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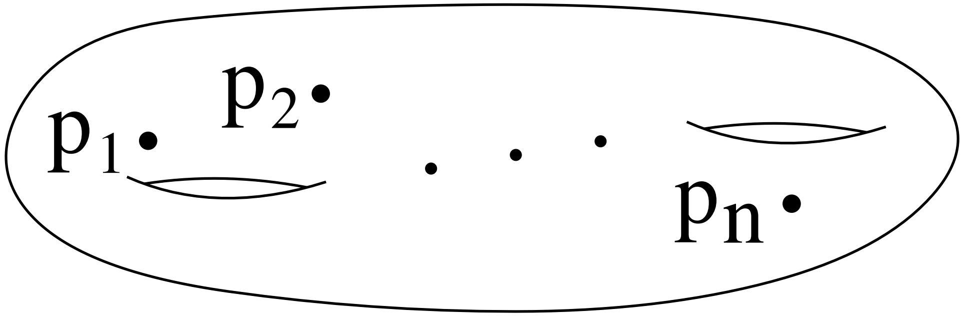
# Cohomology of moduli spaces of stable $n$-pointed curves

**Especially** in low genus or low degree

**Emphasis:** elementary, algebro-geometric methods

Everything $/\mathbb{C}$

A smooth $n$-pointed curve of genus $g$

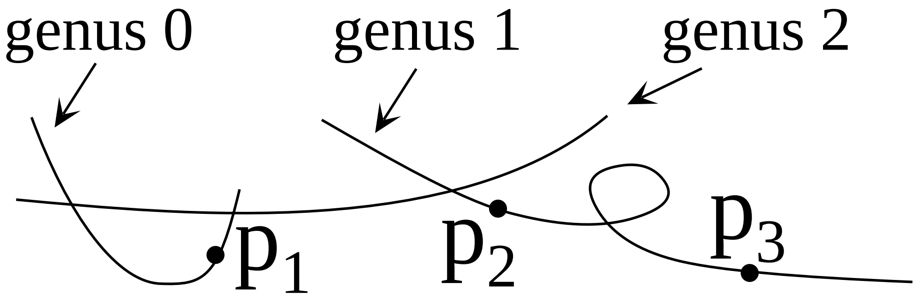


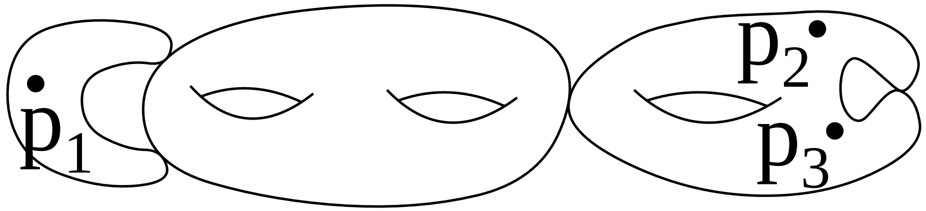

pts. $p_i$ distinct, numbered.

$\mathcal{M}_{g,n}$ parametrizes isomorphism classes of such objects (for $2g-2+n>0$).

$\mathcal{M}_{g,n}$ is: connected, quasi-projective orbifold of $\dim_{\mathbb{C}} 3g-3+n$.

# A stable 3-pointed curve $C$ of genus 5 (two pictures)

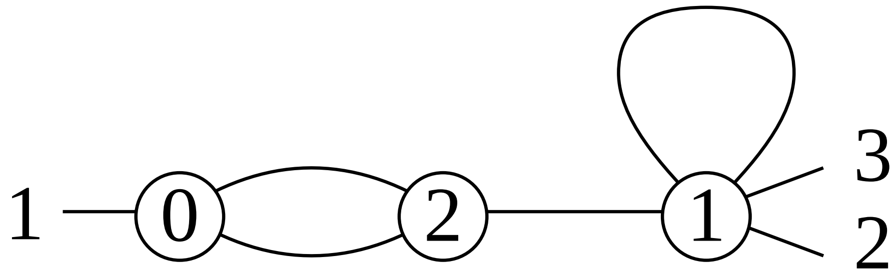



Stable, $n$-pointed means:

- reduced, connected, nodal
- $n$ numbered, distinct, smooth points
- stability: $2g_v - 2 + l_v > 0, \forall v$

and its graph:




**vertices:** components of normalization of $C$
**edges:** nodes of $C$
**legs:** marked points

$\forall$ vertex $v$:
- $g_v$ = genus of component
- $l_v$ = # of half-edges (legs included) stemming from $v$

genus of $C = \sum g_v + \#\,\text{edges} - \#\,\text{vertices} + 1$

$\overline{\mathcal{M}}_{g,n}$ = isomorphism classes of stable, genus $g$, $n$-pointed curves

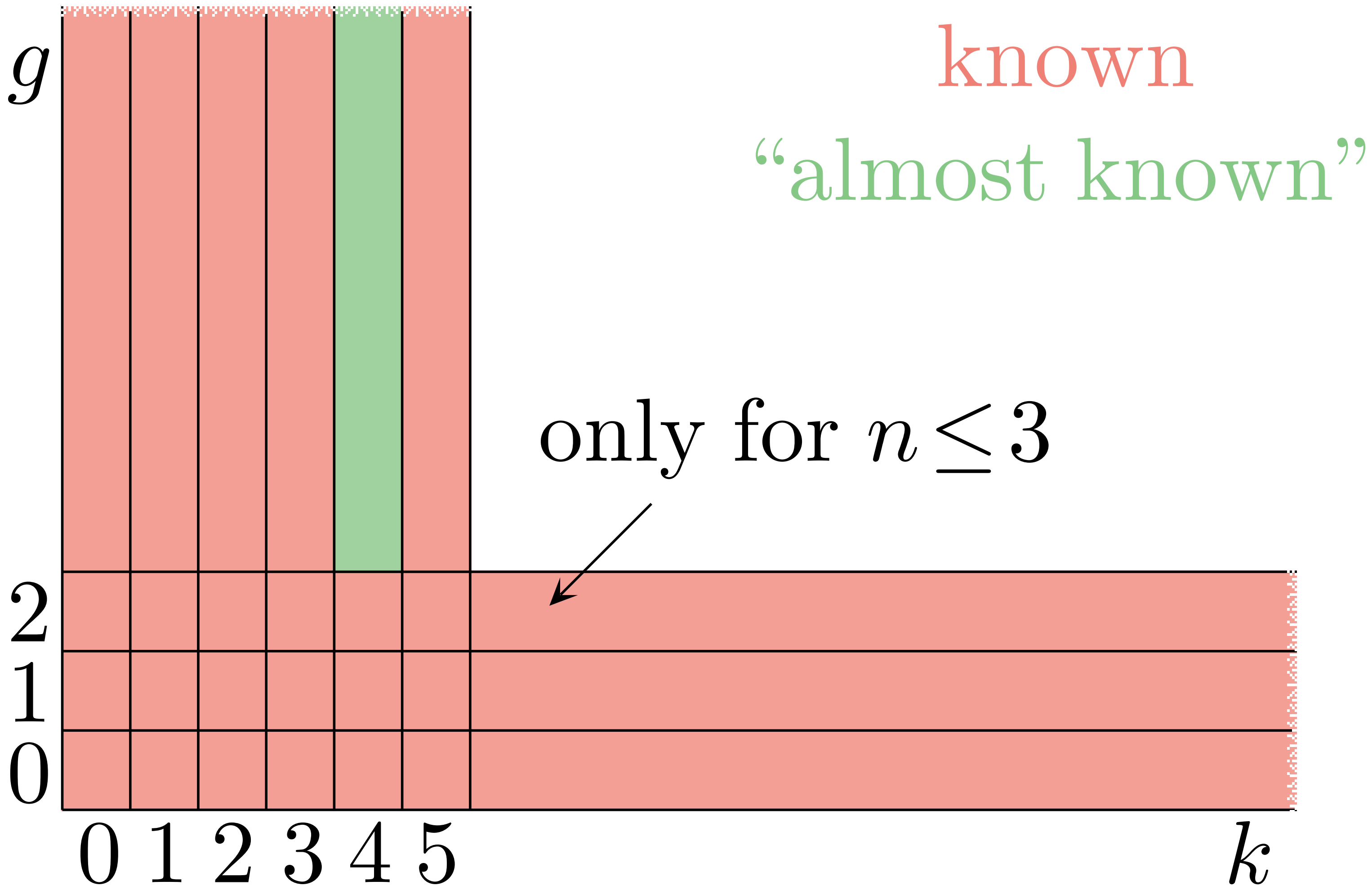
$\overline{\mathcal{M}}_{g,n}$ is a connected, projective orbifold, $\mathcal{M}_{g,n}$ dense Zariski open. $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$

**Goal:** compute rational cohomology of $\overline{\mathcal{M}}_{g,n}$

$\overline{\mathcal{M}}_{g,n}$ complete orbifold implies:

- Poincaré duality holds
- $H^k(\overline{\mathcal{M}}_{g,n})$ has pure Hodge structure of weight $k$

# What we know on $H^k(\overline{\mathcal{M}}_{g,n})$



# Building blocks of $\overline{\mathcal{M}}_{g,n}$

$\Gamma$ connected stable graph of genus $g$ with $n$ legs

$$\xi_\Gamma : X_\Gamma = \prod \overline{\mathcal{M}}_{g_v,l_v} \to \overline{\mathcal{M}}_{g,n}$$

(identify points corresponding to halves of same edge)

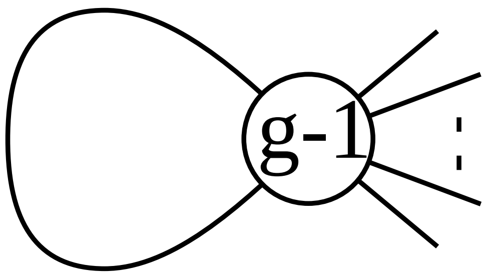
- $\xi_\Gamma$ is finite
- $\dim(LHS) = 3g - 3 + n - \#\,\text{edges}$
- restriction to $\prod \mathcal{M}_{g_v,l_v}$ of $\xi_\Gamma$ is quotient by $\mathrm{Aut}(\Gamma)$; image $\mathcal{M}(\Gamma)$ is an orbifold

Topological stratification: $\overline{\mathcal{M}}_{g,n} = \coprod \mathcal{M}(\Gamma)$

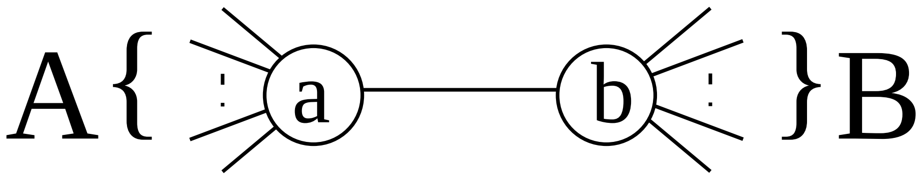
Components of $\partial\mathcal{M}_{g,n}$ = closed strata $\overline{\mathcal{M}(\Gamma)}$ s.t. $\Gamma$ has 1 edge

Types of such $\Gamma$:

- $\Gamma_{irr}$




- $\Gamma_{a,A} = \Gamma_{b,B}$, $\quad (a+b=g, \quad A \coprod B = \{1,\ldots,n\})$






M. Cornalba, slides of ICM98 talk

## Use of building blocks – Strategy A

Spectral sequence of cohomology with compact support:

$$E_2^{p,q} = \bigoplus_{\Gamma \text{ has } q \text{ edges}} H_c^{p+q}(\mathcal{M}(\Gamma))$$

Consequence

$$\chi(\overline{\mathcal{M}}_{g,n}) = \sum \chi(\mathcal{M}(\Gamma))$$

Same for Serre characteristics (Euler char. of $H_c^*(-)$ in Grothendieck group of mixed Hodge structures)

NB. $H^k(\overline{\mathcal{M}}_{g,n})$ pure $\forall k$ implies: if Serre characteristic known then Hodge numbers known

Applications:

- generating function for Serre characteristics of $\overline{\mathcal{M}}_{1,n}$ (Getzler '96);
- Serre characteristic of $\overline{\mathcal{M}}_{2,n}, n \leq 3$ (Getzler '98); (both using modular operads)
- generating function for $\chi(\overline{\mathcal{M}}_{2,n})$ (Bini, Gaiffi, Polito '98)

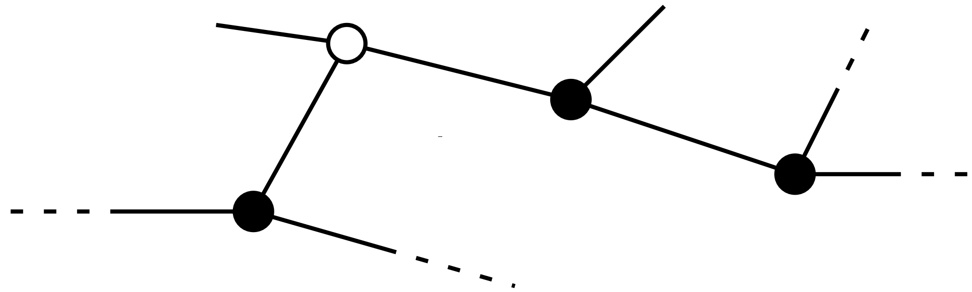
NB. Harer ('98): recursive computation of $\chi(\overline{\mathcal{M}}_{g,n})$

**Example:** $\chi(\overline{\mathcal{M}}_{1,n})$ (following Bini et al.)

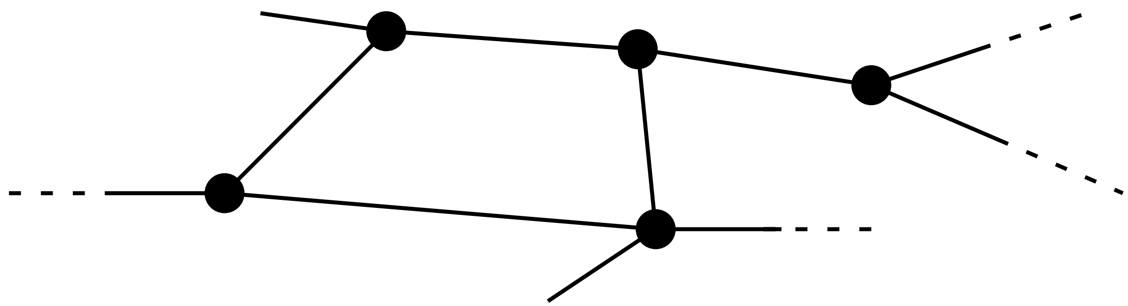
Goal: compute $K_1(t) = \sum \chi(\overline{\mathcal{M}}_{1,n}) t^n/n!$

Graph patterns in genus 1:

a)



b) ("necklace")



(solid dot: genus 0; hollow dot: genus 1)

$\Gamma$ pattern a) graph; $\mathrm{Aut}(\Gamma)$ always trivial.

$$\chi(\mathcal{M}(\Gamma)) = \chi(\mathcal{M}_{1,m}) \prod_{i=1}^{h} \chi(\mathcal{M}(\Gamma_i))$$

- $\Gamma_i$ stable, genus 0, $(k_i + 1)$-pointed;
- $h \leq m$;
- $n = m - h + \sum k_i$

Contribution of all these $\Gamma$

$$\sum \chi(\mathcal{M}_{1,m}) A^m / m!,$$

$$A(t) = t + \sum_{n \geq 2, G} \chi(\mathcal{M}(G)) \frac{t^n}{n!},$$

(sum over all stable $(n+1)$-pointed genus 0 graphs $G$)

Recursion for $A$:

$$A = t + \sum_{n \geq 2} \chi(\mathcal{M}_{0,n+1}) \frac{A^n}{n!}$$

$$\chi(\mathcal{M}_{0,n+1}) = (-1)^n (n-2)!$$

($\mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n}$ fibration; fiber $\mathbb{P}^1$ minus $n$ points)

$\chi(\mathcal{M}_{1,1}) = \chi(\mathcal{M}_{1,2}) = 1; \quad \chi(\mathcal{M}_{1,3}) = \chi(\mathcal{M}_{1,4}) = 0;$

$\chi(\mathcal{M}_{1,5}) = -2$

$\mathcal{M}_{1,n+1} \to \mathcal{M}_{1,n}$ fibration for $n \geq 5 \implies$

$$\chi(\mathcal{M}_{1,n}) = (-1)^n (n-1)!/12\,, \qquad n \geq 4$$

Similar argument for pattern b), but $\#\,\mathrm{Aut}(\Gamma) = 2$ when "necklace" consists of 1 or 2 edges. Final result:

$$K_1(t) = \frac{19}{12}A + \frac{23}{24}A^2 + \frac{5}{18}A^3 + \frac{1}{24}A^4 - \frac{1}{12}\log(1+A) - \frac{1}{2}\log(1-\log(1+A))$$

| | | | | |
|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | $\cdots$ |
| $\chi(\overline{\mathcal{M}}_{1,n})$ | 2 | 4 | 12 | $\cdots$ |

**Serre chars.:** calculating those of $\mathcal{M}_{g,n}$ is **hard**

## Use of building blocks – Strategy B

(Arbarello – M.C. '98)

**Basic result** (Harer '86)
Coho. dimension of $\mathcal{M}_{g,n}$ for constructible sheaves is:

$$\begin{array}{ll} \leq n-3 & g=0 \\ \leq 4g-5 & n=0 \\ \leq 4g-4+n & \text{otherwise} \end{array}$$

(Check: $\mathcal{M}_{g,n}$ affine for $g = 0, 1$ or $g = 2, n = 0$; in this case bound is $\dim_{\mathbb{C}} \mathcal{M}_{g,n}$)

**Challenge:** prove this algebro-geometrically

Corollary.

$$H^k(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^k(\partial\mathcal{M}_{g,n})$$

$$\text{if } k \leq d(g,n) = \begin{cases} n-4 & g=0 \\ 2g-2 & n=0 \\ 2g-3+n & \text{otherwise} \end{cases}$$

$$X_\Gamma \to \overline{\mathcal{M}}_{g,n} \ ; \qquad X = \coprod_{\Gamma \text{ has } 1 \text{ edge}} X_\Gamma \xrightarrow{\xi} \overline{\mathcal{M}}_{g,n}$$

Corollary. *If* $k \leq d(g,n)$

$$\xi^* : H^k(\overline{\mathcal{M}}_{g,n}) \hookrightarrow \bigoplus_{\Gamma \text{ has } 1 \text{ edge}} H^k(X_\Gamma)$$

*Pf.*
$W_{k-1}(H^k(\partial\mathcal{M}_{g,n})) = \ker(H^k(\partial\mathcal{M}_{g,n}) \to \bigoplus H^k(X_\Gamma))$.
If $\xi^*(\alpha) = 0$, $\alpha$ maps to $W_{k-1}(H^k(\partial\mathcal{M}_{g,n}))$, hence $\alpha \in W_{k-1}H^k(\overline{\mathcal{M}}_{g,n}) = 0$ by strictness ($\rho$ injective).

THEOREM. $H^k(\overline{\mathcal{M}}_{g,n}) = 0$, $k = 1, 3, 5$, $\forall g, n$.

(NB: $\overline{\mathcal{M}}_{g,n}$ is known to be simply connected)

*Pf.* Induction on $k, g, n$. $X_\Gamma$ product of $\overline{\mathcal{M}}_{\gamma,\nu}$ s.t. $\gamma < g$ or $\gamma = g$, $\nu < n$. Enough to do **finite** # of cases s.t. $k > d(g,n)$, i.e.,

- for $k = 1$: $\overline{\mathcal{M}}_{0,3} =$ point, $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$, $\overline{\mathcal{M}}_{1,1} = \mathbb{P}^1$;

- for $k = 3, 5$ all initial cases doable "by hand" save:
  - $\overline{\mathcal{M}}_{1,5}$, $\overline{\mathcal{M}}_{2,2}$, $\overline{\mathcal{M}}_{2,3}$, covered by Getzler's results
  - $\overline{\mathcal{M}}_3$, $\overline{\mathcal{M}}_{3,1}$ can be deduced from knowledge (Looijenga '93) of cohomology of $\mathcal{M}_3, \mathcal{M}_{3,1}$.

  **Examples:** $\overline{\mathcal{M}}_2$, $\overline{\mathcal{M}}_{2,1}$ dominated by $\overline{\mathcal{M}}_{0,6}$, $\overline{\mathcal{M}}_{0,7}$, $h^3(\overline{\mathcal{M}}_{1,3}) = 2 - 2h^1(\overline{\mathcal{M}}_{1,3}) + 2h^2(\overline{\mathcal{M}}_{1,3}) - \chi(\overline{\mathcal{M}}_{1,3}) = 0$ since $h^2(\overline{\mathcal{M}}_{1,3}) = 5$.

NB: method works for odd cohomology, if one can handle initial cases

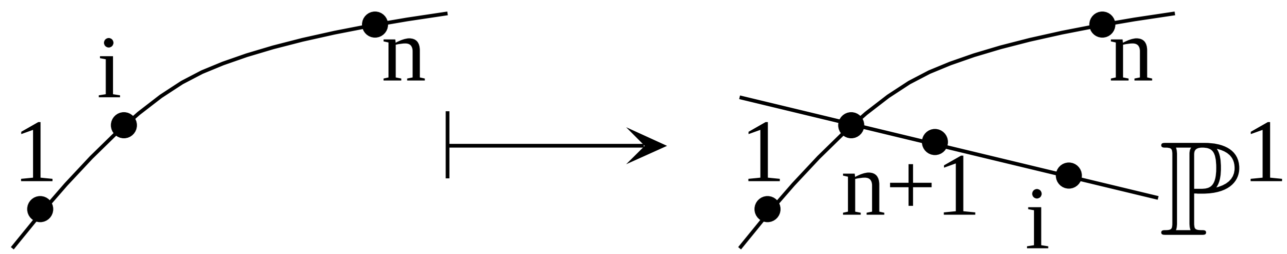
Upper limit of applicability $k = 9$ ($H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$)

**Question:** $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ for $k = 7, 9$ and all $g, n$?

## Computing $H^2(\overline{\mathcal{M}}_{g,n})$

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

(forget last pt.). Section $\sigma_i$:




$D_i$ = divisor on $\overline{\mathcal{M}}_{g,n+1}$ corresponding to $\sigma_i$

$\kappa_a = \pi_*(K^{a+1}) \in H^{2a}(\overline{\mathcal{M}}_{g,n})$ ; $\quad K = c_1(\omega_\pi(\sum D_i))$

$\psi_i = c_1(\sigma_i^*(\omega_\pi)) \in H^2(\overline{\mathcal{M}}_{g,n})$

$\delta_\Gamma =$ orbifold fundamental class of $\overline{\mathcal{M}(\Gamma)}$.
$\delta_\Gamma \in H^{2l}(\overline{\mathcal{M}}_{g,n})$, where $l = \#$edges of $\Gamma$

THEOREM. *(Harer, essentially) $H^2(\overline{\mathcal{M}}_{g,n})$ generated by $\kappa_1$, $\psi_i$'s, $\delta_\Gamma$'s, $\Gamma$ with 1 edge (natural classes). No relations if $g \geq 3$, explicit relations if $g < 3$.*

**Strategy of inductive pf.** (Arbarello-M.C.):

a) direct check for $2 > d(g, n)$
b) $2 \leq d(g, n)$: supp. result known for $\overline{\mathcal{M}}_{\gamma,\nu}$, $\gamma < g$ or $\gamma = g$, $\nu < n$.
$\alpha \in H^2(\overline{\mathcal{M}}_{g,n})$
$\alpha_\Gamma = \xi_\Gamma^*(\alpha)$ ($\Gamma$ with 1 edge). Then:

i) $\alpha_\Gamma$ is natural
(induction hypothesis)

ii) pullbacks of $\alpha_\Gamma$, $\alpha_{\Gamma'}$ to $X_\Gamma \times_{\overline{\mathcal{M}}_{g,n}} X_{\Gamma'}$ are equal

iii) explicit formulas for pullbacks of natural classes via maps $\xi_\Gamma$

Using iii), linear algebra shows that all collections $\beta_\Gamma$ of classes satisfying compatibility ii) come from a natural class $\beta \in H^2(\overline{\mathcal{M}}_{g,n})$. Conclusion by injectivity of

$$H^2(\overline{\mathcal{M}}_{g,n}) \to \bigoplus_{\Gamma \text{ has 1 edge}} H^2(X_\Gamma)$$

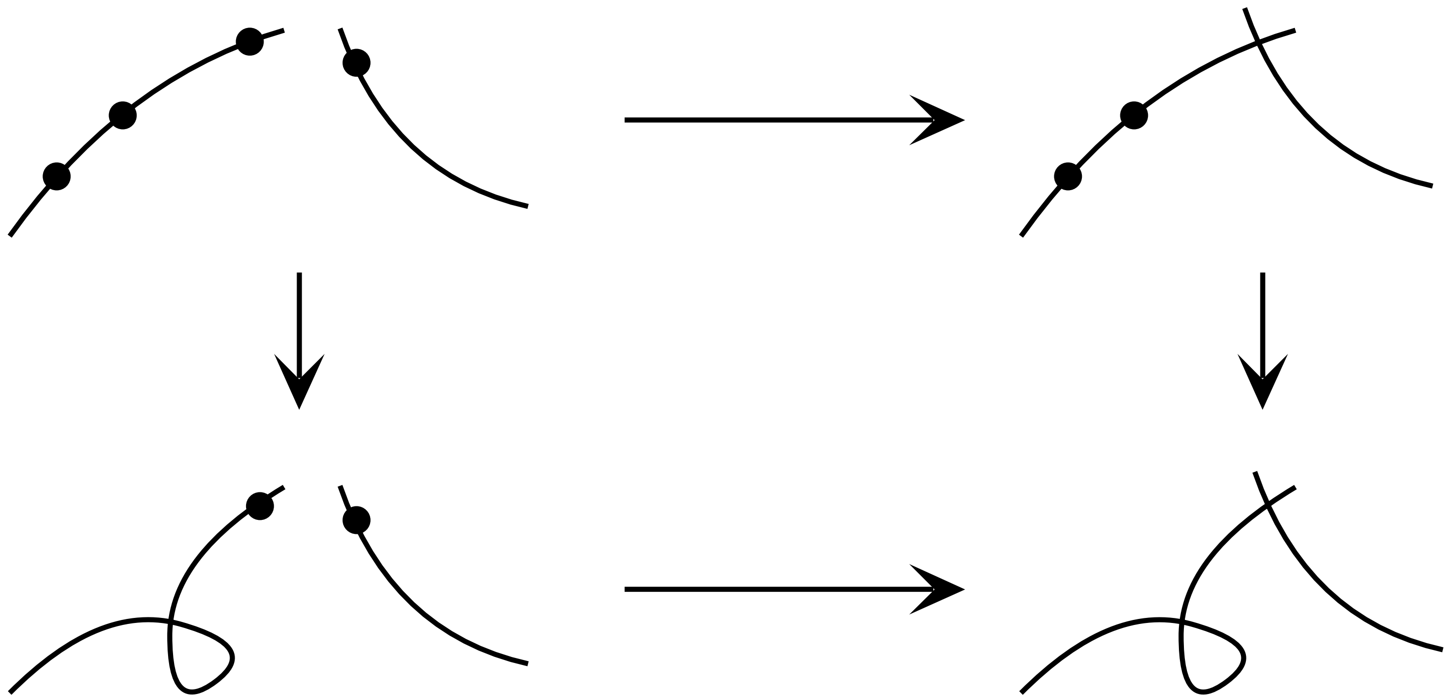
## A sample compatibility computation

THEOREM. $\xi_{irr}^* : H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\overline{\mathcal{M}}_{g-1,n+2})$ *is injective if* $k \leq \min(2g-2, g+5)$

**Toy case:** $k = 2, g = 3, n = 0$

$\xi_{irr}^*(x) = 0$. Must show that $\xi_\Gamma^*(x) = 0$ for all other $\Gamma$ with one edge. There is only $G = \Gamma_{1,\emptyset}$

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{1,3} \times \overline{\mathcal{M}}_{1,1} & \longrightarrow & \overline{\mathcal{M}}_{2,2} \\
\alpha \downarrow & & \downarrow \xi_{irr} \\
\overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{1,1} & \xrightarrow{\ \xi_G\ } & \overline{\mathcal{M}}_3
\end{array}
$$



$\xi_G^*(x) = (y, z)$, $0 = \alpha^*(y, z) = (y', z)$, hence $z = 0$. Same argument shows that $y = 0$.

# $H^k(\overline{\mathcal{M}}_{g,n})$ for higher even $k$

Problems:

1. Do initial cases of induction (OK for $k = 4$)
2. Do linear algebra.
   New hurdle: relations among natural classes not fully known, even for $k = 4$. New relations in degree 4 recently discovered by Getzler and Belorousski-Pandharipande.

# A list of wishes

1. Compute $H^4(\overline{\mathcal{M}}_{g,n})$; show it is generated by natural classes and find all relations among these.
2. Decide whether $H^k(\overline{\mathcal{M}}_{g,n})$ is zero for $k = 7, 9$.
3. Prove Harer's bound on the cohomological dimension of $\mathcal{M}_{g,n}$ algebro-geometrically.
4. Understand conceptually why natural classes on boundary components of $\overline{\mathcal{M}}_{g,n}$ which patch together come from a natural class on $\overline{\mathcal{M}}_{g,n}$.
5. Decide whether the even cohomology of $\overline{\mathcal{M}}_{g,n}$ consists entirely of natural classes.
6. ...