - $\mathrm{Aut}(C/X)$ *is a normal subgroup of* $\mathrm{Aut}(C)$, *except possibly in case $g' = 0$, $n = 3$, or $g' = 1$, $n = 2$.*

We shall first show how to deduce Theorem 1 from Lemma 3, and then prove the lemma. The case when $g' = 0$, $n = 3$ is covered by Lemma 2. We next show that the exceptional cases iii), iv), and v) do indeed occur. In case iii) we can normalize things so that $\{q_1, q_2, q_3, q_4\} = \{0, 1, \infty, \zeta\}$, where ζ is a complex number different from 0 and 1. We let σ be the linear fractional transformation

$$\sigma(z) = \zeta / z .$$

It follows from Lemma 1 that σ lifts to an automorphism of C.

To see that case iv) does occur, choose a group law on X such that q_1 and q_2 add to zero, and let σ be multiplication by -1. Since $a_2 = p-1$, L has degree 1, hence is of the form $\mathcal{O}(q)$, for some point q of X. An easy application of Lemma 1 says that, in order for σ to be liftable to an automorphism τ of C, q must satisfy the relation

$$\mathcal{O}(\sigma(q)) \cong \mathcal{O}((p-1)q - (p-2)q_2) \, .$$

Since $q + \sigma(q)$ is linearly equivalent to $q_1 + q_2$, this is a formal consequence of the fact that $\mathcal{O}(pq)$ is isomorphic to $\mathcal{O}(q_1 + (p-1)q_2)$.

To see that case v) does occur, in view of Lemma 1 it suffices to show that there exists a non-trivial line bundle L on X such that L^p is trivial, and such that, if σ is the hyperelliptic involution of X, $\sigma^*(L)$ is a power of L. Since the Jacobian of X consists entirely of anti-invariants under the action of σ, any non-zero p-torsion point in it will do.

Our next task is to show that the automorphism group of C is different from $\mathbf{Z}/(p)$ only in cases i) through v). The case $g' = 0$, $n = 3$ has already been dealt with, while the case $g' = 1$, $n = 2$ presents no problems; we therefore exclude them from our considerations. Suppose Aut (C/X) is different from Aut (C), let τ be an element of Aut (C) not belonging to Aut (C/X), and σ the automorphism of X it induces. By the generality of X and of the q_i, the existence of σ excludes the cases when $g' \geqslant 3$, $g' = 0$ and $n > 4$, $g' \geqslant 2$ and $n > 0$, or $g' = 1$ and $n > 2$. The cases when $g' = 0$ and $n = 2$, or $g' = 1$ and $n = 0$ are excluded by the requirement that $g \geqslant 2$. There remain two cases:

a) $g' = 0$, $n = 4$;

b) $g' = 2$, $n = 4$.

Case *b*) corresponds to case v) of the theorem. By the generality of the q_i's, in case *a*) σ must act on $\{q_1, q_2, q_3, q_4\}$ as the product of two disjoint transpositions. Suppose, for example, that it interchanges q_1 and q_2. Then a_2^2 is congruent to 1 modulo p, so it equals 1 or $p-1$. Since the a_i's are non-decreasing and add to a multiple of p, and σ interchanges q_3 and q_4, the only possibility is that $a_2 = 1$ and $a_3 = a_4 = p-1$. The other cases are similar.

It is clear that we can always choose τ to have the same order as σ, except possibly when the order of σ equals p. This never happens in cases i), ii), iii) since $g \geqslant 2$. Suppose then that we are in case iv) or v), and that $p = 2$. In both cases σ fixes at least one point Q that is not a branch point of $f\colon C \to X$. Let Q_1 and Q_2 be the points of C lying over Q, and denote by γ the non-trivial covering transformation of C over X. Since τ^2 covers the identity of X, it must be either the identity or γ. The latter case cannot occur; in fact τ either fixes or interchanges Q_1 and Q_2, so τ^2 fixes Q_1 and Q_2, while γ interchanges them. This concludes the proof of the theorem, if we assume Lemma 3.

PROOF OF LEMMA 3. – We first deal with the case $g' = 0$. The proof is by induction on n and relies on a degeneration argument. Assume that $n \geqslant 4$. Set

$$C_1 = C\Big(p, \boldsymbol{P}^1, \sum_{i=1}^{3} b_i r_i, \mathcal{O}(1)\Big); \qquad C_2 = C\Big(p, \boldsymbol{P}^1, \sum_{i=1}^{n-1} c_i s_i, L\Big),$$

where the s_i are general points of $\boldsymbol{P}^1$, $b_1 = c_1 = 1$, and $L^p = \mathcal{O}\big(\sum c_i s_i\big)$. Let R, S be the points of C_1 and C_2 that lie above r_1 and s_1. Let D be the stable curve obtained from the union of C_1 and C_2 by identifying R with S. The curve D is an admissible covering of the union E of two copies of $\boldsymbol{P}^1$ with r_1 on the first copy identified with s_1 on the second (cf. [1] or [3] for a discussion of admissible coverings(1)). Let $\alpha\colon D \to E$ be the natural projection. The group Aut (D/E) is equal to Aut $(C_1/\boldsymbol{P}^1) \times$ $\times$ Aut $(C_2/\boldsymbol{P}^1)$. Let G be the group of automorphisms of D sending C_1 to itself (and hence C_2 to itself). Clearly

$$G = \operatorname{Aut}(C_1, R) \times \operatorname{Aut}(C_2, S).$$

The first factor is described by Lemma 2. Moreover, G equals Aut (D) unless $n = 4$ and $\{b_2, b_3\} = \{c_2, c_3\}$, in which case G has index 2 in Aut (D).

Consider a family of admissible coverings, i.e. a commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\;\xi\;} & \mathcal{E} \\ & {\scriptstyle\vartheta}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & T & \end{array}$$

such that, for any $t \in T$, $\xi|_{\vartheta^{-1}(t)}\colon \vartheta^{-1}(t) \to \eta^{-1}(t)$ is an admissible covering. Set

$$\xi_t = \xi|_{\vartheta^{-1}(t)}, \qquad D_t = \vartheta^{-1}(t), \qquad E_t = \eta^{-1}(t).$$

It is possible to construct a family of admissible coverings as above in such a way that T is smooth, connected and one-dimensional, there is a distinguished point $0 \in T$ such that

$$(\xi_0\colon D_0 \to E_0) = (\alpha\colon D \to E),$$

and, for $t \neq 0$, D_t is a p-sheeted cyclic covering of $E_t = \boldsymbol{P}^1$. Moreover we can arrange things so that, near the singular points of D and E, the surfaces $\mathcal{D}$ and $\mathcal{E}$ are of the form

$$xy = t; \qquad uv = t^p,$$

(1) The admissible coverings of [3] have simple ramification, while ours have total ramification everywhere. The two notions agree for the degree two coverings considered in [1].

respectively, where t is a local coordinate on T centered at 0, and ξ is given by

$$u = x^p ; \qquad v = y^p .$$

We let $\beta\colon \mathcal{G} \to T$ be the group scheme over T of fiberwise automorphisms of $\mathcal{D} \to T$. The morphism β is proper. Possibly after a base change, if t is a general point of T, for any point h of $\mathrm{Aut}\,(D_t)$ there is a section χ of β passing through h. We get a homomorphism

$$\mathrm{Aut}\,(D_t) \to \mathrm{Aut}\,(D)$$

by sending h to $\chi(0)$. We claim that this is injective. In fact, suppose that $\chi(0) = 1$. We can view χ as an automorphism of $\mathcal{D}$ over T restricting to the identity on D. Since χ has finite order, if w and z are suitable coordinates at a smooth point of D, then $D = \{w = 0\}$ and χ sends (z, w) to $(\zeta z, w)$, where ζ is a root of unity. Thus χ can preserve the fibers of θ only if $\zeta = 1$, i.e., if χ is the identity everywhere. Notice that this argument does not depend on the particular nature of $\mathcal{D} \to T$, but only on the fact that we are dealing with a family of stable curves.

Suppose $n = 4$. It follows from the preceding considerations that $\mathrm{Aut}\,(D_t)$ is abelian for general t, unless $\{b_2, b_3\} = \{c_2, c_3\}$. In the latter case, let α be an isomorphism of C_1 onto C_2 carrying r_1 to s_1, and let $\tau = (\alpha, \alpha^{-1})$ be the corresponding order two element of $\mathrm{Aut}\,(D)$. The group $\mathrm{Aut}\,(D)$ is the semidirect product of the abelian normal subgroup G with the order two subgroup generated by τ. Recall that $\mathcal{D}$, $\mathcal{E}$, ξ are locally of the form

$$xy = t ; \qquad uv = t^p ; \qquad u = x^p , \qquad v = y^p ,$$

respectively. Then, if ζ is a non-trivial p-th root of unity,

$$(x, y) \to (\zeta x, \zeta^{-1} y)$$

extends to an automorphism of $\mathcal{D}$ over $\mathcal{E}$ that restricts to a non-trivial element γ of $\mathrm{Aut}\,(D_t/E_t)$ for any t. It is clear that $\tau\gamma\tau^{-1} = \gamma^{-1}$. Thus τ normalizes $\mathrm{Aut}\,(D_t/E_t)$ for any $t \neq 0$. This shows that $\mathrm{Aut}\,(D_t/E_t)$ is normal in $\mathrm{Aut}\,(D_t)$ for any t, and concludes the proof of the lemma in case $g' = 0$, $n = 4$. In fact, we can set $C = D_t$ for general t; thus C belongs to

$$S(p, 0; b_2, b_3, p - c_2, p - c_3) ,$$

and it is immediate to check that all possible $S(p, 0; a_1, \ldots, a_4)$ can be gotten by varying the b's and the c's. Notice also that the analysis of case iii) in the proof of Theorem 1 shows that, whenever Q is a point of C lying over one of the q_i's, $\mathrm{Aut}\,(C, Q)$ is abelian.

Now let n be strictly larger than 4, and assume the lemma proved for coverings of P^1 branched at $n-1$ points. The proof of the induction step is similar to the proof of the case $n = 4$, but simpler. In fact Aut (C) is a subgroup of

$$\mathrm{Aut}\,(D) = \mathrm{Aut}\,(C_1, R) \times \mathrm{Aut}\,(C_2, S)$$

and we may assume, inductively, that Aut (C_2, S) is abelian, so Aut (C) is abelian, too.

The same degeneration argument used for $g' = 0$, namely «attaching tails belonging to $S(p, 0; 1, b_2, b_3)$», proves the lemma for $g' = 1$.

We now prove the lemma for an unramified p-fold covering of a genus 2 curve. This is done by degeneration to $\pi\colon D \to E$, where π, D, and E are as follows. Choose two general elliptic curves E_1 and E_2 and points $e_1 \in E_1$, $e_2 \in E_2$: Let $\pi_1\colon D_1 \to E_1$ and $\pi_2\colon D_2 \to E_2$ be two unramified p-sheeted cyclic coverings. Pick points $d_1 \in \pi_1^{-1}(e_1)$, $d_2 \in \pi_2^{-1}(e_2)$ and generators γ_1, γ_2 of Aut (D_1/E_1) and Aut (D_2/E_2). Then let E be the union of E_1 and E_2 with e_1 and e_2 identified, and let D be the union of D_1 and D_2 with $\gamma_1^n(d_1)$ identified to $\gamma_2^n(d_2)$ for every n. Let π be the unique map that restricts to π_i on each D_i. Then Aut (D) is a subgroup of

$$\mathrm{Aut}\,\big(D_1, \pi_1^{-1}(e_1)\big) \times \mathrm{Aut}\,\big(D_2, \pi_2^{-1}(e_2)\big)\,.$$

On the other hand Aut $\big(D_i, \pi_i^{-1}(e_i)\big)$ is the dihedral group generated by the multiplication by -1 with respect to the origin d_i, which we denote by δ_i, and by γ_i. Thus Aut (D) is the dihedral group of order $2p$ generated by (δ_1, δ_2) and (γ_1, γ_2), unless $p = 2$, in which case it is the abelian group of order 8 generated by $(\delta_1, 1)$, $(1, \delta_2)$, and (γ_1, γ_2). In any case Aut (D/E) is normal in Aut (D).

We next prove the lemma for a p-fold covering of a genus 2 curve branched at two points. This is done by degenerating to an admissible covering $\pi\colon D \to E$ which we shall now describe. Let $\pi_1\colon D_1 \to E_1$ be an unramified cyclic p-fold covering of a general genus 2 curve. Let D_2, E_2 be two copies of P^1 and let $\pi_2\colon D_2 \to E_2$ be the p-th power morphism. Fix a general point e on E_1, a point d in $\pi_1^{-1}(e)$, a generator γ for Aut (D_1/E_1), and a primitive p-th root of unity ζ. Let E be the union of E_1 and E_2 with $e \in E_1$ identified to $1 \in E_2$. Let D be the union of D_1 and D_2 with $\gamma^n(d) \in D_1$ identified to $\zeta^n \in D_2$ for every n. Let π be the unique map that restricts to π_i on each D_i. Suppose first that $p \geqslant 3$, so that D is stable. Then Aut $\big(D_2, \{1, \zeta, \ldots, \zeta^{p-1}\}\big)$ is the dihedral group of order $2p$ generated by multiplication by ζ and by the inversion $z \mapsto z^{-1}$. On the other hand, by the generality of e and by the lemma applied to $\pi_1\colon D_1 \to E_1$, Aut $\big(D_1, \pi_1^{-1}(e)\big)$ equals Aut (D_1/E_1). Thus Aut (D) is isomorphic to Aut $\big(D_2, \{1, \zeta, \ldots, \zeta^{p-1}\}\big)$ and Aut (D/E) is normal in it. This takes care of the case $p \geqslant 3$. If $p = 2$, then D is no more stable. To be able to apply our degeneration argument we must blow down D_2. Thus we have to examine Aut (D'), where D' is obtained from D_1 by identifying the two points of $\pi_1^{-1}(e)$. By the generality of e, Aut (D') equals $\mathbf{Z}/(2)$. This concludes the proof of the lemma in the case at hand. Notice moreover that in the proof of Theorem 1 it was shown that the lemma implies that the automorphism group of a general cyclic p-sheeted covering of a genus two curve branched at two points is $\mathbf{Z}/(p)$.

We may now conclude the proof of Lemma 3 by yet another degeneration argument. This time we shall use induction on g' and keep the number of branch points fixed. The induction starts with the cases $g'=1$, $n\geqslant 3$ and $g'=2$, $n=0$ or $n=2$. Notice that in all these cases, except when $g'=2$, $n=0$, the full automorphism group is $\mathbf{Z}/(p)$. When $g'=2$, $n=0$, Aut (C/X) has index two in Aut (C), the quotient being generated by the hyperelliptic involution of X. Fix a cyclic p-sheeted covering $\pi_1\colon D_1\to E_1$ branched at n general points, where E_1 is a general curve of genus $g'\geqslant 1$ ($g'\geqslant 2$ if $n\leqslant 2$). Let E_2 be a general elliptic curve. Choose a point e_2 on E_2 and a general point e_1 on E_1. We let D be the union of D_1 and of p copies of E_2, attached by e_2 to the p points of $\pi_1^{-1}(e_1)$. Obviously, D is an admissible covering of the union of E_1 and E_2 with e_1 and e_2 identified, which we denote by E. Clearly Aut (D) is the semidirect product of Aut $(D_1, \pi_1^{-1}(e_1)) = \mathbf{Z}/(p)$ and Aut $(E_2, e_2)^p$, the first group acting on the second one by permuting the factors. The infinitesimal first order deformations of D are in one-to-one correspondence with the elements of $\mathrm{Ext}^1(\Omega^1_D, \mathcal{O}_D)$. This group, in turn, fits into an exact sequence

$$0\to H^1(\mathcal{H}om(\Omega^1_D, \mathcal{O}_D))\to \mathrm{Ext}^1(\Omega^1_D, \mathcal{O}_D)\to H^0(\mathcal{E}xt^1(\Omega^1_D, \mathcal{O}_D))\to 0\,,$$

where the sheaf $\mathcal{E}xt^1(\Omega^1_D, \mathcal{O}_D)$ consists of p copies of $\mathbf{C}$ concentrated at the singular points of D. The vector space $H^1(\mathcal{H}om(\Omega^1_D, \mathcal{O}_D))$ classifies the infinitesimal locally trivial deformations of D. The automorphisms of D act on $\mathrm{Ext}^1(\Omega^1_D, \mathcal{O}_D)$: an automorphism η extends along a first order deformation v if and only if v is η-invariant. In particular, in order to survive smoothing of the singular points of D, η must act trivially on $\mathcal{E}xt^1(\Omega^1_D, \mathcal{O}_D)$. Now Aut (E_2, e_2) is the group of order 2 generated by multiplication by -1 with respect to the origin e_2. Denote by $\delta_1, \delta_2, \ldots, \delta_p$ the generators of the p copies of Aut (E_2, e_2) in Aut (D), and by $Q_1, Q_2, \ldots, Q_p$ the corresponding singular points of D. The automorphism δ_i acts as multiplication by -1 on the stalk of $\mathcal{E}xt^1(\Omega^1_D, \mathcal{O}_D)$ at Q_i, hence does not survive smoothing of the singular points of D. Thus, if $D_t\to E_t$ is a one-parameter family of cyclic p-sheeted coverings such that $(D_0\to E_0)=(D\to E)$ and D_t, E_t are smooth for $t\neq 0$, then, for general t, Aut $(D_t/E_t)=\mathbf{Z}/(p)$. This completes the proof of the induction step from cyclic p-sheeted coverings of genus g' curves to cyclic p-sheeted coverings of genus $g'+1$ curves. q.e.d.

COROLLARY 1. – *If $g\geqslant 3$, the components of S_g are the subvarieties $S(p, g'; a_1, \ldots, a_n)$ with $1=a_1\leqslant\ldots\leqslant a_n<p$ such that*

$$2g-2=p(2g'-2)+n(p-1)\,,$$

with the exclusion of those satisfying one of the following conditions:

i) $g' = 0$, $n = 3$, $a_2 = 1$ *(or* $(a_2 = a_3)$.

ii) $g' = 0$, $n = 3$, a_2 *is a non-trivial cubic root of* 1 *modulo* p.

iii) $g' = 0$, $n = 4$, $a_2 = 1$, $a_3 = a_4 = p - 1$.

iv) $g' = 1$, $n = 2$.

v) $g' = 2$, $n = 0$.

If $S(p, g'; a_1, \ldots, a_n)$ *and* $S(p, g'; a_1', \ldots, a_n')$ *do not satisfy* i), ii), iii), iv), *or* v), *and are equal, then there are an integer* b *and a permutation* j *such that* $a'_{j(i)}$ *is congruent to* ba_i *modulo* p *for every* i.

PROOF. – The only point that requires some explanation concerns the exclusion of the components of type v) in genus 3. In fact, if $f\colon C \to X$ is an unramified covering, X has genus 2, and C has genus 3, the Riemann-Hurwitz formula yields $p = 2$. So, if τ is an order two automorphism of C covering the hyperelliptic involution of X, it might a priori be possible that the quotient of C by τ has genus 2, i.e. that τ has no fixed points. This, however, is not the case. Let γ be the generator of Aut (C/X). The fixed points of τ, if any, come in pairs of points lying above the Weierstrass points of X. Moreover, if Q is a Weierstrass point of X and Q_1, Q_2 are the points of C above it, either τ fixes Q_1 and Q_2 and $\gamma\tau$ interchanges them, or viceversa. Thus, if τ has ν fixed points, $\gamma\tau$ has $12 - \nu$ fixed points. The Riemann-Hurwitz formula implies that an order 2 automorphism of C can only have 0, 4, or 8 fixed points. Thus, either τ has four fixed points and $\gamma\tau$ has eight, or viceversa. In particular, C is hyperelliptic and belongs to $S(2, 1; 1, 1, 1, 1)$. q.e.d.

COROLLARY 2. – *a) The singular locus of* M_g *equals* S_g *if* $g \geqslant 4$.

b) The components of the singular locus of M_3 *are:*

$$S(3, 0; 1, 1, 1, 1, 2)\,,$$

$$S(7, 0; 1, 1, 5)\,,$$

$$S(2, 1; 1, 1, 1, 1)\,.$$

PROOF. – We first recall how the singularities of M_g arise. We assume that $g \geqslant 3$ throughout. Let Q be a point of M_g, corresponding to a curve C. Let

$$f\colon \mathcal{C} \to B$$

be the universal deformation of C; thus there is a distinguished point $b \in B$ such that $f^{-1}(b) = C$. The action of Aut (C) on C extends to an equivariant action of

Aut (C) on $\mathbb{C}$ and B, and the quotient $B/\mathrm{Aut}\,(C)$ is isomorphic to a neighbourhood of Q in M_g. Moreover, the action of Aut (C) on the tangent space to B at b is faithful. The covering $B \to B/\mathrm{Aut}\,(C)$ is unramified off the locus of curves with non-trivial automorphisms, so the singular locus of M_g is contained in S_g. By the purity of the branch locus theorem, any component of S_g of codimension two or more consists entirely of singular points. If $g \geq 4$, we know from Corollary 1 that every component of S_g has codimension at least two: this proves a). Now suppose that $g = 3$. The only divisor component of S_3 is the hyperelliptic locus. Therefore a non-hyperelliptic curve corresponds to a singular point of M_3 if and only if it has non-trivial automorphisms. Suppose instead that C is hyperelliptic, and let τ be the hyperelliptic involution. The quotient of B by the action of the normal subgroup of Aut (C) generated by τ is a smooth manifold B', and B is a two-sheeted covering of B' ramified along the locus of hyperelliptic curves. The moduli space M_3 is, locally, the quotient of B' by the action of Aut $(C)/\langle\tau\rangle$. Using again the purity of the branch locus theorem, we conclude that the singular points of M_3 lying on the hyperelliptic locus correspond to the hyperelliptic curves with extra automorphisms.

The varieties $S(p, g'; a_1, \dots, a_n)$ contained in M_3 are the hyperelliptic locus and

$$S(3, 0; 1, 1, 1, 1, 2)\,,$$
$$S(7, 0; 1, 1, 5)\,,$$
$$S(7, 0; 1, 2, 4)\,,$$
$$S(2, 1; 1, 1, 1, 1)\,,$$
$$S(3, 1; 1, 2)\,,$$
$$S(2, 2)\,.$$

We know from Theorem 1 that $S(3, 0; 1, 1, 1, 1, 2)$ and $S(2, 1; 1, 1, 1, 1)$ are contained in no other component of S_3: In the proof of Corollary 1 we have seen that

$$S(2, 2) \subset S(2, 1; 1, 1, 1, 1)\,.$$

We now show that

$$S(3, 1; 1, 2) \subset S(2, 1; 1, 1, 1, 1)\,.$$

Let C be a three-sheeted cyclic covering of the elliptic curve X, branched at two points q_1 and q_2. Denote by γ a generator of Aut (C/X) and by τ an order two automorphism of C covering an automorphism σ of X. Let Q be a fixed point of σ and Q_1, Q_2, Q_3 the points of C above it. If γ commutes with τ, then τ fixes Q_1, Q_2, Q_3, otherwise τ fixes one among them and interchanges the other two. Since σ has four fixed points, the first alternative would imply that τ has twelve fixed points, which would contradict the Riemann-Hurwitz formula. Hence τ has four fixed points, which proves our assertion. A similar argument shows that

$$S(7, 0; 1, 2, 4) \subset S(3, 1; 1, 2) .$$

The unique point of $S(7, 0; 1, 1, 5)$ is the double covering of $\boldsymbol{P}^1$ branched at 0 and at the seventh roots of unity. Its only automorphism of order two is the hyperelliptic involution. Hence $S(7, 0; 1, 1, 5)$ is contained in the hyperelliptic locus and in no other variety $S(p, g'; a_1, ..., a_n)$. This finishes the proof of *b)* and of the corollary. q.e.d.

Corollary 2 describes the components of the singular locus of M_g when $g \geqslant 3$. When g equals zero or one, M_g is smooth, while it has been shown by IGUSA [4] that the singular locus of M_2 is $S(5, 0; 1, 1, 3)$; thus M_2 has only one singular point.

We conclude by noticing that the results proved in this paper make it possible to algorithmically calculate the components of the singular locus of moduli space. The results of these calculations, for genus up to 50, are summarized in the tables that follow the bibliography. Due to space limitations, for genus greater than 13 only the number of components for each dimension and the total number of components are given.

REFERENCES

[1] A. BEAUVILLE, *Prym varieties and the Schottky problem*, Invent. Math., **41** (1977), pp. 149-196.

[2] P. DELIGNE - D. MUMFORD, *The irreducibility of the space of curves of given genus*, I.H.E.S. Publ. Math., **36** (1969), pp. 75-110.

[3] J. HARRIS - D. MUMFORD, *On the Kodaira dimension of the moduli space of curves*, Invent. Math., **67** (1982), pp. 23-86.

[4] J.-I. IGUSA, *Arithmetic variety of moduli for genus two*, Ann. Math., **72** (1960), pp. 612-649.

[5] M. SUZUKI, *Group theory II*, Springer-Verlag, New York, 1986.

The components of the singular locus of M_g, $4 \leqslant g \leqslant 13$, arranged by dimension (x: n stands for a sequence of n x's).

genus 4:
dimension 1: S(5,0;1:3,2) S(5,0;1,2,3,4)
3: S(3,0;1:6) S(3,0;1:3,2:3) S(3,1;1:3)
5. S(2,2;1:2)
6: S(2,1;1:6)
7: S(2,0;1:10)

genus 5:
0: S(11,0;1,2,8)
4: S(3,0;1:5,2:2) S(3,1;1:2,2:2)
6: S(2,3)
7: S(2,2;1:4)
8: S(2,1;1:8)
9: S(2,0;1:12)

genus 6:
0: S(13,0;1,2,10)
1: S(7,0;1:3,4) S(7,0;1:2,2,3) S(7,0;1,2,5,6)
2. S(5,0;1:5) S(5,0,1:3,3,4) S(5,0,1:2,2:2,4)
5: S(3,0;1:7,2) S(3,0;1:4,2:4) S(3,1;1:4,2) S(3,2;1,2)
8: S(2,3;1:2)
9. S(2,2;1:6)
10: S(2,1;1:10)
11: S(2,0;1:14)

genus 7:
3. S(5,1;1:2,3)
6: S(3,0;1:9) S(3,0;1:6,2:3) S(3,1;1:6) S(3,1;1:3,2:3) S(3,2;1:3) S(3,3)
S(5,0;1:2,2,3,4:2)
9: S(2,4)
10: S(2,3,1:4)
11 S(2,2,1:8)
12: S(2,1;1:12)
13. S(2,0;1:16)

genus 8:
0: S(17,0;1,2,14) S(17,0;1,3,13)
3: S(5,0;1:4,2,4) S(5,0;1:4,3:2) S(5,0;1:3,2:2,3) S(5,0;1:3,4:3)
7: S(3,0;1:8,2:2) S(3,0;1:5,2:5) S(3,1;1:5,2:2) S(3,2;1:2,2:2)
11: S(2,4;1:2)
12: S(2,3;1:6)
13: S(2,2;1:10)
14: S(2,1;1:14)
15: S(2,0;1:18)

genus 9:
0: S(19,0;1,2,16) S(19,0;1,3,15)
2: S(7,0;1:4,3) S(7,0;1:3,2:2) S(7,0;1:3,5,6) S(7,0;1:2,2,4,6) S(7,0;1:2,2,5:2)
S(7,0;1:2,3,4,5)
4: S(5,1,1:3,2) S(5,1;1:2,4:2) S(5,1;1,2,3,4)
8: S(3,0,1:10,2) S(3,0;1:7,2:4) S(3,1;1:7,2) S(3,1;1:4,2:4) S(3,2;1:4,2)
S(3,3;1,2)
12. S(2,5)
13: S(2,4;1:4)
14: S(2,3;1:8)
15: S(2,2,1:12)
16: S(2,1;1:16)
17: S(2,0;1:20)

genus 10:
1: S(11,0;1:3,8) S(11,0;1:2,2,7) S(11,0;1:2,3,6) S(11,0;1:2,4,5)
S(11,0;1,2,3,5) S(11,0;1,2,9,10) S(11,0;1,3,8,10)
3: S(7,1;1:2,5) S(7,1;1,2,4)
4: S(5,0;1:6,4) S(5,0;1:5,2,3) S(5,0,1:4,2:3) S(5,0;1:4,3,4:2) S(5,0;1:3,2:2,4:2)
S(5,0;1:3,2,3:2,4)
5: S(5,2;1,4)
9: S(3,0;1:12) S(3,0;1:9,2:3) S(3,0;1:6,2:6) S(3,1;1:9) S(3,1;1:6,2:3)
S(3,2;1:6) S(3,2;1:3,2:3) S(3,3;1:3) S(3,4)
14: S(2,5;1:2)
15: S(2,4;1:6)
16: S(2,3;1:10)
17: S(2,2;1:14)
18: S(2,1;1:18)
19: S(2,0;1:22)

genus 11:
0: S(23,0;1,2,20) S(23,0;1,3,19)
S(23,0;1,4,18)
5: S(5,1;1:5) S(5,1;1:3,3,4)
S(5,1;1:2,2:2,4)
6. S(5,3)
10: S(3,0;1:11,2:2) S(3,0;1:8,2:5)
S(3,1,1:8,2:2) S(3,1;1:5,2:5)
S(3,2;1:5,2:2) S(3,3;1:2,2:2)
15 S(2,6)
16: S(2,5;1:4)
17: S(2,4,1:8)
18. S(2,3;1:12)
19: S(2,2;1:16)
20: S(2,1,1:20)
21 S(2,0;1:24)

genus 12:
1: S(13,0;1:3,10) S(13,0;1:2,2,9) S(13,0;1:2,3,8) S(13,0;1:2,4,7) S(13,0;1:2,5,6) S(13,0;1,2,3,7) S(13,0;1,2,11,12)
S(13,0;1,3,4,5) S(13,0;1,3,10,12) S(13,0;1,5,6,12)
3: S(7,0;1:5,2) S(7,0;1:4,4,6) S(7,0;1:4,5:2) S(7,0;1:3,2,3,6) S(7,0;1:3,2,4,5) S(7,0;1:3,3:2,5) S(7,0;1:3,3,4:2)
S(7,0;1:3,6:3) S(7,0;1:2,2:2,3,5) S(7,0;1:2,2:2,4:2) S(7,0;1:2,2,3:2,4) S(7,0,1:2,2,5,6:2) S(7,0;1:2,3,4,6:2)
S(7,0;1,2,3,4,5,6)
5: S(5,0;1:7,3) S(5,0;1:6,2:2) S(5,0;1:5,2,4:2) S(5,0;1:5,3:2,4) S(5,0;1:4,2:2,3,4) S(5,0;1:4,2,3:3) S(5,0;1:4,4:4)
S(5,0,1:3,2:3,3:2) S(5,0;1:3,2,3,4:3) S(5,0;1:2,2:2,3:2,4:2)
6: S(5,2;1:2,3)
11: S(3,0;1:13,2) S(3,0;1:10,2:4) S(3,0;1:7,2:7) S(3,1;1:10,2) S(3,1;1:7,2:4) S(3,2;1:7,2) S(3,2;1:4,2:4) S(3,3;1:4,2)
S(3,4;1,2)
17: S(2,6;1:2)
18: S(2,5;1:6)
19: S(2,4;1:10)
20: S(2,3;1:14)
21: S(2,2,1:18)
22: S(2,1;1:22)
23: S(2,0;1:26)

genus 13:
4 S(7,1,1:3,4) S(7,1;1:2,2,3) S(7,1;1:2,6:2) S(7,1;1,2,5,6)
6: S(5,1;1:4,2,4) S(5,1;1:4,3:2) S(5,1;1:3,2:2,3) S(5,1;1:3,4:3) S(5,1;1:2,2,3,4:2)
12 S(3,0;1:15) S(3,0;1:12,2:3) S(3,0;1:9,2:6) S(3,1;1:12) S(3,1;1:9,2:3) S(3,1;1:6,2:6) S(3,2;1:9) S(3,2,1:6,2:3) S(3,3;1:6) S(3,3;1:3,2:3) S(3,4;1:3) S(3,5)
18 S(2,7)
19 S(2,6;1:4)
20 S(2,5;1:8)
21 S(2,4;1:12)
22 S(2,3,1:16)
23 S(2,2;1:20)
24. S(2,1;1:24)
25 S(2,0,1:28)

The number of components of the singular locus of M_g, $2 \leqslant g \leqslant 50$, by dimension
(x: n means « n components of dimension x »).

genus	
2	0:1 total 1
3	0:1 2:1 4:1 total 3
4	1:2 3:3 5:1 6:1 7:1 total 8
5	0:1 4:2 6:1 7:1 8:1 9:1 total 7
6	0:1 1:3 2:3 5:4 8:1 9:1 10:1 11:1 total 15
7	3:1 6:6 9:1 10:1 11:1 12:1 13:1 total 12
8	0:2 3:5 7:4 11:1 12:1 13:1 14:1 15:1 total 16
9	0:2 2:6 4:3 8:6 12:1 13:1 14:1 15:1 16:1 17:1 total 23
10	1:7 3:2 4:6 5:1 9:9 14:1 15:1 16:1 17:1 18:1 19:1 total 31
11	0:3 5:3 6:1 10:6 15:1 16:1 17:1 18:1 19:1 20:1 21:1 total 20
12	1:10 3:14 5:10 6:1 11:9 17:1 18:1 19:1 20:1 21:1 22:1 23:1 total 51
13	4:4 6:5 12:12 18:1 19:1 20:1 21:1 22:1 23:1 24:1 25:1 total 29
14	0:4 5:1 6:11 7:3 13:9 20:1 21:1 22:1 23:1 24:1 25:1 26:1 27:1 total 36
15	0:4 2:19 4:19 6:1 7:6 8:1 14:12 21:1 22:1 23:1 24:1 25:1 26:1 27:1 28:1 29:1 total 71
16	1:16 3:2 5:6 7:16 8:3 9:1 15:16 23:1 24:1 25:1 26:1 27:1 28:1 29:1 30:1 31:1 total 69
17	0:2 8:10 9:1 16:12 24:1 25:1 26:1 27:1 28:1 29:1 30:1 31:1 32:1 33:1 total 35
18	0:5 1:19 2:28 5:33 8:16 9:5 17:16 26:1 27:1 28:1 29:1 30:1 31:1 32:1 33:1 34:1 35:1 total 134
19	3:3 6:14 9:11 10:3 18:20 27:1 28:1 29:1 30:1 31:1 32:1 33:1 34:1 35:1 36:1 37:1 total 62
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