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Complex Variable Functions — *Jenkins-Strebel differentials*, by ENRICO ARBARELLO and MAURIZIO CORNALBA, communicated on 12 February 2010.[1]

ABSTRACT. — In this mostly expository paper we revisit a fundamental result of Strebel, asserting the existence and uniqueness, on Riemann surfaces of finite type, of Jenkins-Strebel differentials having double poles with prescribed "residues" at prescribed points. In particular, we give a self-contained and somewhat shortened proof of Strebel's result.

KEY WORDS: Quadratic differentials, closed trajectories, Teichmüller space.

2000 MATHEMATICS SUBJECT CLASSIFICATION: Primary 30F30, Secondary 30F60.

1. INTRODUCTION

In this largely expository paper we revisit a rather astonishing result, due to Strebel [19], which we briefly describe. Let $(C, x_1, \dots, x_n)$ be a stable pointed Riemann surface of genus $g \geq 0$. Consider the space of quadratic differentials $\omega \in H^0(C, K_C^2(2x_1 + \cdots + 2x_n))$ having poles of order two at the points x_i. Among these quadratic differentials, look at the ones having the following property: for each $i = 1, \dots, n$, one can find a suitable coordinate z_i vanishing at x_i for which the local expression of ω is

$$\omega = \left(\frac{a_i}{2\pi \mathrm{i}} d \log z_i\right)^2 = -\left(\frac{a_i}{2\pi}\right)^2 \frac{dz_i^2}{z_i^2}, \tag{1}$$

where the a_i are positive real numbers. Let us recall (see for instance section 5 of Chapter 16 in [1]) that, away from its zeros and poles, the differential ω defines a hermitian metric, the ω-metric, which is defined to be the one with local expression

$$|f|\, dz\, d\bar{z},$$

where $f\, dz^2$ is a local expression for ω. If we look at one of the points x_i and set $x = x_i$, $a = a_i$, $z = z_i$ and $z = re^{\mathrm{i}\theta}$, we see that the Riemannian metric associated to the ω-metric is just $\left(\frac{a}{2\pi}\right)^2 \left(\frac{dr^2}{r^2} + d\theta^2\right)$. The concentric circles $r = \textit{constant}$, which are geodesics with respect to the ω-metric, all have length equal to a, so that a punctured disc around x, in the ω-metric, looks like a semi-infinite cylinder:

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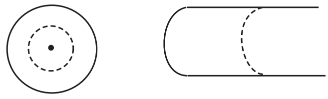

Figure 1

The question Strebel addresses is whether there exist quadratic differentials for which these local pictures fit into a nice global one. The answer is quite beautiful: there exists a unique quadratic differential ω with poles of order two at the points x_i and with local expressions given by (1), having the following additional property. We can choose the coordinates z_i so that their codomains are disks centered at the origin and, furthermore, if U_i stands for the domain of z_i,

$$\Gamma = C \setminus \bigcup_{i=1}^{n} U_i$$

is a graph (i.e., a 1-dimensional complex), called the *Strebel graph,* having a vertex of valency $n+2$ for each n-th order zero of ω. Moreover the edges of Γ are horizontal ω-geodesics for ω. The following picture illustrates the case $g = 0$, $n = 3$, $a_1 = a_2 = a_3$.

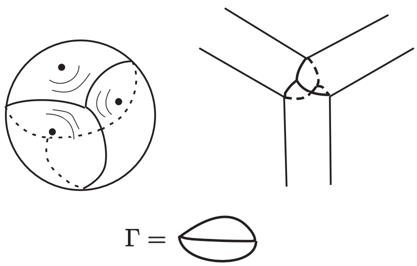

Figure 2

In this paper we will give a self-contained and somewhat shorter proof of Strebel's result by using ideas already partly contained in [13].

Perhaps the most striking application of Strebel's theorem is the one to the cellular decomposition of moduli spaces of pointed curves. Let $M_{g,n}$ be the moduli space of n-pointed, genus g, smooth complete curves. The idea of using Jenkins-Strebel differentials to define a cellular decomposition of $M_{g,n}$ is due to (unpublished) work of Mumford and Thurston and to Harer [6, 5]. Actually, from the point of view of Strebel differentials, the natural space to work with is the space $M_{g,n} \times \mathbb{R}_+^n$. Given a point $y = [C; x_1, \ldots, x_n; a_1, \ldots, a_n] \in M_{g,n} \times \mathbb{R}_+^n$, take the Jenkins-Strebel differential ω associated to it and consider the Strebel graph Γ equipped with the metric induced by ω. Then y can be viewed as a point of the orbi-cell

$$e_\Gamma = \mathbb{R}_+^{N_\Gamma}/\mathrm{Aut}(\Gamma)$$

where N_Γ is the number of edges of Γ. Moving from one cell to another corresponds to a Feynman move. Moving inside a cell e_Γ corresponds to changing the ω-length of the edges of Γ. Using this cell decomposition one can easily prove vanishing theorems for the homology of $M_{g,n}$ (see [5]). Following Harer and Zagier [4] one can also compute the virtual Euler-Poincaré characteristic of $M_{g,n}$ by first expressing it as

$$\chi_{\mathrm{virt}}(M_{g,n}) = \sum_{\Gamma \in \mathscr{G}_{g,n}} \frac{(-1)^{N_\Gamma}}{\mathrm{Aut}(\Gamma)},$$

where $\mathscr{G}_{g,n}$ is the set of isomorphism classes of Strebel graphs of genus g with n boundary components. Kontsevich, in proving Witten's conjecture, shows that a similar method can be used to compute the intersection numbers of tautological classes on $M_{g,n}$ (see [10, 11, 3, 12]).

2. Annular regions and their moduli

Throughout this paper we shall make free use of the uniformization theorem, which says that, up to isomorphism, there are just three simply connected Riemann surfaces, namely the Riemann sphere, the complex plane, and the unit disk. Every Riemann surface inherits from its universal covering a hermitian metric of constant curvature. In particular, the Poincaré metric induces on the *hyperbolic* surfaces, that is, those whose universal covering is the disk, a hermitian metric of constant curvature -1; we shall often refer to this as the *hyperbolic metric*. From a conformal point of view, a Riemann surface whose fundamental group is infinite cyclic is isomorphic to the punctured plane $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$ or to an annulus

$$T_R = \{z \in \mathbb{C} : R < |z| < 1\}, \quad 0 \le R < 1.$$

The *modulus* of T_R is defined by

$$M(T_R) = -\frac{\log R}{2\pi}.$$

When $R = 0$, the annulus T_R coincides with the punctured unit disk and is said to be *degenerate*; its modulus equals $+\infty$. An open region Ω in a Riemann surface is said to be *annular* if it is isomorphic to T_R, with $R \ge 0$. As we shall presently see, R is completely determined by Ω, and one defines the modulus $M(\Omega)$ of Ω to be the one of T_R. For example, it is immediate to check that the modulus of the annulus $T_{r_1,r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ is given by

$$M(T_{r_1,r_2}) = \frac{\log(r_2/r_1)}{2\pi}.$$

In general it is not so easy to work with the above definition of modulus. However, it is possible to give an intrinsic definition of $M(\Omega)$, that is, one which clearly depends only on the isomorphism class of Ω as an abstract complex manifold. To this end, we will consider the family $\mathscr{F}$ of all simple closed curves in Ω that are not homotopic to the identity and we will introduce the conformal invariant notion of extremal length $\Lambda_\Omega(\mathscr{F})$ for curves in this family. We shall then see that

$$M(\Omega) = \frac{1}{\Lambda_\Omega(\mathscr{F})}. \tag{2}$$

Let S be a Riemann surface, and consider all the metrics of finite area on S which are compatible with the conformal structure. We call these metrics *admissible*. For a given admissible metric ρ, we denote by the symbol $l_\rho(\gamma)$ the length of a curve γ and by $A_\rho(\Omega)$ the area of a region $\Omega \subset S$. Now let $\mathscr{F}$ be a family of closed curves in S. We set

$$l_\rho(\mathscr{F}) = \inf_{\gamma \in \mathscr{F}} l_\rho(\gamma).$$

The quantity

$$\Lambda_S(\mathscr{F}) = \sup_{\textit{admissible } \rho} \frac{l_\rho^2(\mathscr{F})}{A_\rho(S)}$$

is clearly invariant upon multiplication of ρ by a positive constant and is called the *extremal length* of $\mathscr{F}$. There are two ways of rescaling data which lead to useful expressions for $\Lambda_S(\mathscr{F})$. First of all one could rescale the area of S to be equal to 1 and get

$$\Lambda_S(\mathscr{F}) = \sup_{\rho \textit{ s.t. } A_\rho(S)=1} l_\rho^2(\mathscr{F}). \tag{3}$$

On the other hand, in case $l_\rho(\mathscr{F}) < +\infty$ for all admissible ρ, one has

$$\Lambda_S(\mathscr{F}) = \sup_{\rho \textit{ s.t. } l_\rho(\mathscr{F})=1} \frac{1}{A_\rho(S)}. \tag{4}$$

From the definition it follows that $\Lambda_S(\mathscr{F})$ is a conformal invariant in the sense that, given an isomorphism $\varphi : S \to S'$ transforming the family $\mathscr{F}$ into a family $\mathscr{F}'$, one has $\Lambda_{S'}(\mathscr{F}') = \Lambda_S(\mathscr{F})$. Let us look at an annulus T_R with $R > 0$, and let us show that, if $\mathscr{F}$ is the family of all closed simple curves contained in T_R and not homotopic to the identity, then

$$\Lambda_{T_R}(\mathscr{F}) = -\frac{2\pi}{\log R} = \frac{1}{M(T_R)}. \tag{5}$$

This will show that indeed we can use (2) as a definition of the modulus of an annular region. To prove (5), let $\rho(z)|dz|$ be the length element associated to an admissible metric on T_R. The function $\rho(z)$ is square integrable. By abuse of notation we shall write l_ρ and A_ρ to designate the length and area functions associated to the metric. Looking at a circle of radius r in T_R we get

$$l_\rho(\mathscr{F}) \leq \int_0^{2\pi} \rho(re^{\mathrm{i}\vartheta})r\,d\vartheta.$$

Dividing by r and integrating we get

$$l_\rho(\mathscr{F}) \log R \leq \int_R^1 \int_0^{2\pi} \rho(re^{\mathrm{i}\vartheta})\,d\vartheta\,dr,$$

so that, by the Schwarz inequality,

$$(l_\rho(\mathscr{F}) \log R)^2 \leq \int_{T_R} \rho^2 r\,d\vartheta\,dr \int_{T_R} \frac{1}{r}\,d\vartheta\,dr = -2\pi \log(R) A_\rho(T_R).$$

Hence

$$\frac{l_\rho^2(\mathscr{F})}{A_\rho(T_R)} \leq -\frac{2\pi}{\log R}$$

showing that

$$\Lambda_{T_R}(\mathscr{F}) \leq -\frac{2\pi}{\log R}$$

On the other hand, if γ is a homotopically non-trivial, simple curve, and if $\rho(r, \vartheta) = \frac{1}{2\pi r}$, then

$$l_\rho(\gamma) = \int_\gamma \frac{\sqrt{dr^2 + r^2\,d\vartheta^2}}{2\pi r} \geq \int_0^{2\pi} \frac{d\vartheta}{2\pi} = 1.$$

Hence $l_\rho(\mathscr{F}) \geq 1$. Also

$$A_\rho(T_R) = \int_{T_R} \frac{r\,dr\,d\vartheta}{(2\pi r)^2} = -\frac{\log R}{2\pi}.$$

Therefore

$$\Lambda_{T_R}(\mathscr{F}) \geq -\frac{2\pi}{\log R},$$

proving (5).

Lemma 1. *Let Ω and Ω_1 be two regions on a Riemann surface S. Let $\mathscr{F}$ (resp., $\mathscr{F}_1$) be a family of closed simple curves in Ω (resp., Ω_1) none of which is homotopic to the identity. Assume that $\Omega_1 \subset \Omega$ and $\mathscr{F}_1 \subset \mathscr{F}$. Then*

$$\Lambda_\Omega(\mathscr{F}) \leq \Lambda_\Omega(\mathscr{F}_1) \leq \Lambda_{\Omega_1}(\mathscr{F}_1).$$

Proof. Since

$$l_\rho(\mathscr{F}) = \inf_{\gamma \in \mathscr{F}} l_\rho(\gamma) \leq \inf_{\gamma_1 \in \mathscr{F}_1} l_\rho(\gamma_1) = l_\rho(\mathscr{F}_1),$$

the first inequality is clear. The second inequality follows from (4) and from the fact that any admissible metric ρ on Ω such that $l_\rho(\mathscr{F}_1) = 1$ restricts to an admissible metric on Ω_1 having the same property, so that

$$\inf_{l_{\rho_1}(\mathscr{F}_1)=1} A_{\rho_1}(\Omega_1) \leq \inf_{l_\rho(\mathscr{F}_1)=1} A_\rho(\Omega). \qquad \square$$

We will use the following terminology. Let γ be a simple closed curve in S. An annular region $\Omega \subset S$ *has the homotopy type of* γ if γ is freely homotopic to a simple closed curve in Ω which is not homotopically trivial.

Lemma 2. *Let Ω be an annular region. Let Ω_1 and Ω_2 be two disjoint annular regions contained in Ω, both having the homotopy type of Ω. Then $M(\Omega) \geq M(\Omega_1) + M(\Omega_2)$. In particular $M(\Omega) \geq M(\Omega_i)$, for $i = 1, 2$.*

Proof. Let $\mathscr{F}$ (resp., $\mathscr{F}_1$, $\mathscr{F}_2$) be the family of simple closed curves in Ω (resp., Ω_1, Ω_2) which are not homotopic to the identity. By assumption $\mathscr{F}_1 \cup \mathscr{F}_2 \subset \mathscr{F}$. By the previous lemma we have

$$\Lambda_{\Omega_1 \cup \Omega_2}(\mathscr{F}_1 \cup \mathscr{F}_2) \geq \Lambda_\Omega(\mathscr{F}_1 \cup \mathscr{F}_2) \geq \Lambda_\Omega(\mathscr{F}).$$

It then suffices to prove that

$$\frac{1}{\Lambda_{\Omega_1 \cup \Omega_2}(\mathscr{F}_1 \cup \mathscr{F}_2)} = \frac{1}{\Lambda_{\Omega_1}(\mathscr{F}_1)} + \frac{1}{\Lambda_{\Omega_2}(\mathscr{F}_2)}. \tag{6}$$

Given any admissible metric ρ on $\Omega_1 \cup \Omega_2$, with $l_\rho(\mathscr{F}_1 \cup \mathscr{F}_2) = 1$, denote by ρ_i its restriction to Ω_i, for $i = 1, 2$ and let $\rho_i' = \rho_i / l_{\rho_i}(\mathscr{F}_i)$. Since $l_{\rho_i}(\mathscr{F}_i) \geq 1$, we have

$$A_\rho(\Omega_1 \cup \Omega_2) \geq A_{\rho_1'}(\Omega_1) + A_{\rho_2'}(\Omega_2).$$

It then follows from (4) that the left hand side of (6) is greater or equal than the right hand side. Conversely, given admissible metrics ρ_i on Ω_i, with $l_{\rho_i}(\mathscr{F}_i) = 1$, one can simply define an admissible metric ρ on $\Omega_1 \cup \Omega_2$ with $l_\rho(\mathscr{F}_1 \cup \mathscr{F}_2) = 1$ by setting $\rho_{|\Omega_i} = \rho_i$, so that $A_\rho(\Omega_1 \cup \Omega_2) = A_{\rho_1}(\Omega_1) + A_{\rho_2}(\Omega_2)$. This implies the reverse inequality, proving (6). $\square$

LEMMA 3. *Let Ω be an annular region and suppose that $\Omega = \bigcup_{n \geq 1} \Omega_n$, where $\Omega_1 \subset \Omega_2 \subset \cdots$ is an increasing sequence of annular domains having the same homotopy type of Ω. Then $M(\Omega) = \lim_{n\to\infty} M(\Omega_n)$.*

PROOF. Lemma 2 implies that the sequence $\{M(\Omega_n)\}$ is non-decreasing and that $M(\Omega_n) \leq M(\Omega)$ for any n. We must show that $\lim_{n\to\infty} M(\Omega_n)$ is not strictly less than $M(\Omega)$. It suffices to treat the case when Ω is a standard annulus T_R. If k is any number such that $\sqrt{R} < k < 1$, the closure of the subannulus $T_{R/k,k} = \{z \in \mathbb{C} : R/k < |z| < k\}$ is contained in Ω_n for large enough n, by compactness. Thus the limit of $\{M(\Omega_n)\}$ is not less than $M(T_{R/k,k}) = \frac{\log(k^2/R)}{2\pi}$. The conclusion follows by taking the limit of this inequality for $k \to 1$. □

We now turn our attention to degenerate annular regions. A degenerate annular region, or a *punctured disk*, on a Riemann surface S is a region $\dot{\Omega}$ which is analytically equivalent to the punctured unit disk $\dot{\Delta} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. We fix a specific isomorphism $\varphi : \dot{\Delta} \to \dot{\Omega}$. Suppose that, for any sequence $\{x_n\}$ in $\dot{\Delta}$ converging to the origin of $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, the image sequence $\{\varphi(x_n)\}$ does not converge in S. Then, if we use φ to glue the unit disk Δ to S, the resulting surface $\overline{S}$ is Hausdorff and is obtained by adding to S a point p, corresponding to the origin of Δ. We shall refer to the point $p \in \overline{S}$ as a *puncture of* S. Suppose instead that there is a sequence $\{x_n\}$ in $\dot{\Delta}$ which converges to 0 and has the property that its image in S also converges to some point p. We claim that in this case $\Omega = \dot{\Omega} \cup \{p\}$ is open in S, and that the analytic isomorphism $\varphi : \dot{\Delta} \to \dot{\Omega}$ extends to a biholomorphism between Δ and Ω. We may assume that S is connected. It suffices to show that φ extends to a holomorphic map from Δ to S, or even, by the uniqueness of the limit, from Δ to some Riemann surface T containing S as an open subset. When S is an algebraic curve, we may take as T a smooth completion of S. Let q be a point of $\dot{\Omega}$, and V a small neighborhood of q in S. We may choose V in such a way that it does not meet $\varphi(\dot{\Delta}_r)$ for some $r < 1$, where $\dot{\Delta}_r = \{z \in \mathbb{C} : 0 < |z| < r\}$. There is a projective embedding $T \hookrightarrow \mathbb{P}^N$ such that there exists a hyperplane H meeting T only at q. Hence $T \backslash V$ is a bounded subset of $\mathbb{C}^N = \mathbb{P}^N \backslash H$, and the restriction of φ to $\dot{\Delta}_r$ extends to a holomorphic map from $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ to T, by the Riemann extension theorem. This argument takes care, in particular, of all the non-hyperbolic Riemann surfaces. In fact, these are just the Riemann sphere, the complex plane, the punctured complex plane, and the one-dimensional complex tori, which are all algebraic. The argument for general hyperbolic surfaces is quite different, and uses the following fundamental result.

LEMMA 4. (*Generalized Schwarz lemma*) *Let Δ be the unit disk, endowed with the Poincaré metric of constant curvature -1, let T be a Riemann surface endowed with a hermitian metric whose curvature is everywhere ≤ -1, and let $f : \Delta \to T$ be holomorphic. Then f is distance-decreasing, in the sense that, for any pair of points $x, y \in \Delta$, the distance between $f(x)$ and $f(y)$ does not exceed the one between x and y.*

We postpone the proof of Lemma 4, and assume its validity. Put the hyperbolic metric on both $\dot{\Delta}$ and S. Composing φ with the universal covering map $\Delta \to \dot{\Delta}$ we get a holomorphic map from Δ to S, which is distance-decreasing by the generalized Schwarz lemma; hence φ is also distance-decreasing. A consequence is the following. Set $C_r = \{z : |z| = r\}$. Since the length of C_r in the hyperbolic metric of $\dot{\Delta}$ goes to zero with r, the distance-decreasing property of φ implies that the same is true for the length of $\varphi(C_r)$ in S. Let D be a geodesically convex geodesic disk of radius 3ε centered at p, where ε is a small positive number. If n is a large enough integer, the distance between p and $\varphi(x_n)$ and the length of $\varphi(C_r)$ are both less than ε, for every $r \leq r_0 = |x_n|$. Thus $\varphi(C_{r_0})$ lies entirely inside the geodesic disk of radius 2ε centered at p. Now let $r_1 < r_0$ be such that $\varphi(C_{r_1})$ is contained in D. By the Jordan curve theorem, either $\varphi(C_{r_1})$ is enclosed by $\varphi(C_{r_0})$, or $\varphi(C_{r_0})$ is enclosed by $\varphi(C_{r_1})$. In the first case, $\varphi(C_{r_1})$ is clearly contained in the disk of radius 2ε centered at p. The same is true in the second case, since the length of $\varphi(C_{r_1})$ is less than ε, and $\varphi(C_{r_1})$ encloses a point whose distance from p is less than ε. The conclusion is that, for any $r < r_0$, the curve $\varphi(C_r)$ cannot escape outside D. By the Riemann extension theorem, then, φ extends to a map $\Delta \to S$, as desired.

Proof of Lemma 4. In local coordinates, a metric on S is of the form $h\,d\zeta\,d\bar{\zeta}$, where h is positive. If $\psi = h\,d\zeta \wedge d\bar{\zeta}$ is the corresponding exterior form, the curvature of the metric is $-2\partial\bar{\partial}\log h/\psi$. If $\alpha = a\,d\zeta \wedge d\bar{\zeta}$ and $\beta = b\,d\zeta \wedge d\bar{\zeta}$ are $(1,1)$-forms with a and b real, we shall, somewhat improperly, write $\alpha \geq \beta$ to indicate that $a \geq b$. With this convention, the assumption on the curvature of S translates into $2\partial\bar{\partial}\log h \geq \psi$. The hyperbolic metric on Δ_r corresponds to the form

$$\eta_r = k_r\,dz \wedge d\bar{z} = \frac{4r^2}{(r^2 - |z|^2)^2}\,dz \wedge d\bar{z}.$$

What must be shown is that $\varphi^*(\psi) \leq \eta_1$; clearly, it suffices to show that $\varphi^*(\psi) \leq \eta_r$ on Δ_r for every $r < 1$. Notice that $\varphi^*(\psi)$ is bounded on Δ_r, while η_r goes to infinity at the boundary. Thus, if we write $\varphi^*(\psi) = u\eta_r$, then u goes to zero at the boundary, and hence has an interior maximum at some point z_0. This implies that $\partial\bar{\partial}\log u \leq 0$ at z_0. On the other hand, $uk_r = |\partial\zeta/\partial z|^2 h \circ \varphi$, and hence $\partial\bar{\partial}\log(h \circ \varphi) = \partial\bar{\partial}\log u + \partial\bar{\partial}\log k_r$. Putting everything together we find that

$$\varphi^*(\psi) \leq 2\partial\bar{\partial}\log(h \circ \varphi) = 2\partial\bar{\partial}\log u + 2\partial\bar{\partial}\log k_r \leq 2\partial\bar{\partial}\log k_r = \eta_r$$

at z_0, that is, $u(z_0) \leq 1$. Since u has a maximum at z_0, $u \leq 1$ everywhere, hence $\varphi^*(\psi) \leq \eta_r$ everywhere. □

Now let $\dot{\Omega}$ be a punctured disk in S. Write $\dot{\Omega} = \Omega \setminus \{p\}$, where Ω is a disk in S or in a Riemann surface $\bar{S} = S \cup \{p\}$ containing S as an open subset. Let v be a non-zero tangent vector to Ω at p, let $F : \Omega \to \Delta$ be a biholomorphic map with $F(p) = 0$, and set

$$r = \frac{1}{|v(F)|}. \tag{7}$$

Since the automorphisms of the unit disk carrying the origin to itself are just the rotations, the number r does not depend on the isomorphism F, but only on $\dot{\Omega}$ and on the choice of v. It could be equivalently defined as the radius of a disk $\Delta_r = \{z \in \mathbb{C} : 0 < |z| < r\}$ for which there exists an isomorphism $f : \Omega \to \Delta_r$ with

$$f(p) = 0, \quad |v(f)| = 1. \tag{8}$$

The *reduced modulus of* $\dot{\Omega}$ *with respect to* v is defined by:

$$\dot{M}_v(\dot{\Omega}) = \frac{\log r}{2\pi} = \frac{-\log|v(F)|}{2\pi}. \tag{9}$$

Sometimes, if no confusion is likely, we will omit the reference to the tangent vector v in the notation for the reduced modulus. To connect the notion of reduced modulus with the notion of modulus of an annular region, pick a local coordinate ζ centered at p and with $v = \frac{\partial}{\partial \zeta}$, denote by γ_ρ the preimage in Ω of a circle of radius ρ centered at the origin in the ζ-plane, and denote by Ω_ρ the annular region which is the connected component of $\Omega \setminus \gamma_\rho$ not containing the point p.

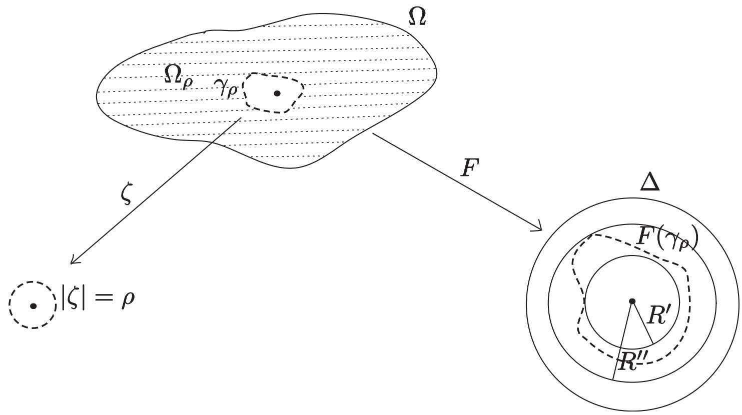

Figure 3

We want to show that

$$\lim_{\rho \to 0} \left(M(\Omega_\rho) + \frac{\log \rho}{2\pi} \right) = \dot{M}_v(\dot{\Omega}). \tag{10}$$

Let r be as in (7). Then

$$F(\zeta) = \frac{\zeta}{r} + a_2 \zeta^2 + \cdots.$$

Let R', (resp., R'') be the minimum (resp., the maximum) of $|F|$ over γ_ρ. Then

$$\left(\frac{\rho}{r} - |a_2|\rho^2 - \cdots\right) \le R' \le |w(\zeta)| \le R'' \le \left(\frac{\rho}{r} + |a_2|\rho^2 + \cdots\right).$$

By Lemma 2, we have

$$-\frac{1}{2\pi}\log\left(\frac{\rho}{r} + |a_2|\rho^2 + \cdots\right) \le M(T_{R''}) \le M(\Omega_\rho) \le M(T_{R'})$$
$$\le -\frac{1}{2\pi}\log\left(\frac{\rho}{r} - |a_2|\rho^2 - \cdots\right),$$

which implies (10). A corollary of (10) is the following analogue of Lemma 2.

Lemma 5. *Let $\dot\Omega_1 \subset \dot\Omega_2$ be two concentric punctured disks on a Riemann surface S. Denote by Ω the annular region $\dot\Omega_2 \backslash \dot\Omega_1$. Then $\dot M_v(\dot\Omega_2) \ge M(\Omega) + \dot M_v(\dot\Omega_1)$.*

Proof. Consider the annuli

$$\Omega_{i,\rho} = \dot\Omega_i \backslash \{|\zeta| < \rho\}, \quad i = 1, 2.$$

By Lemma 2 we have $M(\Omega_{2,\rho}) \ge M(\Omega) + M(\Omega_{1,\rho})$. The result follows from (10). □

Before proving further results on annular regions, we turn our attention to *annular coverings* of Riemann surfaces. Denote by $\eta : \tilde S \to S$ the universal cover of a Riemann surface S. Fix points $x \in S$, $\tilde x \in \tilde S$ with $\eta(\tilde x) = x$, and identify $\pi_1(S, x)$ with the group of deck transformations of η. Fix a *simple, closed, homotopically non-trivial loop* γ in S, based at x, and denote by S_γ the quotient of $\tilde S$ by the cyclic subgroup of $\pi_1(S, x)$ generated by $[\gamma]$. Denote by $\omega : \tilde S \to S_\gamma$ the quotient map and by $\pi : S_\gamma \to S$ the topological cover corresponding to the subgroup $\langle[\gamma]\rangle$, so that $\pi\omega = \eta$. Let $\tilde\gamma$ be the image under ω of the lifting of γ to $\tilde S$ with initial point $\tilde x$. Set $y = \omega(\tilde x)$. The curve $\tilde\gamma$ is a loop and is the lifting of γ with initial point y. Clearly, $\pi_1(S_\gamma, y) = \langle[\tilde\gamma]\rangle$. We are going to prove that, when S is a hyperbolic surface, then $\tilde\gamma$ is the only closed curve in S_γ which is a lifting of γ.

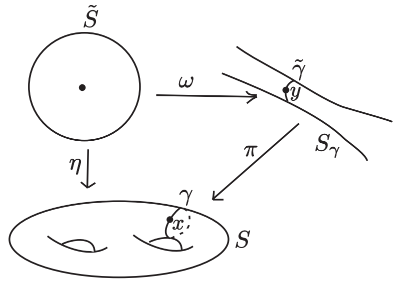

Figure 4

LEMMA 6. *If S is hyperbolic, the loop $\tilde{\gamma}$ is the only closed lifting of the simple loop γ. More generally, if a closed curve σ in S has a closed lifting in S_γ, then this lifting is unique* (*and σ is freely homotopic to γ^n, for some n*).

PROOF. Let σ be a closed loop in S. Suppose σ_1 is a closed lifting of σ to S_γ. The loop σ_1 is freely homotopic to $\tilde{\gamma}^n$, for some n. As a consequence, σ is freely homotopic to γ^n. We want to prove that σ_1 is the only closed lifting of σ. Suppose there is another closed lifting σ_2 of σ. Let z be the base point of σ_1 and w the base point of σ_2. Join y to z with a path τ_1. Let τ be the projection of τ_1 and τ_2^{-1} the lifting of τ^{-1} with initial point w. Then $\tau_1\sigma_1\tau_1^{-1}$ and $\tau_2\sigma_2\tau_2^{-1}$ are two closed liftings of $\tau\sigma\tau^{-1}$ and the base point of $\tau_1\sigma_1\tau_1^{-1}$ is y. Hence, we may as well assume that σ is based at x and σ_1 is based at y. In particular σ_1 is homotopic to $\tilde{\gamma}^n$. By the same argument we used above, we deduce that σ_2 is freely homotopic to $\tilde{\gamma}^n$ and hence to σ_1. So, if σ_2 is based at $y' \in \pi^{-1}(x)$, there is a path α from y to y' such that $\alpha\sigma_1\alpha^{-1} \sim \sigma_2$. The projection of α via π is a closed loop β, and $[\beta]$ commutes with $[\gamma^n]$. Now, any abelian subgroup of $\mathrm{Aut}(\Delta) \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ is contained in a one-parameter subgroup. Moreover, as $\pi_1(S,x)$ is contained in $\mathrm{Aut}(\Delta)$ as a discrete subgroup, there must be a cyclic subgroup containing $[\beta]$ and $[\gamma^n]$. Since γ is simple, $\langle[\gamma]\rangle$ is the largest cyclic subgroup of $\pi_1(S,x)$ containing $[\gamma^n]$, and hence $[\beta]$ is a power of $[\gamma]$. This implies that $y' = y$. But then, by the uniqueness of liftings, $\sigma_1 = \sigma_2$. □

COROLLARY 7. *Let S be a hyperbolic surface. Let γ be a simple, homotopically non-trivial loop in S. Let Ω be an annular region in S with the same homotopy type as γ. Then there is a unique annular region $\tilde{\Omega} \subset S_\gamma$ such that the covering $\pi : S_\gamma \to S$ restricts to an isomorphism between $\tilde{\Omega}$ and Ω. Moreover $\tilde{\Omega}$ has the same homotopy type as $\tilde{\gamma}$.*

PROOF. Let F be an isomorphism between an annulus T_R and Ω, and let ρ and ϑ be polar coordinates in T_R. By the preceding lemma there is a unique closed lifting $\tilde{\gamma}_\rho$ of the curve γ_ρ defined by $\gamma_\rho(\vartheta) = F(\rho, \vartheta)$. Since π is holomorphic, this lifting depends holomorphically on the point (ρ, ϑ) and we define $\tilde{\Omega}$ to be the image of the holomorphic map $\tilde{F}$ given by $\tilde{F}(\rho, \vartheta) = \tilde{\gamma}_\rho(\vartheta)$. □

LEMMA 8. *Let γ be a simple closed curve on a hyperbolic surface S. Let $\mathscr{A}(\gamma)$ be the set of annular subregions of S with the same homotopy type as γ, and define*

$$M(\gamma) = \sup_{\Omega \in \mathscr{A}(\gamma)} \{M(\Omega)\}. \tag{11}$$

Then $M(\gamma) = +\infty$ if and only if γ can be contracted to a point of S or to a puncture.

PROOF. Only one implication is non-trivial. Let us then assume that $M(\gamma) = +\infty$ and that γ is not contractible in S. We keep the notation of the previous lemma. Since S is hyperbolic, S_γ is either an annulus or a punctured disk. Because of the previous corollary, we have $M(\tilde{\gamma}) = +\infty$. By Lemma 2, $M(\tilde{\gamma}) = M(S_\gamma)$. Thus S_γ

is a punctured disk. Let Ω be the connected component of $S_\gamma \backslash \{\tilde{\gamma}\}$ which is isomorphic to a punctured disk. The lemma will be proved if we can show that $\pi : \Omega \to \pi(\Omega)$ is an isomorphism. Of course it is enough to prove that this map is injective. We contend that for this it is enough to prove that there is no pair of points $x \in \tilde{\gamma} = \partial\Omega$ and $y \in \Omega$ such that $\pi(x) = \pi(y)$. Suppose in fact that there are $z, w \in \Omega$ with $\pi(z) = \pi(w)$. Join z to a point of $\tilde{\gamma}$ with a smooth path α in Ω not passing through w, and let β be the lifting of $\pi(\alpha)$ with initial point w. The end points of β and α lie in the same fiber of π, so that the end point of β cannot belong to $\tilde{\gamma}$. If β stays inside Ω denote by y its final point and by x the final point of α (picture on the left in Figure 5).

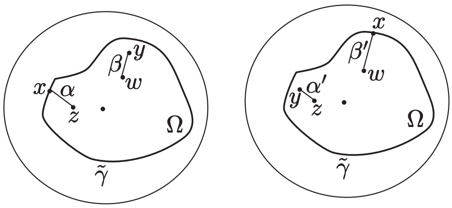

Figure 5

Evidently $\pi(x) = \pi(y)$. If, on the other hand, β does intersect $\tilde{\gamma}$, denote by x the first point of intersection, by β' the portion of β going from w to x, and by y the final point of the lifting α' of $\pi(\beta')$ with initial point z (picture on the right in Figure 5). The point y cannot coincide with x by the uniqueness of lifting. Since x and y lie in the same fiber of π, they cannot both belong to $\tilde{\gamma}$. It follows that α' is a proper sub-path of α, so that y lies in Ω. But then again $\pi(x) = \pi(y)$.

We claim that $\pi^{-1}\pi(x) \cap \Omega$ is a finite set. If this were not the case, since the fiber of π over x is discrete, we could find a sequence $\{x_n\}_{n>0} \subset \pi^{-1}\pi(x)$ converging to the puncture of Ω. Identify Ω with the standard punctured disk $\dot{\Delta}$. Assuming, as we may, that the sequence $\{|x_n|\}$ is strictly decreasing, we let C_n denote the circle centered at zero with radius $|x_n|$. Both S_γ and S inherit the hyperbolic metric from the Poincaré disk. In this metric, the length of C_n tends to zero as n tends to infinity. Since π is a local isometry, the same is true for the length of the lifting σ_n of $\pi(C_n)$ with initial point x. By Lemma 6 this lifting is not closed and we denote by $y_n \in \pi^{-1}(x)$ its final point.

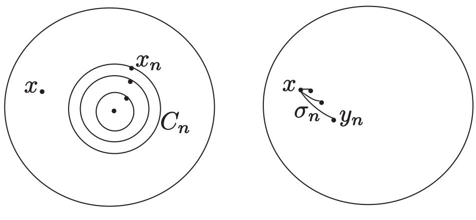

Figure 6

Since the length of σ_n tends to zero as n tends to infinity, we have $\lim_{n\to\infty} y_n = x$, which is absurd, since the fiber over x is discrete. Our claim on the finiteness of $\pi^{-1}\pi(x) \cap \Omega$ is proved.

Now we shall show that $\pi^{-1}\pi(x) \cap \Omega$ is empty. Suppose not, and let x_1 be one of its points. By construction, $x \neq x_1$. The lift of γ with initial point x_1, which we denote γ_1, does not intersect $\tilde{\gamma}$, by the uniqueness of liftings. In particular, its final point x_2 belongs to $\pi^{-1}\pi(x) \cap \Omega$. Now let γ_2 be the lift of γ with initial point x_2, denote by x_3 its endpoint, then lift γ to a path γ_3 with initial point at x_3, and so on. Denote by τ the curve obtained by joining all the γ_i. As we observed, τ does not intersect $\tilde{\gamma}$. On the other hand, since $\pi^{-1}\pi(x) \cap \Omega$ is finite, it must contain a simple closed loop $\sigma \subset \Omega$. As σ is simple and S_γ is isomorphic to a punctured disk, either σ is contractible or it is freely homotopic to $\tilde{\gamma}$. Since the projection of σ is γ^k, with $k \neq 0$, only the second possibility may occur. But then, by Lemma 6, $\sigma = \tilde{\gamma}$, contradicting the fact that σ is entirely contained in Ω. □

It is useful to introduce a notion of convergence for annular regions.

Definition 9. A sequence $\{\Omega_n\}$ of annular regions on a Riemann surface S is said to converge to an annular region $\Omega \subset S$, if there exist a sequence of non-negative numbers $R_n < 1$ converging to a number $R_0 < 1$ and a sequence of isomorphisms $f_n : T_{R_n} \to \Omega_n$, converging uniformly on compact subsets of T_{R_0} to an isomorphism $f : T_{R_0} \to \Omega$.

Lemma 10. *Let γ be a simple closed loop on a hyperbolic Riemann surface S. Let $\{\Omega_n\}$ be a sequence of annular regions with the same homotopy type as γ. Assume that $M(\gamma) < +\infty$ and that $\lim_{n\to\infty} M(\Omega_n) = M > 0$. Then there exists an annular region Ω with $M(\Omega) = M$ such that a suitable subsequence of $\{\Omega_n\}$ converges to Ω.*

Proof. Since $M(\gamma) < +\infty$, there is an isomorphism $\varphi : T_R \to S_\gamma$ with $R > 0$. Choose isomorphisms $f_n : T_{R_n} \to \Omega_n$. By Corollary 7, there are holomorphic maps $g_n : \Omega_n \to S_\gamma$ such that $\pi \circ g_n$ is the inclusion $\Omega_n \hookrightarrow S$, whence commutative diagrams

$$
\begin{array}{ccc}
T_R & \xrightarrow{\varphi} & S_\gamma \\
h_n \Big\uparrow & & \Big\uparrow g_n \\
T_{R_n} & \xrightarrow{f_n} & \Omega_n \subset S
\end{array}
\tag{12}
$$

Clearly, each h_n is an isomorphism onto its image. Set $R_0 = e^{-2\pi M}$. The sequence $\{h_n\}$ is a uniformly bounded sequence of holomorphic maps. Therefore, passing to a subsequence if necessary, we may suppose that $\{h_n\}$ converges uniformly on compact subsets of T_{R_0} to a conformal map $h : T_{R_0} \to T_R$, and hence that $\{f_n\}$ converges uniformly on compact subsets of T_{R_0} to $f = \pi \circ \varphi \circ h : T_{R_0} \to S$. The map h is not constant, since h_n is a homotopy equivalence for every n. Thus h and f are open. To see that f is injective, suppose that $f(x) = f(y)$, with $x \neq y$. Choose small disjoint disks A and B around x and y, respectively. We claim that $f_n(A) \ni f(x)$ for large n, and similarly for $f_n(B)$, which contradicts the injectivity of f_n. To prove the claim, we argue by contradiction, and suppose that $f(x) \notin f_n(A)$ for every n. The distance between $f(x)$ and $f_n(A)$ which, by our assumption, equals the distance between $f(x)$ and $f_n(\partial A)$, goes to zero as $n \to \infty$. On the other hand, the distance between $f(x)$ and $f(\partial A)$ is strictly positive, a contradiction. It follows that f is an isomorphism between T_{R_0} and $\Omega = f(T_{R_0})$, and the lemma is proved. □

Corollary 11. *Let γ be a simple closed curve on a hyperbolic surface S. Suppose $M(\gamma) < +\infty$. Then there exists an annular region $\Omega \subset S$ with $M(\Omega) = M(\gamma)$.*

The next result is an analogue, for punctured disks, of the preceding lemma.

Lemma 12. *Let γ be a homotopically non-trivial simple closed loop on a hyperbolic Riemann surface S. Suppose that $M(\gamma) = +\infty$. Let $\{\dot{\Omega}_n\}$ be a sequence of punctured disks with the same homotopy type as γ. Let v be a non-zero tangent vector to the completed surface $S \cup \{p\}$ at the puncture p. Assume that the sequence of reduced moduli $\dot{M}_v(\dot{\Omega}_n)$ converges to $M \neq -\infty$ as n tends to infinity. Then a subsequence of $\{\dot{\Omega}_n\}$ converges to a punctured disk $\dot{\Omega}$ with reduced modulus equal to M.*

Proof. Set $\Omega_n = \dot{\Omega}_n \cup \{p\}$. Choose isomorphisms $f_n : \Delta_n = \{z \in \mathbb{C} : |z| < r_n\} \to \Omega_n$ such that $f_n(0) = p$ and $v(f_n^{-1}) = 1$. Then $\dot{M}_v(\dot{\Omega}_n) = \frac{1}{2\pi} \log r_n$. Set $r = e^{2\pi M}$. Now one looks at the annular cover S_γ of S, which is a punctured disk, and proceeds exactly as in the proof of Lemma 10. □

We end this section by proving two topological lemmas that we will need in the sequel.

Lemma 13. *Let γ be a homotopically trivial simple closed curve on a Riemann surface S. Then there is a region $\Omega \subset S$ isomorphic to a disk such that $\partial\Omega = \gamma$. This region is unique, unless $S = \mathbb{P}^1$.*

Proof. The case $S = \mathbb{P}^1$ is the classical theorem of Jordan. If S is not $\mathbb{P}^1$, let $\pi : \tilde{S} \to S$ be the universal cover. Since $\tilde{S}$ is either $\mathbb{C}$ or the unit disk, the curve γ lifts to a simple closed curve $\tilde{\gamma}$ which, again by Jordan's theorem, bounds a disk $D \subset \tilde{S}$. It now suffices to show that $\pi : \bar{D} \to S$ is injective. If two points of $\bar{D}$ map to the same point of S, there is a deck transformation T carrying one to the other. Thus $\bar{D}$ and $T(\bar{D})$ are not disjoint. By the uniqueness of liftings, $T(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset$. But then either $T(\bar{D}) \subset \bar{D}$ or $\bar{D} \subset T(\bar{D})$. In the first case T has a fixed point in $\bar{D}$, by the Brouwer fixed point theorem, and hence is the identity. In the second case, replacing T with T^{-1}, we reach the same conclusion. Thus $\bar{D} \to S$ is injective. □

Lemma 14. *Let γ and δ be non-intersecting, freely homotopic and homotopically non-trivial simple closed curves on a Riemann surface S. Then there is a an annular region $\Omega \subset S$ such that $\partial\Omega = \gamma \cup \delta$.*

Proof. The cases when S is not hyperbolic can be dealt with directly and are left to the reader. We thus assume that S is hyperbolic. Consider the annular covering $\pi : S_\gamma \to S$, and identify S_γ with T_R, where $R \geq 0$. Let $\tilde{\gamma}$ and $\tilde{\delta}$ be liftings of γ and δ, respectively. Clearly, $\tilde{\gamma}$ and $\tilde{\delta}$ bound an annular region $\tilde{\Omega} \subset T_R$. It now suffices to show that $\pi : \tilde{\Omega} \to S$ is injective, and this is done exactly as in the beginning of the proof of Lemma 8. □

3. Trajectories of quadratic differentials

Let us briefly recall the geometry associated to a quadratic differential $\omega \in H^0(S, K_S^2)$ on a Riemann surface S. Let Z be the set of zeroes of ω. Away from Z, one can define a hermitian metric, the so-called *ω-metric*, which is the metric with local expression

$$|f|\, dz\, d\bar{z},$$

where $f\, dz^2$ is a local expression for ω. On a neighborhood of each point $p \in S \backslash Z$ the quadratic differential ω defines a set of distinguished coordinates, any two of which differ at most by a sign and the addition of a constant. In a neighborhood of a point $p \in S \backslash Z$, such a coordinate ζ is simply defined by

$$\zeta = \int \sqrt{\omega}.$$

If we impose its vanishing at p, a distinguished coordinate is fixed up to a sign. These distinguished coordinates are called the ω-coordinates. In terms of these coordinates, the ω-metric has local expression $d\zeta\, d\bar{\zeta}$ and therefore $S_0 = S \backslash Z$, equipped with the ω-metric, looks locally like the euclidean plane. Geodesics for the ω-metric will be called *ω-geodesics*. Clearly, a curve is an ω-geodesic if and only if, at each one of its points, it is a straight line in ω-coordinates. There is also an intrinsic notion of horizontal (resp. vertical) geodesic. In terms of an ω-coordinate $\zeta = \xi + i\eta$, the horizontal (resp. vertical) geodesics are the curves $\eta =$ constant (resp., $\xi =$ constant). Figure 7 shows the structure of the horizontal and vertical geodesics in the neighborhood of a point $p \in S$ which is a zero of order n, with n equal to 1, 2 and 3, respectively.

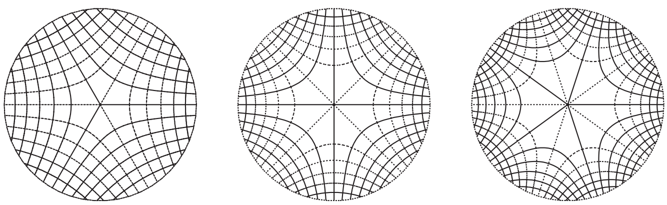

Figure 7

A geodesic arc α in S_0 *locally minimizes distances*. In fact, for arcs entirely contained in S_0, the local minimizing property characterizes geodesics. Since it makes sense to talk about the length of an arc in S (and not only in S_0), one defines an *ω-geodesic in S* to be a path in S having the property of locally minimizing distances. Geodesics passing through a zero of ω are called *singular*. It can be shown that, on a compact Riemann surface of genus $g > 1$, any two points can be joined by an ω-geodesic, and that such a geodesic is unique within its homotopy class. In proving this fact the main ingredient is a Gauss-Bonnet-type result which we are now going to state.

Consider an ω-geodesic polygon P in a connected Riemann surface S, so that P is homeomorphic to a disk and the boundary of P is the union of finitely many ω-geodesic arcs whose interiors do not contain zeroes of ω. Let $q_1, \dots, q_s$ be the points of the boundary of P where two of these arcs meet, let μ_j be the multiplicity of q_j as a zero of ω, and denote by ϑ_j the interior angle formed by the sub-arcs of ∂P adjoining q_j, for $j = 1, \dots, s$. By compactness, P contains a finite number of zeroes of ω; let them be $p_1, \dots, p_r$, and let $v_i > 0$ be the multiplicity of p_i as a zero of ω. The situation is illustrated in Figure 8.

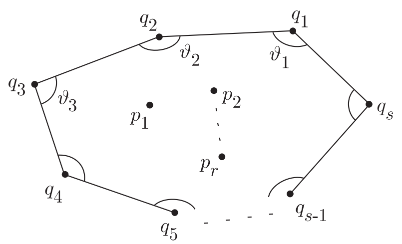

Figure 8

Set $v = \sum_{i=1}^{r} v_i$. Then the following Gauss-Bonnet formula holds:

$$(v+2)2\pi = \sum_{j=1}^{s}(2\pi - (\mu_j + 2)\vartheta_j). \tag{13}$$

Let us finally recall that the *critical points of ω* are the zeroes of ω and the punctures of S. A horizontal ω-geodesic is said to be a *trajectory* of ω if it does not pass through any critical point and is maximal with respect to this property. A trajectory is said to be *closed* if it is a (simple) loop. A horizontal ω-geodesic is said to be a *critical trajectory* if it joins two critical points of ω. It is a trivial but important observation that distinct trajectories do not intersect, and that a trajectory crosses itself only if it is closed. We refer the reader to section 5 of Chapter 16 in [1] for the proof of all the above facts.

In our study, we are going to restrict ourselves to the case of *admissible* quadratic differential on *Riemann surfaces of finite type.* Recall that a Riemann surface S is said to be of *finite type* if it is obtained from a compact surface $\bar{S}$ by deleting a finite number of points

$$S = \bar{S} \backslash \{y_1, \dots, y_m\}. \tag{14}$$

On the other hand, a quadratic differential on a Riemann surface of finite type is said to be *admissible* if S has finite ω-area:

$$\int_S |\omega| < \infty. \tag{15}$$

It is straightforward to show that an admissible quadratic differential on S extends meromorphically to a differential on $\bar{S}$, having at worst simple poles at the points y_i. Since these simple poles are naturally brought into the picture let us examine the trajectory structure of ω near one of them, call it p. We can always find a local coordinate ζ on $\bar{S}$ near p and vanishing at p, such that, locally

$$\omega = \frac{1}{4\zeta} d\zeta^2 = (d\zeta^{1/2})^2.$$

It is then clear that the trajectories of ω near p can be described by the picture in Figure 9.

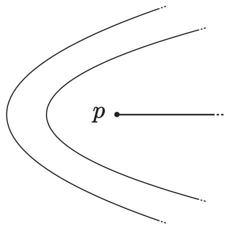

Figure 9

In this section we are going to prove two main results. The first one concerns closed trajectories. We will prove that any such trajectory is contained in a *maximal annular region which is swept out by trajectories*. The second one concerns trajectories that are neither closed nor critical. We will prove that the closure of such a trajectory is a set of *positive measure*.

We fix, for the remainder of this section, a Riemann surface $S = \bar{S} \backslash \{y_1, \dots y_m\}$ of finite type and an admissible quadratic differential ω on it.

Let α be a trajectory of ω. We view α as a path $\rho : I \to S$ parametrized by arclength, where I is an open interval which may be finite or infinite. Let ζ be a distinguished coordinate at a point $\rho(t)$. Recall that ζ is well defined only up to sign and translation. We can get rid of these ambiguities by asking that $\zeta(\rho(t)) = t$. Denote by v_t the unit tangent vector to S at $\rho(t)$ orthogonal to α and pointing in the direction of increasing $\mathrm{Im}(\zeta)$; by the way ζ has been chosen, v_t depends smoothly on t. Then let $s \mapsto \beta_t(s)$ be the geodesic with initial point $\rho(t)$ and initial tangent vector v_t. It is a vertical geodesic, which we think of as extending in both directions as far as, and excluding, the first critical point encountered, or indefinitely, if none is met. We denote by J_t the interval on which β_t is defined, and by u_t, ℓ_t its upper and lower endpoints. It is useful to notice that, if ζ is a distinguished coordinate as above, then, by definition, $\zeta(\beta_t(s)) = t + \mathrm{i}s$ for small s. We now regard t and s as the real and imaginary parts of a complex coordinate $z = t + \mathrm{i}s$, and set $\Psi(z) = \beta_t(s)$. This defines a map from a connected subset D of the z-plane to S. By what we just observed, Ψ is defined and holomorphic on a neighborhood of I. Moreover, near any point of I, it has a local inverse which is a distinguished coordinate.

We now wish to show that this is true on all of D. We shall prove that D is open and that, locally near any point of D, the map Ψ is the inverse of a distinguished coordinate, so that in particular it is holomorphic. For any $t \in I$ let u_t' (resp., ℓ_t') be the supremum (resp., the infimum) of all points $s \in J_t$, $s \geq 0$ (resp., $s \leq 0$) such that Ψ is defined and equal to the inverse of a distinguished coordinate on a neighborhood of $t + i\sigma$ for every σ between 0 and s. We must show that $u_t' = u_t$ and $\ell_t' = \ell_t$ for all $t \in I$. We shall deal only with u_t', the argument for ℓ_t' being just the same. We begin by noticing that $u_t' > 0$, by what we observed above. Suppose $u_{t_0}' < u_{t_0}$ for some $t_0 \in I$; then $p = \Psi(t_0 + \mathrm{i}u_{t_0}')$ is not critical. Let ζ be a distinguished coordinate in a geodesically convex neighborhood U of p. If $s_0 < u_{t_0}'$ is large enough, $\Psi(t_0 + \mathrm{i}s_0) \in U$. On the other hand, by compactness, Ψ is defined and locally equal to the inverse of a distinguished coordinate at every point of a rectangle $R = \{t + \mathrm{i}s : t_0 - \varepsilon \leq t \leq t_0 + \varepsilon, 0 \leq s \leq s_0\}$; moreover, if ε is small enough, Ψ maps the entire top edge of R into U. Since distinguished coordinates are unique up to sign and translation, on this edge the composition $\zeta \circ \Psi$ is of the form $z \mapsto \pm z + c$, where c is a constant; changing ζ if necessary, we may thus suppose that $\zeta(\Psi(z)) = z$ for all z belonging to the top edge of R. But then the vertical geodesic $\mathrm{Re}(\zeta) = t$ is part of the geodesic β_t for $t_0 - \varepsilon \leq t \leq t_0 + \varepsilon$. Hence Ψ is defined and equal to ζ^{-1} on $\zeta(U) \cap \{z : t_0 - \varepsilon \leq \mathrm{Re}(z) \leq t_0 + \varepsilon\}$, which is a neighborhood of $t_0 + \mathrm{i}u_{t_0}'$. This contradicts the definition of u_{t_0}', and establishes our claim.

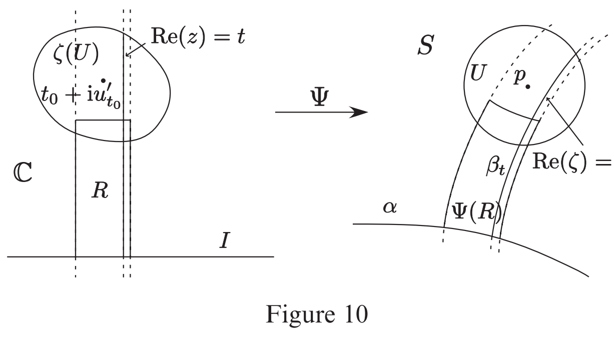

Figure 10

It is important to notice that Ψ sends segments in D to pieces of geodesics and horizontal segments to pieces of trajectories. Moreover, $\Psi^*(\omega) = dz^2$. To better understand the nature of the domain D we need to make a preliminary remark. By construction, D is contained in the infinite strip

$$T = \{t + \mathrm{i}s \in \mathbb{C} : t \in I\}.$$

If $u_t < +\infty$ (resp., if $\ell_t > -\infty$) we set $\mathbf{z}_t = t + \mathrm{i}u_t$ (resp., $\mathbf{w}_t = t + \mathrm{i}\ell_t$). The set D is just the complement, inside T, of the vertical closed half-lines extending upwards from each $\mathbf{z}_t$ and downwards from each $\mathbf{w}_t$. Since D is open, the functions $t \mapsto u_t$ and $t \mapsto \ell_t$ are, respectively, lower and upper semicontinuous. We can be even more precise.

Lemma 15. *The points $\mathbf{z}_t$ and $\mathbf{w}_t$ are isolated in T.*

Proof. We shall deal only with $\mathbf{z}_t$; the proof for $\mathbf{w}_t$ is no different. Suppose $u_{t_0} < +\infty$. Then $\Psi(t_0 + \mathrm{i}s)$ has a limit p as $s \to u_{t_0}$. By the very definition of u_{t_0}, the pp is a critical point of ω. As we know, there is a coordinate ζ centered at p such that $\omega = \zeta^n\, d\zeta^2$ near p, where either $n = -1$ or $n > 0$. Let U be the disk $\{|\zeta| < r\}$, for some small r. If s_0 is close enough to u_{t_0}, then $\beta_{t_0}(s_0) = \Psi(t_0 + \mathrm{i}s_0)$ belongs to U. Hence, if ε is small enough, the horizontal geodesic segment $L = \{\Psi(t + \mathrm{i}s_0) : t_0 - \varepsilon < t < t_0 + \varepsilon\}$ is entirely contained in U. We set $\delta = u_{t_0} - s_0$. We claim that the only point of the form $\mathbf{z}_t$ contained in the rectangle $\{z \in \mathbb{C} : |\mathrm{Re}(z) - t_0| < \varepsilon, |\mathrm{Im}(z) - u_{t_0}| < \delta\}$ is $\mathbf{z}_{t_0}$. This will clearly follow if we can show that $u_t \geq u_{t_0} + \delta$ when $|t - t_0| < \varepsilon$ and $t \neq t_0$. We shall give a proof "by pictures". The case in which $n = -1$ is clear. We can then assume that $n > 0$. Recall that the pattern of horizontal and vertical geodesics in the ζ coordinate is as shown in Figure 7. Then look at Figure 11. This illustrates what happens for $n = 1$, but things are no different for arbitrary positive n.

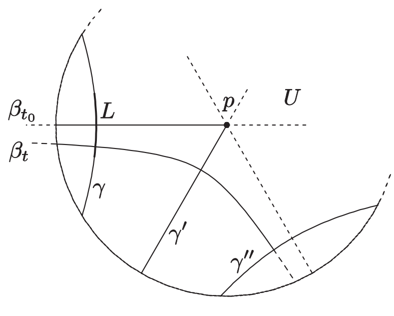

Figure 11

The curves γ, γ' and γ'' are horizontal geodesics, and L is contained in γ. The geodesic γ'' is chosen so that its distance from p equals the distance between p and γ. Thus the piece of the vertical geodesic β_t between γ' and γ'' is just as long as the one between γ and γ'. This, in turn, has the same length as the part of β_{t_0} between γ and p, that is, δ. This means that $u_t \geq s_0 + 2\delta$ or, equivalently, that $u_t \geq u_{t_0} + \delta$. $\square$

Figure 12 is the picture of a domain $D = \mathbb{C} \backslash \{\text{dotted lines}\}$ when $I = \mathbb{R}$.

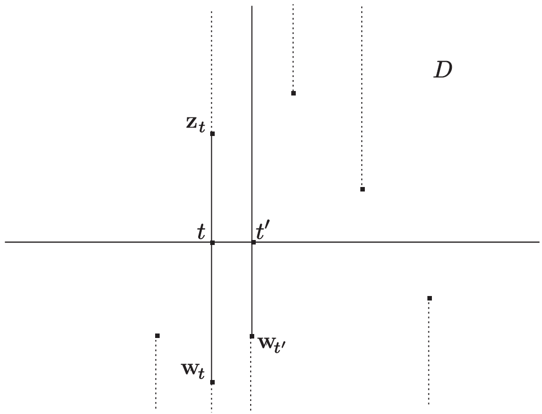

Figure 12

It is important to notice that Ψ cannot be extended holomorphically (or for that matter even continuously) to a neighborhood of $\mathbf{z}_t$ or $\mathbf{w}_t$, for any t.

Our next task is to examine the case in which α is a *closed trajectory*. In this case, we have $I = \mathbb{R}$ and the map $\Psi : I \to \alpha$ is periodic. We let $a \in \mathbb{R}_+$ be its period. Keeping the notation introduced above, we set

$$b_1 = \max\{\ell_t : 0 \leq t \leq a\}, \quad b_2 = \min\{u_t : 0 \leq t \leq a\},$$

and we consider the infinite strip

$$R = \{t + \mathrm{i}s \in \mathbb{C} : b_1 < s < b_2\} \subset D. \tag{16}$$

LEMMA 16. *Let α be a closed trajectory for an admissible quadratic differential ω on a Riemann surface S of finite type. Two cases can occur.*

i) *S is a genus 1 curve, that is, the quotient of $\mathbb{C}$ by a lattice Λ, $R = D = \mathbb{C}$, and Ψ can be identified with the quotient map $\mathbb{C} \to \mathbb{C}/\Lambda$.*
ii) *If z_1, z_2 belong to R, then $\Psi(z_1) = \Psi(z_2)$ if and only if $z_2 - z_1$ is an integral multiple of a. Moreover, $\Omega = \Psi(R)$ is an annular region, or a copy of $\mathbb{C}^*$, containing α; it is swept out by closed trajectories of ω, and is the maximal region with these properties.*

In either case, $\Psi^*(\omega) = dz^2$.

PROOF. If $x \in \mathbb{C}$, we denote by τ_x the translation by x. Since $\Psi(t+a) = \Psi(t)$ for all real t, and $D \cap \tau_{-a}(D)$ is connected, analytic continuation implies that $\Psi(z+a) = \Psi(z)$ for all $z \in D \cup \tau_{-a}(D)$. On the other hand, $D = \tau_{-a}(D)$, since otherwise $\Psi(z+a)$ would provide a holomorphic extension of Ψ across some of the $\mathbf{z}_t$ or $\mathbf{w}_t$. Now suppose that there are points $z_1 \neq z_2$ of R such that $\Psi(z_1) = \Psi(z_2)$ but $u = z_2 - z_1$ is not an integral multiple of a. We can assume that these two points have minimal distance with respect to this property, so that in particular $|\mathrm{Re}(u)| < a$. We denote by L the piece of the line joining z_1 and z_2 lying inside D. The image of L under Ψ is a closed geodesic. Hence, if z and $z+u$ both belong to L, then $\Psi(z) = \Psi(z+u)$; this implies, in particular, that $\Psi(z) = \Psi(z+u)$ for any point z belonging to L and sufficiently close to z_1. Arguing as for a, we conclude that $D = \tau_{-u}(D)$ and that $\Psi(z+u) = \Psi(z)$ for any $z \in D$. Since $|\mathrm{Re}(u)| < a$ and a is a period, u cannot be a real number. Since D is equal to $\tau_{-u}(D)$, a consequence is that D must be the whole plane. If Λ is the lattice generated by a and u, the minimality of $|u|$ shows that Ψ induces an isomorphism $\mathbb{C}/\Lambda \to S$, as desired. We can now assume to be in case ii). From the construction it follows that Ω is an annular region, or a copy of $\mathbb{C}^*$, containing α, that it is swept out by trajectories, and that it is maximal with respect to these properties. $\square$

The region Ω in the statement of the preceding lemma is called the *maximal annular domain* associated to the closed trajectory α. This is a slight departure from our customary usage of the word "annular", since it may well be that Ω is isomorphic to $\mathbb{C}^*$. Clearly, Ω is completely determined by α. Setting

$$r_j = \exp(-2\pi b_j), \qquad \Phi(\zeta) = \Psi\Big(\frac{a}{2\pi \mathrm{i}} \log\zeta\Big),$$

then Φ is a biholomorphic map between the annulus T_{r_1,r_2} and Ω, and $\Phi^*(\omega)$ is of the form

$$\Big(\frac{a}{2\pi \mathrm{i}} d\log\zeta\Big)^2. \tag{17}$$

It may well be that $r_1 = 0$ and $r_2 = +\infty$. In this case S is the Riemann sphere and $\Omega = \mathbb{C}^*$. On the other hand, when $r_1 = 0$, $r_2 < +\infty$ or $r_1 > 0$, $r_2 = +\infty$, the region Ω is a punctured disk.

From the proof of the lemma it follows that both boundary components of Ω must contain critical points of ω, otherwise one could continue Ψ either above the line $y = b_2$ or below the line $y = b_1$, or both. The following can also be deduced from the proof of the lemma. Suppose that α' is another closed trajectory of ω, and let Ω' be its associated maximal annular domain. Then either $\Omega' = \Omega$ or $\Omega \cap \Omega' = \emptyset$.

As we anticipated, we shall now study the case of a trajectory which is neither closed nor critical.

PROPOSITION 17. *Let ω be an admissible quadratic differential on a Riemann surface S of finite type. Let α be a trajectory of ω. Assume that α is neither closed nor critical, and denote by $\bar{\alpha}$ the closure of α. Then the measure of $\bar{\alpha}$ is strictly positive.*

Before proving the proposition we need a couple of remarks. We keep the notation introduced so far in this section. Since α is not closed, Ψ gives a bijection between I and α. Since α is not critical, I cannot be finite. Without loss of generality, we may then assume that $b = +\infty$. We set $\alpha^+ = \Psi[0, +\infty)$, $\alpha^- = \Psi(a, 0]$, and $p_0 = \Psi(0)$. Recall that Ψ maps horizontal and vertical segments in D to horizontal and vertical ω-geodesics. By *positive direction* along a horizontal segment in D or its image in S we shall mean the direction of increasing $\mathrm{Re}(z)$, $z \in D$. We need three lemmas.

LEMMA 18. *Let β be the portion of vertical ω-geodesic which is the image of a vertical segment $J = \{\mathrm{i}s : s \in \mathbb{R}, 0 \le s \le c\}$ under Ψ. Then, for every $t \in [0, +\infty)$ there exists $t_1 \in [0, +\infty)$, with $t_1 > t$, such that α^+ cuts β in $\Psi(t_1)$ in the positive direction.*

PROOF. Clearly, if the conclusion of the lemma holds for a certain value of c, it also holds for all larger values of c. Hence we may assume that the map Ψ gives a bijection between J and β. Also, it suffices to prove that there exists $t_1 > 0$ such that α^+ cuts β in $\Psi(t_1)$ in the positive direction. Let $K \subset J$ be defined by

$$K = \left\{ z \in J : \begin{array}{l} \text{the horizontal ray starting at } \Psi(z) \text{ in the positive direction} \\ \text{hits a critical point before hitting } \beta \text{ in the positive direction} \end{array} \right\}$$

We claim that K is finite. Since there are only a finite number of critical points and a finite number of trajectories leading to any one of them, it suffices to show that each trajectory leading to a critical point contains only one ray with the above property. In fact, suppose that γ^+ and δ^+ are two rays contained in the same trajectory. Then either $\gamma^+ \subset \delta^+$ or $\gamma^+ \supset \delta^+$. Say that the former holds, and say that δ^+ starts at $q \in \Psi(K)$ while γ^+ starts at $p \in \Psi(K)$. This means that δ^+ starts at q (in the positive direction), passes through p (in the positive direction) changing its name into γ^+, and then hits a critical point. But then $q \notin \Psi(K)$, contrary to our assumption.

In conclusion, there exists a subinterval $J' \subset J$ containing p_0 and having empty intersection with K. We let β' be the image of J' under Ψ. We claim that there exists a point $p \in \beta'$ such that the horizontal ray starting at p comes back to β in the positive direction. We argue by contradiction. Suppose this is not the case. Since $J' \cap K = \emptyset$, every ray starting at a point of β' can be continued indefinitely. Fix a point $p = \Psi(\mathrm{i}b) \in \beta'$. Look at the infinite strip $\Sigma = \{z = t + \mathrm{i}s \in \mathbb{C} : 0 < s < b\}$. By what we just observed, the map Ψ is defined on all of Σ. Since we are assuming that no ray starting at a point of β' comes back to β, and a fortiori to β', the restriction of Ψ to Σ must be injective. But this is absurd, since $\Psi(\Sigma)$ would be a region in S with infinite ω-area. Let us denote by

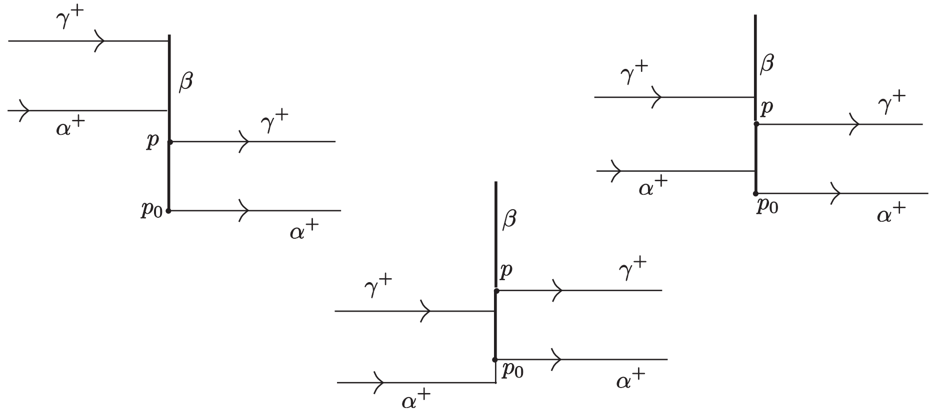

Figure 13

γ^+ a horizontal ray starting at a point $p \in \beta'$ and coming back to β in the positive direction. Figure 13 illustrates the three cases that can occur; in each, one sees the rays α^+ and γ^+ departing from the right of β and then coming back to the left of β. If the top two cases occur, the ray α^+ hits β in the positive direction and we are done. In the third case, we denote by a' the length of portion of γ^+ between the initial point and the point where it hits β in the positive direction, and by b' the vertical distance between the endpoints of this portion. We also set $R_a = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < a, 0 < \mathrm{Im}(z) < b'\}$, for any positive a. We then look at the portion of the ray α^- inside the image of the strip Σ, and we trace it back to the point of β where it started. This portion of α^-, a segment of α^+, a segment of β and its vertical mirror bound a region which is the image of $R_{a'}$ under Ψ, as shown in Figure 14.

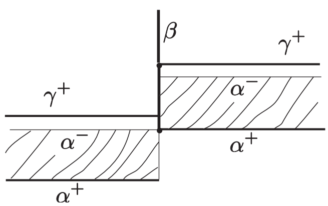

Figure 14

Exactly as before, we look at the map $\Psi : R_a \to S$ as a tends to infinity, and we observe that it cannot be injective for all values of a. So we are back to the three cases of Figure 14, but now, in all cases, we get that ray α^+ comes back to β in the positive direction. $\square$

The second lemma we need is the following.

Lemma 19. *The closure of* α *is*

$$\bar{\alpha} = \left\{ p \in S : \exists \{t_n\} \subset \mathbb{R} \text{ with } \lim_{n\to\infty} t_n = +\infty \text{ and } \lim_{n\to\infty} \Psi(t_n) = p \right\}. \tag{18}$$

Proof. Denote by A the set on the right-hand side of (18). Clearly, $A \subset \bar{\alpha}$. The set A is closed. In fact, suppose that $p = \lim_{n\to\infty} p_n$, $p_n \in A$. Then, for each n, there is a real number $t_n > n$ such that the distance between $\Psi(t_n)$ and p_n is less than $1/n$. But then $\lim_{n\to\infty} \Psi(t_n) = p$ and $\lim_{n\to\infty} t_n = +\infty$. It then suffices to show that $\alpha \subset A$. This follows immediately from the preceding lemma. In fact if p is any point in α, the lemma tells us that there exists a sequence of points $p_n = \Psi(t_n)$, lying in the vertical ω-geodesic through p, such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} t_n = +\infty$. □

The third lemma is the following.

Lemma 20. *If* $p \in \bar{\alpha}$ *is a regular point, then* $\bar{\alpha}$ *contains the trajectory through* p.

Proof. Let γ be the trajectory through p. Consider an arbitrary point $q \in \gamma$ and denote by a the ω-length of the closed subinterval of γ going from p to q. This subinterval is contained in the middle interval of a small rectangle $T \subset S$ swept out by segments of trajectories (Figure 15). By Lemma 19, there is a sequence $\{t_n\} \subset \mathbb{R}$ such that $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty} \Psi(t_n) = p$. But then, possibly after replacing $\{t_n\}$ with a subsequence, either $\lim_{n\to\infty} \Psi(t_n + a) = q$ or $\lim_{n\to\infty} \Psi(t_n - a) = q$, proving that $q \in \bar{\alpha}$. □

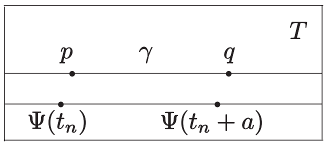

Figure 15

We are now ready to prove Proposition 17. We keep the notation of the preceding lemmas. Let p be a point in α. We are going to prove that there is a rectangle $T \subset \bar{\alpha}$ swept out by trajectories and having as one side a subinterval of α centered at p (Figure 16). Let β be a segment of a vertical ω-geodesic starting at

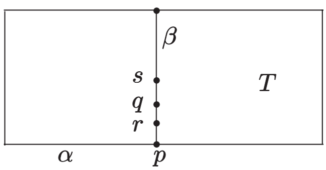

Figure 16

p and not containing critical points. If the vertical interval β is entirely contained in $\bar{\alpha}$, we are done, since by Lemma 20 all trajectories passing through a point of β are entirely contained in $\bar{\alpha}$.

If the vertical interval β is not entirely contained in $\bar{\alpha}$, let q be a point of β which is not in $\bar{\alpha}$. Since $\bar{\alpha}$ is closed, there is a largest subinterval of β containing q and not contained in $\bar{\alpha}$. Let r be the endpoint of this interval lying between p and q, and s the other endpoint. Since $r \in \bar{\alpha}$, the trajectory γ through r, can not be closed, otherwise α would be closed too. By Lemma 18, γ crosses the interval (r, s). This is absurd, since $\gamma \subset \bar{\alpha}$ by Lemma 20.

4. Holomorphic Jenkins-Strebel differentials

Throughout this section, $\overline{S}$ will denote a compact Riemann surface. A holomorphic quadratic differential on a Riemann surface is called a *holomorphic Jenkins-Strebel differential* if all its non-critical trajectories are closed. Suppose a holomorphic Jenkins-Strebel differential ω is given on a Riemann surface $S = \overline{S} \setminus \{y_1, \dots, y_m\}$ of finite type with $\chi(S) < 0$. There are finitely many critical trajectories of ω, each joining a pair of critical points of ω. Their union forms a graph Γ which is called the *critical graph* of ω. Let

$$S \setminus \Gamma = \Omega_1 \cup \cdots \cup \Omega_N$$

be the decomposition of its complement into connected components. The assumption that all trajectories are closed, together with Lemma 16, tells us that each Ω_i is a maximal annular region. Moreover, if $\Omega = \Omega_i$ is one of these regions and if

$$f : T_r \to \Omega$$

is a biholomorphic map from a standard annulus T_r, then

$$f^*(\omega) = \left(\frac{a}{2\pi \mathrm{i}} d \log z \right)^2 \tag{19}$$

where a is the ω-length of the trajectories in Ω. We recall that the modulus of Ω is given by

$$M(\Omega) = -\frac{1}{2\pi} \log r,$$

while the ω-area is given by

$$A(\Omega) = a^2 M(\Omega).$$

Before stating the main theorem of this section, we need a remark and a couple of definitions.

Remark 21. For a quadratic differential on an annulus Ω, to be of the form (19) is an intrinsic property, in the sense that it does not depend on the choice of the isomorphism f between T_r and Ω. In fact, if h is any automorphism of T_r, then $h^*(d\log z)^2 = (d\log z)^2$; this follows immediately from the observation that any such automorphism is of the form $z \mapsto cz$ or $z \mapsto cr/z$, where c is a constant of absolute value 1. There are several ways of proving the latter assertion; here is a possible argument. We write C_ρ to indicate the circle of radius ρ centered at the origin of $\mathbb{C}$. For any given ε, if δ is small enough, $h(\{z \in \mathbb{C} : r+\delta < |z| < 1-\delta\})$ contains the closed annulus $\{z \in \mathbb{C} : r+\varepsilon \leq |z| \leq 1-\varepsilon\}$, as this is compact and h is onto. Moreover, $h(\{z \in \mathbb{C} : r+\delta < |z| < 1-\delta\})$ is bounded by the images of the circles $C_{r+\delta}$ and $C_{1-\delta}$, since h is injective. The image of one of these two circles is on the outside of $\{z \in \mathbb{C} : r+\varepsilon \leq |z| \leq 1-\varepsilon\}$, and the other on the inside; which side they are on does not depend on δ, again since h is injective. A consequence is that the function $|h|^2$ extends continuously to the closure of T_r, and takes on the value 1 on its outer boundary and the value r^2 on the inner one, or conversely. In the first case we set $g(z) = z$, and in the second $g(z) = r/z$. The function $\log|h|^2 - \log|g|^2$ is harmonic in the interior of T_r, and vanishes at its boundary; therefore, by the maximum principle, it vanishes identically. In other words, $|h/g|$ is identically equal to 1, and hence h/g is a constant of absolute value 1, which is exactly what had to be proved.

Definition 22. Let S be a Riemann surface. An admissible system of curves on S is a collection $(\gamma_1, \ldots, \gamma_k)$ of simple closed curves which are mutually disjoint, homotopically non-trivial, and such that γ_i is not freely homotopic to γ_j if $i \neq j$.

The straightforward proof of following topological lemma is left to the reader.

Lemma 23. *Let $S = \bar{S}\setminus\{y_1, \ldots, y_m\}$ be a Riemann surface of finite type and of genus g. Assume that $\chi(S) < 0$. Let $(\gamma_1, \ldots, \gamma_k)$ be an admissible system of curves on S; then $k \leq 3g-3+m$, and one can find simple closed curves $\gamma_{k+1}, \ldots, \gamma_{3g-3+m}$ such that $(\gamma_1, \ldots, \gamma_{3g-3+m})$ is admissible.*

Definition 24. Given an admissible system of curves $(\gamma_1, \ldots, \gamma_k)$ on a Riemann surface S, a collection $(\Omega_1, \ldots, \Omega_k)$ of disjoint subsets of S is said to be a system of annular regions of type $(\gamma_1, \ldots, \gamma_k)$ if Ω_i is either the empty set or an annular region with the same homotopy type as γ_i, for $i = 1, \ldots, k$.

The result we want to prove in this section is the following.

Theorem 25. *Let S be Riemann surface of finite type with $\chi(S) < 0$. Let $(\gamma_1, \ldots, \gamma_k)$ be an admissible system of curves on S and let $a_1, \ldots, a_k$ be positive real numbers. Then there exists a unique admissible Jenkins-Strebel differential ω having the following properties.*

i) *If Γ is the critical graph of ω, then $S\setminus\Gamma = \Omega_1 \cup \cdots \cup \Omega_k$, where $(\Omega_1, \ldots, \Omega_k)$ is a system of annular regions of type $(\gamma_1, \ldots, \gamma_k)$.*
ii) *If Ω_i is not empty, it is swept out by trajectories whose ω-length is a_i.*

Before turning to the proof of this theorem we need to establish a number of lemmas. In particular, the next three lemmas will provide the essential tool in proving the uniqueness part of Theorem 25.

LEMMA 26. *Let $S = \overline{S} \setminus \{y_1, \dots, y_m\}$ be a Riemann surface of finite type and of genus g. Assume $\chi(S) < 0$. Let ω be an admissible differential on S. Let α be a closed trajectory of ω and let γ be a closed curve on S which is freely homotopic to α. Then $l_\omega(\gamma) \geq l_\omega(\alpha)$, and equality holds if and only if one of the following two cases occurs. Either γ is a trajectory belonging to the maximal annular domain Ω defined by α, or else it coincides with one of the two boundary components of Ω. In particular, in the second case, γ is a cycle inside the critical graph of Ω.*

PROOF. We may of course assume that γ is different from α. First of all we remark that α is not homotopically trivial for, if this were the case, α would bound a disk contained in S, by Lemma 13, which would contradict the Gauss-Bonnet formula (13). We can then consider the annular covering $\pi : S_\alpha \to S$ attached to α. We may identify S_α with an annulus centered at the origin in the complex plane. We set $\omega' = \pi^*(\omega)$, and denote by α', γ' the unique closed liftings of α and γ to S_α. Clearly, ω' is a holomorphic Jenkins-Strebel differential and α' is a closed trajectory for it. We also recall that α' and γ' map isomorphically to α and γ via π. Pick two concentric sub-annuli T and T' of S_α with the property that T contains both α' and γ', the closure of T is contained in T', and the closure of T' is contained in S_α. By compactness, only a finite number of zeros of ω' are contained in the closure of T', and in fact we may assume that they are all contained in T'. We let $\varepsilon > 0$ be the ω'-distance between the boundaries of T and T'. Let B be the subset of α' consisting of all points b such that one of the two vertical geodesic rays beginning at b hits a zero of ω' before leaving T'; clearly, B is finite. Let a be a point of $\alpha' \setminus B$, and let β be one of the vertical geodesic rays starting at a. We observe that β does not meet α' again before leaving T'. Suppose in fact that it did; let a' be the first point of intersection between β and α', and β' the part of β between a and a'. The curve α' bounds a region D in the complex plane which is biholomorphic to a disk. If $\beta' \subset D$, then β' divides D in two connected components. One of these contains the bounded component of the complement of T in $\mathbb{C}$. The other component is a region contained in T, bounded by geodesic segments, and isomorphic to a disk. This is impossible, again by the Gauss-Bonnet formula (13). If $\beta' \cap D = \emptyset$, then β' and a segment of α' between a and a' bound a disk, which is again impossible.

Let I be any segment of length strictly less than ε contained in $\alpha' \setminus B$. Draw all the vertical geodesic segments of given length δ starting at points of I and lying on a given side of α'. If δ is small these segments sweep out a subset $R_\delta \subset T'$. This subset is the locally isometric image of a euclidean rectangle, and is actually a homeomorphic image. In fact, the only way for this not to happen would be if one of the vertical geodesic rays starting on I did meet I, and hence α', again without leaving T'. But we already excluded this. Since T' has finite ω'-area, the rectangle R_δ cannot stay inside T' for every $\delta > 0$. Let δ' be the first value of δ for which $R_\delta \not\subset T'$. Since the length of the top side of $R_{\delta'}$ is strictly less than ε, that is, strictly less than the distance between the boundaries of T and T', this side of $R_{\delta'}$ is entirely outside the closure of T. If δ is sufficiently close to δ', but smaller than it, then R_δ is contained in T', but its top side does not meet the closure of T. Now we perform the same construction on the other side of α'. The end result is a rectangle R entirely contained in T' whose top and bottom sides do not intersect the closure of T and whose intersection with α' is I. Every one of the vertical segments spanning R has one endpoint in each of the two components of the complement of T, and hence must meet γ'. This situation is depicted in Figure 17.

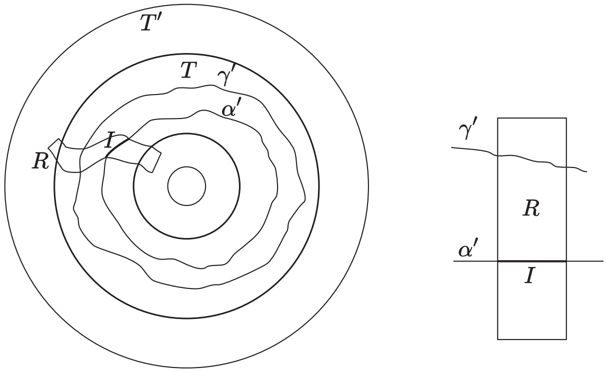

Figure 17

Elementary euclidean geometry then says that the portion of γ' intercepted by R is longer than I, and strictly so unless it is a connected piece of trajectory. Since all this can be done everywhere in $\alpha' \setminus B$, and B is finite, the conclusion is that

$$l_\omega(\gamma) = l_{\omega'}(\gamma') \geq l_{\omega'}(\alpha') = l_\omega(\alpha),$$

and that to have equality in the above γ' must be the union of segments of trajectories. The first statement of the lemma is proved. Now assume that $l_\omega(\gamma) = l_\omega(\alpha)$. As we just observed, γ' must necessarily be a union of pieces of trajectories. A first consequence is that it cannot intersect α', since the latter does not contain zeros of ω'. Since α' divides T in two annular subregions, γ' is contained in one of them, which we denote by T''. One of the boundary components of T'' is α; we denote by η the other one. What we must show is that, for any $a \in \alpha'$, the vertical ray in T'' issuing from a meets γ'. We have seen that this is the case if $a \notin B$. Assume then that $a \in B$. Let I be a small interval centered at a and containing no other point of B. Draw the vertical segments β_1 and β_2 connecting the endpoints of I to points q_1 and q_2 of η. The portion of T'' bounded by I, β_1, β_2, and one of the two arcs into which q_1 and q_2 divide η is topologically a disk. We denote by γ'' the portion of γ' lying inside this region. Notice that the ω'-length of γ'' equals the one of I. Moreover, γ'' is made up of at most two pieces of trajectories, meeting at a point y. The region D bounded by I, γ'' and the two pieces of β_1 and β_2 connecting them is also a disk, and is bounded by geodesic arcs. Moreover, I and γ'' meet β_1 and β_2 at right angles.

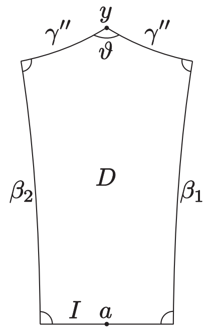

Figure 18

We denote by ϑ and μ the interior angle and the order of vanishing of ω' at y, and by v the cumulated order of zero of ω' in the interior of D. We then apply (13) again to D, and get $2\pi(v+2) = 4\big(2\pi - 2\frac{\pi}{2}\big) + (2\pi - (\mu+2)\vartheta)$, that is

$$2\pi v = 2\pi - (\mu+2)\vartheta.$$

Since $\vartheta > 0$, this says that ω' has no zeros in the interior of D. Thus the vertical geodesic ray based at a and directed towards the interior of D reaches y. In conclusion, two cases can occur. Either $\mu = 0$ and γ' is an unbroken trajectory at y, or y is one of the zeros of ω' lying on the boundary of the maximal annular domain Ω' determined by α' and, near y, γ' consists of two pieces of the boundary of Ω'. The lemma follows by observing that π induces an isomorphism between Ω and Ω'. □

Given an admissible system of curves $(\gamma_1, \dots, \gamma_k)$ on a Riemann surface of finite type, we will denote by $\mathscr{A}(\gamma_1, \dots, \gamma_k)$ the set of all systems of annular regions $(\Omega_1, \dots, \Omega_k)$ of type $(\gamma_1, \dots, \gamma_k)$. Given a system of annular regions $(\Omega_1, \dots, \Omega_k)$ of type $(\gamma_1, \dots, \gamma_k)$ and positive numbers $a_1, \dots, a_k$, an important invariant is the quantity

$$\sum_{i=1}^{k} a_i^2 M(\Omega_i). \tag{20}$$

We recall its significance. The quantity $a^2 M(T_r)$ is the ω-area of the standard annulus T_r, where $\omega = (a/2\pi\mathrm{i})^2 (d\log z)^2$.

LEMMA 27. *Let* $(\gamma_1, \dots, \gamma_k)$ *be an admissible system of curves. Let* $a_1, \dots, a_k$ *be positive numbers. Set*

$$N = \sup_{(\Omega_1,\dots,\Omega_k)\in\mathscr{A}(\gamma_1,\dots,\gamma_k)} \sum_{i=1}^{k} a_i^2 M(\Omega_i).$$

Then there exists $(\Omega_1, \dots, \Omega_k) \in \mathscr{A}(\gamma_1, \dots, \gamma_k)$ *with*

$$N = \sum_{i=1}^{k} a_i^2 M(\Omega_i).$$

PROOF. Choose a sequence $\{(\Omega_1^n, \dots, \Omega_k^n)\}$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{k} a_i^2 M(\Omega_i^n) = N.$$

Recall that $M(\Omega_i^n) \le M(\gamma_i) < +\infty$, by Lemma 8. Possibly passing to a subsequence, we may then assume that $M_i = \lim_{n\to\infty} M(\Omega_i^n)$ exists for each i. If $M_i = 0$, we set $\Omega_i = \emptyset$. Again passing to a subsequence, Lemma 10 tells us that $\{\Omega_i^n\}$ converges to an annular region Ω_i of type γ_i whenever $M_i > 0$, and that $M(\Omega_i) = M_i$. The annular regions Ω_i are disjoint because the regions $\Omega_1^n, \dots, \Omega_k^n$ are disjoint for each n. □

We are now in a position to prove the results which are at the basis of the uniqueness statement in Strebel's theorem.

LEMMA 28. *Let* ω *be an admissible, holomorphic Jenkins-Strebel differential on a Riemann surface* $S = \overline{S}\setminus\{p_1, \dots, p_m\}$ *of finite type. Assume* $\chi(S) < 0$. *Let* Γ *be the critical graph of* ω *and let*

$$\overline{S}\setminus\Gamma = \Omega_1 \cup \cdots \cup \Omega_k$$

be the decomposition of its complement into annular regions. Let a_i *be the* ω*-length of a horizontal trajectory in* Ω_i, *for* $i = 1, \dots, k$. *Let* $(\Xi_1, \dots, \Xi_k)$ *be a system of annular regions with the same homotopy type as* $(\Omega_1, \dots, \Omega_k)$. *Then*

$$\sum_{i=1}^{k} a_i^2 M(\Xi_i) \le \sum_{i=1}^{k} a_i^2 M(\Omega_i). \tag{21}$$

Moreover, equality holds in (21) *if and only if* $\Xi_i = \Omega_i$ *for all* i.

PROOF. For each i, we choose an isomorphisms z_i between Ξ_i and a standard annulus $T_{r_i} = \{z \in \mathbb{C} : r_i < |z| < 1\}$. We also set $\Xi = \bigcup_{i=1}^{n} \Xi_i$, and let z be the complex-valued function on Ξ which restricts to z_i on Ξ_i for each i. On Ξ, we may then write $\omega = \varphi\, dz^2$. For any fixed $i = 1, \dots, k$, let r be such that $r_i < r < 1$, and let C_r^i be the circle $|z_i| = r$ in Ξ_i. From Lemma 26 it follows that

$$a_i \le l_\omega(C_r^i) = \int_{C_r^i} \sqrt{|\varphi|}|dz|. \tag{22}$$

Let a be the function on Ξ whose restriction to Ξ_i is a_i. Using the previous inequality, and writing x and y for the real and imaginary parts of z, we get

$$\sum_{i=1}^k a_i^2 M(\Xi_i) = \sum_{i=1}^k \frac{1}{2\pi} \int_{r_i}^1 a_i^2 \frac{dr}{r} \le \int_\Xi \frac{a}{2\pi|z|} \sqrt{|\varphi|}\, dx \wedge dy.$$

Using the Schwarz inequality we get

$$\begin{aligned} \Big(\sum_{i=1}^k a_i^2 M(\Xi_i)\Big)^2 &\le \Big(\int_\Xi \Big(\frac{a}{2\pi|z|}\Big)^2 dx \wedge dy\Big) \cdot \Big(\int_\Xi |\varphi|\, dx \wedge dy\Big) \\ &\le \Big(\sum_{i=1}^k a_i^2 M(\Xi_i)\Big) \cdot A_\omega(S) \\ &= \Big(\sum_{i=1}^k a_i^2 M(\Xi_i)\Big) \cdot \Big(\sum_{i=1}^k a_i^2 M(\Omega_i)\Big), \end{aligned} \tag{23}$$

proving the first part of the lemma. If equality holds in (21), then it must hold in (23), and in (22) for almost all r. But then, by Lemma 26, C_r^i is a trajectory in Ω_i. This implies that $\Xi_i \subset \Omega_i$, so that $M(\Xi_i) \le M(\Omega_i)$. The equality in (21) now implies that $\Xi_i = \Omega_i$ for all i. □

We now want to study how the basic invariant (20) changes under an admissible diffeomorphism. Recall that, given a diffeomorphism $F : S \to S$, the Beltrami differential μ_F is the vector-valued $(0,1)$-form locally defined by

$$\mu_F = \frac{F_{\bar z}}{F_z} \frac{\partial}{\partial z} \otimes d\bar z$$

and that F is said to be admissible if $\|\mu_F\| < 1$. If μ is a Beltrami differential with local expression $v \frac{\partial}{\partial z} \otimes d\bar z$ and ω is a quadratic differential with local expression $h\, dz^2$, contraction gives a tensor $\omega\mu$ with local expression $hv\, dz\, d\bar z$. We also let $|\omega|$ be the ω-metric, that is, the tensor with local expression $|h|\, dz\, d\bar z$, and we recall that the area form dA_ω associated to the ω-metric can be written locally as $|h|\, dx \wedge dy$.

Lemma 29. *Let S be a Riemann surface. Let $F : S \to S$ be an admissible diffeomorphism and let $\mu = \mu_F$ be the associated Beltrami differential. Let $(\gamma_1, \dots, \gamma_k)$ be an admissible system of curves on S. Let $(\Omega_1, \dots, \Omega_k) \in \mathscr{A}(\gamma_1, \dots, \gamma_k)$. Let ω be the holomorphic differential defined in the region $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ and equal to $(a_i/2\pi i)^2 (d \log z_i)^2$ in Ω_i. Then*

$$\sum_{i=1}^k a_i^2 M(\Omega_i) \le \Big(\int_{\bigcup \Omega_i} \frac{|(1 + \omega\mu/|\omega|)|^2}{1 - |\mu|^2} dA_\omega \Big) \cdot \Big(\sum_{i=1}^k a_i^2 M_i(F(\Omega_i)) \Big). \tag{24}$$

Before proving the lemma it is useful to make a few remarks. First of all, we may notice that the absolute value of μ is a C^∞ function, and so is $\omega\mu/|\omega|$. The second remark is that, when F is biholomorphic, then $\mu = 0$ and (24) is an identity. Finally, as an example, suppose that $i = 1$, that $\Omega = \Omega_1$ is the image of an injective holomorphic map $f : R \to S$, where R is the rectangle $R = \{z = x + iy : 0 \le x < a, 0 < y < 1\}$, and that $f^*(\omega) = dz^2$. Also suppose that the diffeomorphism $F : \Omega \to F(\Omega)$ lifts to the affine map $G : R \to G(R) \subset \mathbb{C}$ given by $G(z) = z + k\bar{z}$ with $k < 1$. Then (24) is nothing but the intuitively clear relation

$$a = A_\omega(\Omega) = a^2 M(\Omega) \le \frac{1+k}{1-k} a = A_\omega(F(\Omega)).$$

Proof of Lemma 29. We set $\Xi_i = F(\Omega_i)$, $i = 1, \dots, n$. For each i we fix, once and for all, isomorphisms $z_i : \Omega_i \to T_{r_i}$ and $w_i : \Xi_i \to T_{s_i}$ where, as usual, T_r denotes the standard annulus $\{z \in \mathbb{C} : r < |z| < 1\}$. We also set $\Omega = \bigcup_{i=1}^n \Omega_i$ and let z be the complex-valued function on Ω which restricts to z_i on Ω_i for each i. We similarly define Ξ and w. By abuse of notation we indicate by F_z and $F_{\bar{z}}$ the corresponding derivatives of $w \circ F$. The restriction of the differential ω to Ω_i is $(a_i/2\pi \mathrm{i})^2 (d \log z_i)^2$. In more compact notation, we can say that the restriction of ω to Ω is $(a/2\pi \mathrm{i})^2 (d \log z)^2$, where a stands for the function on Ω which restricts to the constant a_i on Ω_i for each i. We put on Ξ the hermitian metric

$$\sigma = \alpha^2 \, dw \, d\overline{w} = \frac{1}{(2\pi|w|)^2} dw \, d\overline{w}.$$

Notice that, in this metric, the length of any simple, closed, homotopically nontrivial curve in Ξ_i is at least 1. Thus, if we denote by C_r^i the circle $|z_i| = r$ in Ω_i, we have that

$$1 \le l_\sigma(F(C_r^i)) = \int_{C_r^i} (\alpha \circ F) |F_z \, dz + F_{\bar{z}} \, d\bar{z}|.$$

On the other hand,

$$\frac{\omega\mu}{|\omega|} = -\frac{\bar{z}}{z} \frac{F_{\bar{z}}}{F_z}.$$

Hence

$$2\pi M(\Omega_i) = \int_{r_i}^{1} \frac{1}{r}\,dr \leq \int_{r_i}^{1}\int_{C_r^i} (\alpha\circ F)|F_z\,dz + F_{\bar z}\,d\bar z|\frac{1}{r}\,dr$$

$$= \int_{\Omega_i} (\alpha\circ F)\frac{|F_z|}{|z|}\left|1+\frac{\omega\mu}{|\omega|}\right| dx\wedge dy.$$

Recalling that the Jacobian determinant of F is $J = |F_z|^2 - |F_{\bar z}|^2$, we get

$$\sum_{i=1}^{k} a_i^2 M_i(\Omega_i) \leq \int_{\Omega} \frac{a^2}{2\pi|z|}(\alpha\circ F)\sqrt{J}\left|1+\frac{\omega\mu}{|\omega|}\right|\cdot\frac{|F_z|}{\sqrt{J}}\cdot dx\wedge dy.$$

Using Schwarz's inequality we then get

$$\Big(\sum_{i=1}^{k} a_i^2 M_i(\Omega_i)\Big)^2 \leq \Big(\int_{\Omega} a^2(\alpha\circ F)^2 J\,dx\wedge dy\Big)$$

$$\cdot\Big(\int_{\Omega}\Big(\frac{a}{2\pi|z|}\Big)^2\frac{|(1+\omega\mu/|\omega|)|^2}{1-|\mu|^2}\,dx\wedge dy\Big)$$

$$= \Big(\sum_{i=1}^{k} a_i^2 M_i(\Xi_i)\Big)\cdot\Big(\int_{\bigcup\Omega_i}\frac{|(1+\omega\mu/|\omega|)|^2}{1-|\mu|^2}\,dA_\omega\Big). \qquad \square$$

In the proof of Strebel's theorem, we will use a corollary of the preceding lemma. If in addition to the assumptions of the lemma we also assume that F is homotopically trivial and that $(\Omega_1,\dots,\Omega_k)$ and $(a_1,\dots,a_k)$ are as in the statement of Lemma 27, we immediately get the following result.

COROLLARY 30. *Under the assumptions of the preceding lemma and assuming that*

(1) *F is isotopic to the identity,*
(2) *$(\Omega_1,\dots,\Omega_k)$ and $(a_1,\dots,a_k)$ are as in the statement of Lemma* 27,

then:

$$\sum_{i=1}^{k} a_i^2 M_i(\Omega_i) \leq \int_{\bigcup\Omega_i}\frac{|(1+\omega\mu/|\omega|)|^2}{1-|\mu|^2}\,dA_\omega.$$

PROOF OF THEOREM 25. Using Lemma 27, choose a system of annular regions $(\Omega_1,\dots,\Omega_k)$ of type $(\gamma_1,\dots,\gamma_k)$ maximizing the quantity $\sum_{i=1}^{k} a_i^2 M_i(\Omega_i)$. If $\Omega_i \neq \emptyset$, choose an isomorphism f_i from Ω_i to a standard annulus T_{r_i}. Define a (discontinuous) quadratic differential ω' on S by setting it equal to

$$f_i^*\Big(\Big(\frac{a_i}{2\pi \mathrm{i}}\Big)^2(d\log z)^2\Big)$$

on Ω_i and to zero on $S\setminus\bigcup_{i=1}^k \Omega_i$. If we could show that ω' coincides almost everywhere with a holomorphic quadratic differential ω, we would conclude that $\omega = \omega'$ in $\bigcup_{i=1}^k \Omega_i$ and that $S\setminus\bigcup_{i=1}^k \Omega_i$ has measure zero. From Proposition 17 it would then follow that the trajectories of ω are either closed or critical, and the theorem would be proved. The almost everywhere holomorphicity of ω' is a local question. Let U be an arbitrary coordinate neighborhood in S and write $\omega' = f\,dz^2$ in U, with $f \in L^1(U)$. We shall prove that $f_{\bar z} = 0$ in the sense of distributions. Weyl's lemma, which asserts that a harmonic distribution is necessarily C^∞, then shows that f coincides almost everywhere with a holomorphic function. We must prove that

$$\int_U h_{\bar z} f\,dx\wedge dy = 0 \tag{25}$$

for every C^∞ function h with compact support in U. For any such function, we define an admissible homotopically trivial diffeomorphism $F : S \to S$ in the following way. Fix a small real number ε, set $F(z) = z + \varepsilon h(z)$ for $z \in U$, and extend F to all of S by setting it equal to the identity on $S\setminus U$. For sufficiently small ε, F is an admissible diffeomorphism. The Beltrami differential associated to F has compact support contained in U, and is given by

$$\mu = \frac{\varepsilon h_{\bar z}}{1+\varepsilon h_z}\frac{\partial}{\partial z}\otimes d\bar z.$$

A straightforward computation yields

$$\frac{|1+\omega\mu/|\omega|\,|^2}{1-|\mu|^2}dA_\omega = (|f| + 2\,\mathrm{Re}(\varepsilon h_{\bar z} f) + O(\varepsilon^2))\,dx\wedge dy,$$

where, as usual, dA_ω stands for the area form associated to the ω-metric. From Corollary 30 it follows that

$$\begin{aligned}\sum_{i=1}^k a_i^2 M(\Omega_i) &\le \int_U dA_\omega + \int_U (\mathrm{Re}(\varepsilon h_{\bar z} f) + O(\varepsilon^2))\,dx\wedge dy\\ &\le \sum_{i=1}^k a_i^2 M(\Omega_i) + \int_U \mathrm{Re}(\varepsilon h_{\bar z} f)\,dx\wedge dy + O(\varepsilon^2).\end{aligned}$$

We conclude that

$$\mathrm{Re}\Big(\int_U \varepsilon h_{\bar z} f\,dx\wedge dy\Big) \ge 0,$$

for every ε and every h with compact support in U, proving (25) and the existence part of the theorem.

The uniqueness of ω is a direct consequence of Lemma 28. Suppose in fact that $\tilde\omega$ is another holomorphic Jenkins-Strebel differential satisfying i) and ii), let $\tilde\Gamma$ be its critical graph, and $S\setminus\tilde\Gamma = \tilde\Omega_1 \cup\cdots\cup \tilde\Omega_k$ the decomposition of its complement in annular regions. Lemma 28 implies that $\tilde{\Omega}_i = \Omega_i$ for every i. Thus ω and $\tilde{\omega}$ have the same critical graph, and hence in particular the same zeroes. It follows that ω and $\tilde{\omega}$ are proportional and, since they have the same critical trajectories, that the constant of proportionality is a positive real number. But then ω and $\tilde{\omega}$ must be equal, since they have the same trajectories, and the ω-length of any one of these is equal to the $\tilde{\omega}$-length, by assumption. □

5. Jenkins-Strebel differentials with double poles

In this section we are going to prove the theorem we announced in the introduction. We must first introduce the notion of *admissible meromorphic quadratic differential* on a Riemann surface of finite type $S = \overline{S} \setminus \{y_1, \dots, y_m\}$. Let us fix a meromorphic quadratic differential ω on S. The first requirement for ω to be admissible is that it should extend to a meromorphic quadratic differential on $\overline{S}$ having at worst simple poles at the points p_i. The second is that the only poles of ω on S should be second order poles with negative quadratic residues. To explain this terminology, let p be a pole of order two for ω and let z be a local coordinate centered at p. A local expression for ω is $(h\,dz/z)^2$, where h is holomorphic and $c = h(0)$ is not zero. Write $h = c(1 + zh_1)$, and let k be a primitive of h_1. We then have

$$h\frac{dz}{z} = c(d\log z + h_1\,dz) = c\,d(\log z + k) = c\,d\log(ze^k).$$

Thus, in terms of the local coordinate $\zeta = ze^k$, a local expression for ω is

$$c^2(d\log\zeta)^2.$$

We shall say that ζ a *distinguished parameter* at p; it is unique up to multiplication by non-zero constants. Notice that the constant c^2 is intrinsically attached to ω and does not depend on the choice of coordinate. We will call it the *quadratic residue* of ω at p; in fact, c is nothing but the residue of $\sqrt{\omega}$ at this point.

Definition 31. Let $S = \overline{S} \setminus \{y_1, \dots, y_m\}$ be a Riemann surface of finite type. A meromorphic quadratic differential ω on S is said to be admissible if it satisfies the following conditions:

(1) ω extends to a meromorphic quadratic differentials to $\overline{S}$ having at worst simple poles at the points y_i;
(2) the only poles of ω in S are double poles with negative residue.

The geometrical implication of the negativity of the residue is the following. The local expression of an admissible meromorphic quadratic differential ω near any one of its double poles must be of the form (17), where ζ is a suitable local coordinate and a is a positive real number. The geodesics of ω near a pole p are particularly easy to describe. Consider the neighborhood U of p given by $\{|\zeta| < \varepsilon\}$, and set $\dot{U} = U \setminus \{p\}$. The universal covering of $\dot{U}$ can be identified with the half-plane $H = \{w \in \mathbb{C} : \mathrm{Im}(w) > -(a/2\pi)\log\varepsilon\}$, and the universal covering map η with

$$w \mapsto e^{(2\pi \mathrm{i}/a)w} = \zeta.$$

One checks immediately that $\eta^*(\omega) = dw^2$. Thus η is a local isometry between H, endowed with the euclidean metric, and $\dot{U}$, endowed with the ω-metric. In fact, $\dot{U}$ is biholomorphic and isometric to the half-infinite cylinder which one obtains by taking the quotient of H modulo translations by multiples of a. This has several interesting consequences. The first is that p lies at infinite distance from all other points of S; in particular, no geodesic reaches it. The ω-geodesics in U are the images of pieces of straight line in H. The horizontal geodesics are the images of the horizontal lines, and hence are just the circles $|\zeta| = \text{constant}$. A consequence is that any trajectory of ω is contained in a compact subset of $S \setminus \{\text{poles of } \omega\}$. The vertical geodesics in U are the radii $\arg(\zeta) = \text{constant}$, while the remaining geodesics are the logarithmic spirals in U wrapping around p.

Definition 32. Let S be a Riemann surface of finite type. An admissible meromorphic quadratic differential on S is said to be a meromorphic Jenkins-Strebel differential if its non-critical trajectories are closed.

We are now in a position to state and prove the result we announced in the introduction.

Theorem 33. *Let $S = \overline{S} \setminus \{y_1, \dots, y_m\}$ be a Riemann surface of finite type and of genus g. Let $x_1, \dots, x_n \in S$. Assume that $\chi(S \setminus \{x_1, \dots, x_n\}) < 0$. Let $a_1, \dots a_n$ be positive real numbers. Then there exists a unique meromorphic Jenkins-Strebel quadratic differential ω on S having the following properties.*

i) *ω is holomorphic on $S \setminus \{x_1, \dots, x_n\}$ and has poles of order two at the points x_i, with residues $(a_i/2\pi \mathrm{i})^2$, $i = 1, \dots, n$.*
ii) *If Γ is the critical graph of ω, then $\overline{S} \setminus \Gamma$ is the union of n disjoint disks $\Delta_1, \dots, \Delta_n$, with $x_i \in \Delta_i$, $i = 1, \dots, n$.*

Given a meromorphic Jenkins-Strebel differential ω, the vertices of its critical graph Γ are its critical points. These are the zeros and the simple poles of ω in $\overline{S}$. Moreover, a vertex p of Γ has valency $v(\geq 1)$ if and only if $v = \mathrm{ord}_p\,\omega - 2$. Finally $\mathrm{ord}_{p_i}\,\omega \geq 1$, for $i = 1, \dots, m$ and the only vertices of valency ≤ 2 are among the points $y_1, \dots, y_m$.

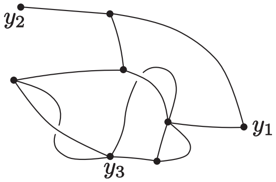

Figure 19

Before proving this theorem we need to recall the construction of the double of a Riemann surface with boundary. A *Riemann surface with boundary* is a connected, 2-dimensional manifold R with boundary ∂R, together with a structure of Riemann surface on the open set $R \setminus \partial R$ and a family $\mathscr{F}$ of homeomorphisms $\varphi : U \to V$, where U is an open subset of R whose intersection with ∂R is non-empty and V is an open subset of the closed upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$, with the following properties.

i) $\varphi(U \cap \partial R) = \varphi(U) \cap \mathbb{R}$ for any $\varphi : U \to V$ in $\mathscr{F}$.
ii) The domains of the homeomorphisms in $\mathscr{F}$ cover ∂R.
iii) If φ is any element of $\mathscr{F}$, then the restriction of φ to $U \setminus (U \cap \partial R)$ is holomorphic.
iv) The family $\mathscr{F}$ is maximal with respect to the above properties.

It should be remarked that property i) is actually redundant, since it is a consequence of the invariance of domain theorem. It is also helpful to notice that, if φ and φ' are two charts in $\mathscr{F}$, then the composition $\varphi' \circ \varphi^{-1}$ is the restriction of a holomorphic function on an open subset of $\mathbb{C}$, and hence is real analytic on the real axis; this will follow from the construction of the double below.

The *conjugate* $(R^*, \partial R^*)$ of $(R, \partial R)$ is the surface whose underlying topological space is the same as the one of $(R, \partial R)$, whose charts in $R^* \setminus \partial R^*$ are the complex conjugates of the charts of $R \setminus \partial R$, and whose charts at boundary points are of the form $-\bar{\varphi}$, where φ varies in $\mathscr{F}$. The identity gives an antiholomorphic map $\sigma : (R, \partial R) \to (R^*, \partial R^*)$.

The *double* of R is the surface (without boundary) $\hat{R}$ obtained from the disjoint union of R and R^* by identifying ∂R and ∂R^* via σ. There is a natural structure of Riemann surface on $\hat{R}$ which agrees with the given ones on $R \setminus \partial R$ and $R^* \setminus \partial R^*$. One can construct charts for this structure at points $p \in \partial R = \partial R^*$ as follows. Let $\varphi : U \to V$ be an element of $\mathscr{F}$ such that $p \in U$. Then, writing V^* for the image of V under complex conjugation, a chart for $\hat{R}$ at p is the homeomorphism $\hat{\varphi} : U \cup \sigma(U) \to V \cup V^*$ which agrees with φ on U and with $\bar{\varphi}$ on $\sigma(U)$. The only thing that has to be shown is that, if $\varphi' : U' \to V'$ is another chart such that $p \in U'$, then $\hat{\varphi}' \circ \hat{\varphi}^{-1}$ is holomorphic. This is obvious everywhere, except along the real axis. We may then appeal to the following well-known fact. Let f be a continuous complex-valued function defined on an open subset A of the complex plane; if f is holomorphic on $A \setminus (A \cap \mathbb{R})$, then it is holomorphic on all of A. This can be proved, for instance, by checking that Cauchy's theorem holds for f.

The map σ extends to an involutive antiholomorphic automorphism τ of $\hat{R}$ whose fixed point set is $\partial R = \partial R^*$.

Proof of Theorem 33. For each $i = 1, \dots, n$, let γ_i be a small simple loop around x_i, and let v_i be a non-zero tangent vector in $T_{x_i}(S)$. We denote by $\dot{\mathscr{A}}(\gamma_1, \dots, \gamma_n)$ the set of n-tuples $\dot{\Omega}_1, \dots, \dot{\Omega}_n$ of disjoint punctured disks of type $(\gamma_1, \dots, \gamma_n)$ in $S \setminus \{x_1, \dots, x_n\}$. Set

$$N = \sup_{(\dot{\Omega}_1,\dots,\dot{\Omega}_k)\in\dot{\mathscr{A}}(\gamma_1,\dots,\gamma_n)} \left(\sum_{i=1}^{n} a_i^2 \dot{M}_{v_i}(\dot{\Omega}_i)\right)$$

Since $2g-2+n>0$, the n-pointed surface $(S, x_1 \dots, x_n)$ is not the twice punctured sphere, and hence $N < +\infty$. It is also evident that $N \neq -\infty$. Using Lemma 12, and proceeding exactly as in the proof of Lemma 27, we find $(\dot{\Omega}_1, \dots, \dot{\Omega}_n) \in \dot{\mathscr{A}}(\gamma_1, \dots, \gamma_n)$ such that

$$\sum_{i=1}^{n} a_i^2 \dot{M}_{v_i}(\dot{\Omega}_i) = N.$$

In particular, the $\dot{\Omega}_i$ are all non-empty. We denote by Ω_i the disk $\dot{\Omega}_i \cup \{x_i\}$, and pick an isomorphism $f_i : \Omega_i \to \Delta_{r_i}$ with $f_i(x_i) = 0$ and $|v_i(f_i)| = 1$, so that the reduced modulus of $\dot{\Omega}_i$ is $\log r_i/2\pi$. As in the proof of Theorem 25, we will be done if we can show that the complement of $\bigcup_{i=1}^n \dot{\Omega}_i$ has measure zero, and that there is a holomorphic quadratic differential on $S\setminus\{x_1, \dots, x_n\}$ which agrees with

$$\omega_i = f_i^*\left(\left(\frac{a_i}{2\pi \mathrm{i}}\right)^2 (d\log z)^2\right) \tag{26}$$

on $\dot{\Omega}_i$, for $i = 1, \dots, n$. We may assume that, for each i, the cycle $\gamma_i \subset \dot{\Omega}_i$ is the preimage under f_i of a circle of radius $r < \min\{r_1, \dots, r_n\}$ centered at the origin of Δ_{r_i}, and we let $B_i \subset \Omega_i$ denote the preimage, under f_i, of the open disk of radius r in Δ_{r_i}, so that $\partial B_i = \gamma_i$. Set $B = \bigcup_{i=1}^n B_i$, $\gamma = \bigcup_{i=1}^n \gamma_i$, and denote by $\hat{R}$ the double of the Riemann surface with boundary $R = S\setminus B$. Then, with a slight abuse of notation, we can write $\hat{R} = R^1 \cup \gamma \cup R^2$, where R^1 and R^2 are the interiors of R and R^*, respectively, and $\partial R^1 = \partial R^2 = \gamma$ (see Figure 20). The antiholomorphic involution τ interchanges R^1 and R^2, and leaves γ fixed. We can view $\Omega_i\setminus B_i$ as a subset of R, and we define the annular region $\hat{\Omega}_i \subset \hat{R}$ by setting

$$\hat{\Omega}_i = (\Omega_i\setminus B_i) \cup \tau(\Omega_i\setminus B_i), \quad i = 1, \dots, n.$$

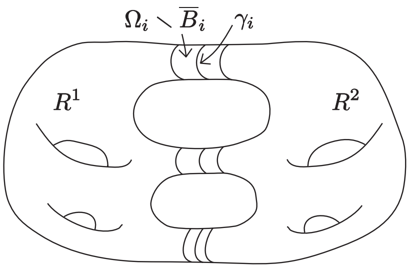

Figure 20

Clearly, $(\hat{\Omega}_1, \dots, \hat{\Omega}_n)$ is a system of annular regions in $\hat{R}$ of type $(\gamma_1, \dots, \gamma_n)$. Since $\hat{R}$ is a compact Riemann surface of genus $2g+n-1>1$, Theorem 25 applies; we denote by φ the holomorphic Jenkins-Strebel differential associated to the admissible system $(\gamma_1, \dots, \gamma_n)$ and to the constants $(a_1, \dots, a_n)$. The trajectory structure of φ defines a system of annular regions $(\Xi_1, \dots, \Xi_n)$ of type $(\gamma_1, \dots, \gamma_n)$ such that

$$\hat{R}\backslash\{\text{critical graph of } \varphi\} = \bigcup_{i=1}^{n} \Xi_i.$$

Moreover, $(\Xi_1, \dots, \Xi_n)$ maximizes the quantity $\sum a_i^2 M(\Omega_i)$ among all $(\Omega_1, \dots, \Omega_k) \in \mathscr{A}(\gamma_1, \dots, \gamma_n)$. In particular,

$$\sum_{i=1}^{n} a_i^2 M(\Xi_i) \geq \sum_{i=1}^{n} a_i^2 M(\hat{\Omega}_i). \tag{27}$$

Observe that, since $\tau(\gamma_i) = \gamma_i$, the uniqueness part of Theorem 25 implies that $\varphi = \overline{\tau^*(\varphi)}$. In particular we must have $\tau(\Xi_i) = \Xi_i$ for $i = 1, \dots, n$. We claim that γ_i is contained in Ξ_i. To prove this, we look at the annular covering $\eta : \hat{R}_{\gamma_i} \to \hat{R}$. Lemma 6 and Corollary 7 say that γ_i has a unique closed lifting $\tilde{\gamma}_i$, that there is a unique annular region $\tilde{\Xi}_i$ in $\hat{R}_{\gamma_i}$ which is mapped isomorphically to Ξ_i by η, and that $\tilde{\Xi}_i$ has the same homotopy type as $\tilde{\gamma}_i$.

Since τ leaves γ_i pointwise fixed, it lifts to an involutory antiholomorphic automorphism $\tilde{\tau} : \hat{R}_{\gamma_i} \to \hat{R}_{\gamma_i}$ carrying $\tilde{\Xi}_i$ to itself. The surface $\hat{R}_{\gamma_i}$ is an annulus, and $\tilde{\gamma}_i$ cuts it into two sub-annuli $R^1_{\gamma_i}$ and $R^2_{\gamma_i}$ which are interchanged by $\tilde{\tau}$. If a boundary component of $\tilde{\Xi}_i$ intersects $\tilde{\gamma}_i$, it is carried to itself by $\tilde{\tau}$, since the points of intersection are fixed by $\tilde{\tau}$. If instead a boundary component of $\tilde{\Xi}_i$ does not intersect $\tilde{\gamma}_i$, the two boundary components are interchanged by $\tilde{\tau}$. Hence one of them lies in $R^1_{\gamma_i}$ and the other in $R^2_{\gamma_i}$, so $\tilde{\Xi}_i$ contains $\tilde{\gamma}_i$. In conclusion, it suffices to exclude the case when each boundary components of $\tilde{\Xi}_i$ is carried to itself by $\tilde{\tau}$, and cuts $\tilde{\gamma}_i$. Suppose this case occurs, and let α be one of the boundary components. Then the portion of α lying between two successive points of intersection with $\tilde{\gamma}_i$, which we call a, and one of the two arcs of $\tilde{\gamma}_i$ bounded by these two points, which we call b, bound a region D homeomorphic to a disk (see Figure 21). Since b is left

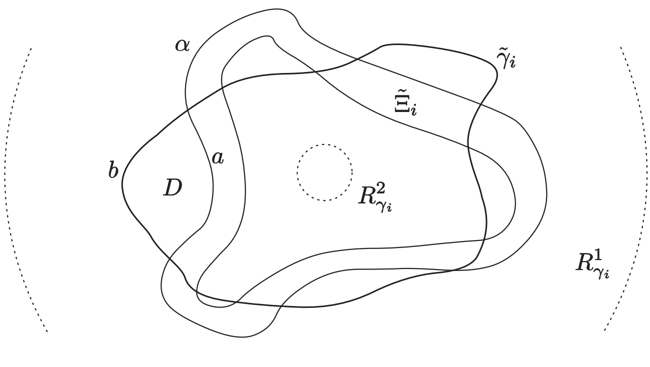

Figure 21

pointwise fixed by $\tilde{\tau}$, $D \cup \tilde{\tau}(D)$ is a disk bounded by a and $\tilde{\tau}(a)$. It follows that α is equal to the union of a and $\tilde{\tau}(a)$ and is homotopically trivial. But this contradicts the fact that α is homotopic to $\tilde{\gamma}_i$.

We have seen that γ_i is contained in Ξ_i. Furthermore, we can find an isomorphism between Ξ_i and a standard annulus under which the quadratic differential φ corresponds to $(a_i/2\pi\mathrm{i})^2(d\log z)^2$. We need the following lemma.

Lemma 34. *Let $T = T_s \subset \mathbb{C}$ be a standard annulus. Let $\gamma \subset T$ be a homotopically non-trivial, simple closed curve. Let $\tau : T \to T$ be an antiholomorphic involution which is the identity on γ, and assume that $\tau^*(d\log z)^2 = (d\log \bar{z})^2$. Then $\tau(z) = s/\bar{z}$, and γ is the circle of radius $\sqrt{s}$ centered at the origin. Thus, if T^1 and T^2 are the connected components of $T\backslash\gamma$, then $M(T^1) = M(T^2)$ and $M(T) = M(T^1) + M(T^2)$.*

Proof. The condition that τ carries $(d\log z)^2$ to its conjugate translates into

$$\frac{\partial \log \tau}{\partial \bar{z}} = \pm \frac{\partial \log \bar{z}}{\partial \bar{z}}.$$

Thus $\tau(z)$ equals either $c\bar{z}$ or $c/\bar{z}$, where c is a constant. The condition that $\tau = \tau^{-1}$ implies that $|c| = 1$ in the first case, and that c is real in the second. In the first case, however, the set of fixed points is contained in the line $2\arg(z) = \arg(c)$, and hence cannot contain γ. On the other hand, in the second case, c must equal $\pm s$ if τ is to carry T to itself. Moreover, if c were equal to $-s$, τ would have no fixed points. The conclusion is that $\tau(z) = s/\bar{z}$ and that γ is the circle of radius $\sqrt{s}$ centered at the origin. The remaining assertions of the lemma follow at once. $\square$

Set $\Xi_i^1 = \Xi_i \cap R^1$. Lemma 34 and formula (27) imply that

$$\sum_{i=1}^{n} a_i^2 M(\Xi_i^1) = \frac{1}{2}\sum_{i=1}^{n} a_i^2 M(\Xi_i) \geq \frac{1}{2}\sum_{i=1}^{n} a_i^2 M(\hat{\Omega}_i) = \sum_{i=1}^{n} a_i^2 M(\Omega_i\backslash B_i). \tag{28}$$

Now let $\dot{\Xi}_i$ be the punctured disk $\Xi_i^1 \cup (\bar{B}_i\backslash\{x_i\})$ in S. From Lemma 5 we get

$$\dot{M}_{v_i}(\dot{\Xi}_i) \geq M(\Xi_i^1) + \frac{\log r}{2\pi},$$

while

$$\dot{M}_{v_i}(\dot{\Omega}_i) = M(\Omega_i\backslash B_i) + \frac{\log r}{2\pi},$$

hence (28) gives

$$\sum_{i=1}^{n} a_i^2 \dot{M}_{v_i}(\dot{\Xi}_i) \geq \sum_{i=1}^{n} a_i^2 \dot{M}_{v_i}(\dot{\Omega}_i).$$

By the very choice of $(\dot{\Omega}_1, \dots, \dot{\Omega}_n)$, this must be an equality, and therefore (28) and (27) must also be equalities. If follows that $\Xi_i = \hat{\Omega}_i$ for all i, by Lemma 28. But then the complement of $\bigcup_{i=1}^n \dot{\Omega}_i$ in $S \backslash \{x_1, \dots, x_n\}$ is equal to the complement of $\bigcup_{i=1}^n \Xi_i \cap R^1$ in $R^1 = S \backslash \bigcup_{i=1}^n \bar{B}_i$, and hence has measure zero. The restrictions to $\Xi_i^1 = \Omega_i \backslash B_i$ of φ and of the differential ω_i defined by (26) are both of the form $(a_i/2\pi\mathrm{i})^2 (d \log z)^2$ and hence, in the light of Remark 21, are equal. The differential ω whose existence is asserted by the theorem can thus be constructed by setting it equal to φ on $R^1 = S \backslash B$ and to ω_i on Ω_i, for $i = 1, \dots, n$.

The existence part of the theorem is now proved. As for uniqueness, recall that in the proof of Theorem 25 this followed directly from Lemma 28. In the case at hand, the same argument applies, provided we replace Lemma 28 with the following analogue.

LEMMA 35. *Let $(S, x_1, \dots, x_n)$ be an n-pointed Riemann surface of finite type with $\chi(S \backslash \{x_1, \dots, x_n\}) < 0$. Let $v_i \in T_{x_i}(S)$, $i = 1, \dots, n$, be non-zero tangent vectors. Let ω be a meromorphic Jenkins-Strebel differential on S, holomorphic on $S \backslash \{x_1, \dots, x_n\}$ and with double poles at the x_i. Let Γ be the critical graph of ω, and assume that its complement is the disjoint union of n disks $\Omega_1, \dots, \Omega_n$, with $x_i \in \Omega_i$ for every i. Set $\dot{\Omega}_i = \Omega_i \backslash \{x_i\}$, and let a_i be the ω-length of a horizontal trajectory in Ω_i, for $i = 1, \dots, n$. Let $\Xi_1, \dots, \Xi_n$ be disjoint disks in S such that $x_i \in \Xi_i$ for every i, and set $\dot{\Xi}_i = \Xi_i \backslash \{x_i\}$. Then*

$$(29) \qquad \sum_{i=1}^n a_i^2 \dot{M}_{v_i}(\dot{\Xi}_i) \leq \sum_{i=1}^n a_i^2 \dot{M}_{v_i}(\dot{\Omega}_i).$$

Moreover, equality holds in (29) *if and only if $\Xi_i = \Omega_i$ for all i.*

We leave to the reader the simple proof of this lemma, which can be deduced from Lemma 28, again considering the double of a suitable Riemann surface. □

One way to rephrase Theorem 33 is the following.

THEOREM 36. *Let S be a compact Riemann surface. Let P be a finite set and $x : P \to S$ an injective map. Assume that $\chi(S \backslash x(P)) < 0$. Let $h : P \to \mathbb{R}_{\geq 0}$ be a non-zero function. Write $h^{-1}(0) = \{p_1, \dots, p_m\}$ and $P \backslash h^{-1}(0) = \{q_1, \dots, q_n\}$. Set $Y = \{x(p_1), \dots, x(p_m)\}$ and $X_\infty = \{x(q_1), \dots, x(q_n)\}$. Then there exists a unique meromorphic Jenkins-Strebel differential ω on $S \backslash Y$ which is holomorphic in $S \backslash x(P)$ and has a double pole at $x(q_i)$ with quadratic residue equal to $-\left(\frac{h(q_i)}{\pi}\right)^2$, for each $i = 1, \dots, n$. In particular, ω is meromorphic in S and has at worst simple poles at the points of Y. Finally, let Γ be the critical graph of ω. The vertices of Γ of valency $v \geq 3$ are the zeros of ω of order $v - 2$. The bivalent and univalent vertices of Γ are among the points of Y. A univalent vertex corresponds to a simple pole of ω. A bivalent vertex is a point of Y which is regular for ω and where ω does not vanish. The points of X_∞ are in one-to-one correspondence with the boundary components of Γ.*

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