The fine-scale Green’s Function
and the construction of variational multiscale methods

T.J.R. Hughes\textsuperscript{1} and G. Sangalli\textsuperscript{2}

\textsuperscript{1}The University of Texas at Austin - ICES Austin, TX
\textsuperscript{2}Dipartimeno di Matematica - Università di Pavia, Pavia, Italy.

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The abstract problem and framework

Given a Hilbert space $V$, with norm $\| \cdot \|_V$ and s.p. $(\cdot, \cdot)_V$, dual space $V^*$, and $\mathcal{L} : V \rightarrow V^*$, we consider the problem:

$$\begin{cases} 
\text{find } u \in V : \\
\mathcal{L}u = f.
\end{cases}$$

We split the space $V$, where the exact solution is, into:

$$\bar{V} = \text{space of coarse scales,}$$
$$V' = \text{space of fine scales,}$$

and then consider:

$$\begin{cases} 
\text{find } \bar{u} \in \bar{V}, u' \in V' : \\
\mathcal{L} (\bar{u} + u') = f.
\end{cases}$$
The abstract problem and framework

Given a Hilbert space \( V \), with norm \( \| \cdot \|_V \) and s.p. \((\cdot, \cdot)_V\), dual space \( V^* \), and \( \mathcal{L} : V \rightarrow V^* \), we consider the problem:

\[
\begin{align*}
\{ \text{find } u \in V : \\
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\}
\end{align*}
\]

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and then consider:

\[
\begin{align*}
\{ \text{find } \bar{u} \in \bar{V}, u' \in V' : \\
\mathcal{L}(\bar{u} + u') &= f.
\}
\]
Variational multiscale (VMS) formulation

The variational formulation of the problem is:

$$\text{find } \bar{u} + u' \in V : \quad V^* \langle L(\bar{u} + u'), v \rangle_V = V^* \langle f, v \rangle_V, \quad \forall v \in V.$$ 

Then, we split the problem as:

$$\begin{align*}
V^* \langle L\bar{u}, \bar{v} \rangle_V + V^* \langle Lu', \bar{v} \rangle_V &= V^* \langle f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}, \\
V^* \langle L\bar{u}, v' \rangle_V + V^* \langle Lu', v' \rangle_V &= V^* \langle f, v' \rangle_V, \quad \forall v' \in V'.
\end{align*}$$

$$\Rightarrow u' = G'(f - L\bar{u}) \Rightarrow$$

**VMS formulation (for \( \bar{u} \))**

Find \( \bar{u} \in \bar{V} \) such that:

$$\begin{align*}
V^* \langle L\bar{u}, \bar{v} \rangle_V - V^* \langle LG'\bar{L}\bar{u}, \bar{v} \rangle_V \\
= V^* \langle f, \bar{v} \rangle_V - V^* \langle LG'f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}.
\end{align*}$$
Variational multiscale (VMS) formulation

The variational formulation of the problem is:

\[
\text{find } \bar{u} + u' \in V : \quad V^*\langle \mathcal{L}(\bar{u} + u'), v \rangle_V = V^*\langle f, v \rangle_V, \quad \forall v \in V.
\]

Then, we split the problem as:

\[
V^*\langle \mathcal{L}\bar{u}, \bar{v} \rangle_V + V^*\langle \mathcal{L}u', \bar{v} \rangle_V = V^*\langle f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V},
\]

\[
V^*\langle \mathcal{L}\bar{u}, v' \rangle_V + V^*\langle \mathcal{L}u', v' \rangle_V = V^*\langle f, v' \rangle_V, \quad \forall v' \in V'.
\]

\[\Rightarrow u' = G'(f - \mathcal{L}\bar{u}) \Rightarrow\]

VMS formulation (for \( \bar{u} \))

Find \( \bar{u} \in \bar{V} \) such that:

\[
V^*\langle \mathcal{L}\bar{u}, \bar{v} \rangle_V - V^*\langle \mathcal{L}G'\mathcal{L}\bar{u}, \bar{v} \rangle_V
= V^*\langle f, \bar{v} \rangle_V - V^*\langle \mathcal{L}G'f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}.
\]
Variational multiscale (VMS) formulation

The variational formulation of the problem is:

\[
\text{find } \bar{u} + u' \in V : \quad \langle L(\bar{u} + u'), v \rangle_V = \langle f, v \rangle_V, \quad \forall v \in V.
\]

Then, we split the problem as:

\[
\begin{align*}
\langle L\bar{u}, \bar{v} \rangle_V + \langle Lu', \bar{v} \rangle_V &= \langle f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}, \\
\langle L\bar{u}, v' \rangle_V + \langle Lu', v' \rangle_V &= \langle f, v' \rangle_V, \quad \forall v' \in V'.
\end{align*}
\]

\[\Rightarrow u' = G'(f - L\bar{u}) \Rightarrow\]

**VMS formulation (for \(\bar{u}\))**

Find \(\bar{u} \in \bar{V}\) such that:

\[
\begin{align*}
\langle L\bar{u}, \bar{v} \rangle_V - \langle LG'L\bar{u}, \bar{v} \rangle_V &= \langle f, \bar{v} \rangle_V - \langle LG'f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}.
\end{align*}
\]
The variational formulation of the problem is:

\[
\text{find } \bar{u} + u' \in V : \quad \langle \mathcal{L}(\bar{u} + u'), v \rangle_V = \langle f, v \rangle_V, \quad \forall v \in V.
\]

Then, we split the problem as:

\[
\begin{align*}
\langle \mathcal{L}\bar{u}, \bar{v} \rangle_V + \langle \mathcal{L}u', \bar{v} \rangle_V &= \langle f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}, \\
\langle \mathcal{L}\bar{u}, v' \rangle_V + \langle \mathcal{L}u', v' \rangle_V &= \langle f, v' \rangle_V, \quad \forall v' \in V'.
\end{align*}
\]

\[
\Rightarrow u' = \mathcal{G}'(f - \mathcal{L}\bar{u})
\]
Variational multiscale (VMS) formulation

The variational formulation of the problem is:

\[
\text{find } \bar{u} + u' \in V : \quad V^* \langle \mathcal{L}(\bar{u} + u'), v \rangle_V = V^* \langle f, v \rangle_V, \quad \forall v \in V.
\]

Then, we split the problem as:

\[
V^* \langle \mathcal{L} \bar{u}, \bar{v} \rangle_V + V^* \langle \mathcal{L} u', \bar{v} \rangle_V = V^* \langle f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V},
\]
\[
V^* \langle \mathcal{L} \bar{u}, v' \rangle_V + V^* \langle \mathcal{L} u', v' \rangle_V = V^* \langle f, v' \rangle_V, \quad \forall v' \in V'.
\]

\[\Rightarrow u' = \mathcal{G}' (f - \mathcal{L} \bar{u}) \quad \Rightarrow\]

\[\text{VMS formulation (for } \bar{u} \text{)}
\]

Find \(\bar{u} \in \bar{V}\) such that:

\[
V^* \langle \mathcal{L} \bar{u}, \bar{v} \rangle_V - V^* \langle \mathcal{L} \mathcal{G}' \mathcal{L} \bar{u}, \bar{v} \rangle_V
\]
\[
= V^* \langle f, \bar{v} \rangle_V - V^* \langle \mathcal{L} \mathcal{G}' f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}.
\]
Variational multiscale (VMS) formulation

The variational formulation of the problem is:

\[ \text{find } \bar{u} + u' \in V : \quad \nu^*\langle L(\bar{u} + u'), v \rangle_V = \nu^*\langle f, v \rangle_V, \quad \forall v \in V. \]

Then, we split the problem as:

\[ \nu^*\langle L\bar{u}, \bar{v} \rangle_V + \nu^*\langle Lu', \bar{v} \rangle_V = \nu^*\langle f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}, \]
\[ \nu^*\langle L\bar{u}, v' \rangle_V + \nu^*\langle Lu', v' \rangle_V = \nu^*\langle f, v' \rangle_V, \quad \forall v' \in V'. \]

\[ \Rightarrow u' = G'(f - L\bar{u}) \]

VMS formulation (for \( \bar{u} \))

Find \( \bar{u} \in \bar{V} \) such that:

\[ \nu^*\langle L\bar{u}, \bar{v} \rangle_V - \nu^*\langle LG' L\bar{u}, \bar{v} \rangle_V \]
\[ = \nu^*\langle f, \bar{v} \rangle_V - \nu^*\langle LG' f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}. \]
VMS formulation

Find $\bar{u} \in \bar{V}$ such that:

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- The fine-scale effect is determined by the fine scale Green’s operator $G' : V^* \rightarrow V'$ which gives $V' \ni u' = G' r$ such that $V^* \langle Lu', v' \rangle_V = V^* \langle r, v' \rangle_V, \quad \forall v' \in V'$,

- $G'$ is not the classical Green’s operator $G \equiv L^{-1} : V^* \rightarrow V$,

- In order to derive a VMS formulation, we need $\bar{V} \cap V' = \{0\}$, that is we need a direct sum of $\bar{V} \oplus V'$. 

VMS formulation

Find $\bar{u} \in \bar{V}$ such that:

$$V^* \langle L\bar{u}, \bar{v} \rangle_V - V^* \langle LG' L\bar{u}, \bar{v} \rangle_V = V^* \langle f, \bar{v} \rangle_V - V^* \langle LG' f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}. $$

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VMS formulation

Find \( \bar{u} \in \bar{V} \) such that:

\[
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\]

- the fine-scale effect is determined by the fine scale Green's operator \( \mathcal{G}' : V^* \rightarrow V' \) which gives \( V' \ni u' = \mathcal{G}' r \) such that \( V^* \langle \mathcal{L} u', v' \rangle_V = V^* \langle r, v' \rangle_V, \forall v' \in V' \),
- \( \mathcal{G}' \) is not the classical Green's operator \( \mathcal{G} \equiv \mathcal{L}^{-1} : V^* \rightarrow V \),
- in order to derive a VMS formulation, we need \( \bar{V} \cap V' = \{0\} \), that is we need a direct sum of \( \bar{V} \oplus V' \).
VMS formulation

Find $\bar{u} \in \bar{V}$ such that:

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$$= V^* \langle f, \bar{v} \rangle_V - V^* \langle \mathcal{L} \mathcal{G}' f, \bar{v} \rangle_V, \quad \forall \bar{v} \in \bar{V}.$$ 

- the fine-scale effect is determined by the fine scale Green’s operator $\mathcal{G}' : V^* \to V'$ which gives $V' \ni u' = \mathcal{G}' r$ such that
  $$V^* \langle \mathcal{L} u', v' \rangle_V = V^* \langle r, v' \rangle_V, \quad \forall v' \in V',$$
- $\mathcal{G}'$ is not the classical Green’s operator $\mathcal{G} \equiv \mathcal{L}^{-1} : V^* \to V,$
- in order to derive a VMS formulation, we need
  $$\bar{V} \cap V' = \{0\},$$
  that is we need a direct sum of $\bar{V} \oplus V'$. 

Example of typical VMS methods

For 2D advection-diffusion problems, various choices for $\bar{V} \oplus V'$ have been proposed in literature:

- $P1 \oplus$ residual-free bubbles: [F. Brezzi and A. Russo, '94], [T. J. R. Hughes, '95], [F. Brezzi, L. P. Franca, T. J. R. Hughes, and A. Russo, '97], ...
- $P2 \oplus$ residual-free bubbles: [M. I. ASENSIO, A. Russo, and G. Sangalli, '04]
- $P1 \oplus (r.-f. \text{ bubbles} + \ldots)$: [F. Brezzi and L.D. Marini, '02][A.Cangiani and E. Süli, 05], [L. P. Franca, A. L. Madureira and F. Valentin, 05], ...

In all cases,

$$\bar{V} \oplus V' \subset V \equiv \text{all-scale space}.$$
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In all cases,

$$\bar{V} \oplus V' \subset V \equiv \text{all-scale space}.$$
New approach to VMS: error optimization

Minimize $\Phi(u')$ subject to

\[
\begin{align*}
\bar{u} & \in \bar{V}, \\
u' & \in V, \\
\mathcal{L}(\bar{u} + u') &= f
\end{align*}
\]

- we do not assume a-priori constraints on the fine scales,
- we minimize the fine scale $u'$ (i.e., the numerical error $u - \bar{u} \equiv u'$) w.r.t. a functional $\Phi(\cdot)$,
- then $\bar{u}$ and $u'$ are uniquely determined, and the numerical solution $\bar{u}$ is optimal ($\Phi(u - \bar{u})$ is minimized) by design,
- here, we consider $\Phi(\cdot) = \| \cdot \|^2$ (for example, $\| \cdot \| = \| \cdot \|_{H_0^1}$ or $\| \cdot \| = \| \cdot \|_{L^2}$).
- other possibilities: $\Phi(\cdot)$ is not a quadratic form.
New approach to VMS: error optimization

\[
\begin{align*}
\text{Minimize } \Phi(u') \text{ subject to } \\
\begin{cases}
\bar{u} \in \bar{V}, \\
u' \in V, \\
\mathcal{L}(\bar{u} + u') = f
\end{cases}
\end{align*}
\]

- we do not assume a-priori constraints on the fine scales,
- we minimize the fine scale \( u' \) (i.e., the numerical error \( u - \bar{u} \equiv u' \)) w.r.t. a functional \( \Phi(\cdot) \),
- then \( \bar{u} \) and \( u' \) are uniquely determined, and the numerical solution \( \bar{u} \) is optimal (\( \Phi(u - \bar{u}) \) is minimized) by design,
- here, we consider \( \Phi(\cdot) = \| \cdot \|^2 \) (for example, \( \| \cdot \| = \| \cdot \|_{H^1} \) or \( \| \cdot \| = \| \cdot \|_{L^2} \)).
- other possibilities: \( \Phi(\cdot) \) is not a quadratic form.
New approach to VMS: error optimization

Minimize $\Phi(u')$ subject to

\[
\begin{cases}
\tilde{u} \in \bar{V}, \\
u' \in V, \\
\mathcal{L}(\tilde{u} + u') = f
\end{cases}
\]

- we do not assume a-priori constraints on the fine scales,
- we minimize the fine scale $u'$ (i.e., the numerical error $u - \tilde{u} \equiv u'$) w.r.t. a functional $\Phi(\cdot)$,
- then $\tilde{u}$ and $u'$ are uniquely determined, and the numerical solution $\tilde{u}$ is optimal ($\Phi(u - \tilde{u})$ is minimized) by design,
- here, we consider $\Phi(\cdot) = \| \cdot \|_2^2$ (for example, $\| \cdot \| = \| \cdot \|_{H^1_0}$ or $\| \cdot \| = \| \cdot \|_{L^2}$).
- other possibilities: $\Phi(\cdot)$ is not a quadratic form.
New approach to VMS: error optimization

Minimize $\Phi(u')$ subject to
\[
\begin{align*}
\bar{u} &\in \bar{V}, \\
u' &\in V, \\
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\end{align*}
\]

- we do not assume a-priori constraints on the fine scales,
- we minimize the fine scale $u'$ (i.e., the numerical error $u - \bar{u} \equiv u'$) w.r.t. a functional $\Phi(\cdot)$,
- then $\bar{u}$ and $u'$ are uniquely determined, and the numerical solution $\bar{u}$ is optimal ($\Phi(u - \bar{u})$ is minimized) by design,
- here, we consider $\Phi(\cdot) = \| \cdot \| ^2$ (for example, $\| \cdot \| = \| \cdot \| _{H^1 _0}$ or $\| \cdot \| = \| \cdot \| _{L^2}$).
- other possibilities: $\Phi(\cdot)$ is not a quadratic form.
New approach to VMS: error optimization

Minimize $\Phi(u')$ subject to

$$\begin{cases}
\bar{u} \in \bar{V}, \\
u' \in V, \\
L(\bar{u} + u') = f
\end{cases}$$

- we do not assume a-priori constraints on the fine scales,
- we minimize the fine scale $u'$ (i.e., the numerical error $u - \bar{u} \equiv u'$) w.r.t. a functional $\Phi(\cdot)$,
- then $\bar{u}$ and $u'$ are uniquely determined, and the numerical solution $\bar{u}$ is optimal ($\Phi(u - \bar{u})$ is minimized) by design,
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New approach to VMS: error optimization

Minimize $\Phi(u')$ subject to
\[
\begin{cases}
\bar{u} \in \bar{V}, \\
u' \in V, \\
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\end{cases}
\]

- we do not assume a-priori constraints on the fine scales,
- we minimize the fine scale $u'$ (i.e., the numerical error $u - \bar{u} \equiv u'$) w.r.t. a functional $\Phi(\cdot)$,
- then $\bar{u}$ and $u'$ are uniquely determined, and the numerical solution $\bar{u}$ is optimal ($\Phi(u - \bar{u})$ is minimized) by design,
- here, we consider $\Phi(\cdot) = \| \cdot \|^2$ (for example, $\| \cdot \| = \| \cdot \|_{H^1_0}$ or $\| \cdot \| = \| \cdot \|_{L^2}$).
- other possibilities: $\Phi(\cdot)$ is not a quadratic form.
\[
\begin{cases}
\bar{u} \in \bar{V}, \\
\bar{u}' \in V, \\
\mathcal{L}(\bar{u} + u') = f
\end{cases}
\iff u' \in u + \bar{V} \leadsto \delta u' \in \bar{V}.
\]

Then, under suitable condition on \(\Phi(\cdot)\):

\[
\Phi(u') = \min_{v' \in u + \bar{V}} \Phi(v') \iff D\Phi(u'; \delta u') = 0, \quad \forall \delta u' \in \bar{V}.
\]

In case of \(\Phi(\cdot) = \| \cdot \|_2^2\), then \(D\Phi(u'; \delta u') = (u', \delta u') = 0\), for all \(\delta u' \in \bar{V}\), that is, \(u'\) is orthogonal to \(\bar{V}\), that is,

\[
\mathcal{P} u' = 0,
\]

where \(\mathcal{P} : V \rightarrow \bar{V}\) is the orthogonal projector on \(\bar{V}\).
\[
\begin{cases}
\bar{u} \in \bar{V}, \\
u' \in V, \\
\mathcal{L}(\bar{u} + u') = f
\end{cases}
\iff u' \in u + \bar{V} \sim \delta u' \in \bar{V}.
\]

Then, under suitable condition on $\Phi(\cdot)$:

$$
\Phi(u') = \min_{v' \in u + \bar{V}} \Phi(v') \iff D\Phi(u'; \delta u') = 0, \quad \forall \delta u' \in \bar{V}.
$$

In case of $\Phi(\cdot) = \| \cdot \|_2^2$, then $D\Phi(u'; \delta u') = (u', \delta u') = 0$, for all $\delta u' \in \bar{V}$, that is, $u'$ is orthogonal to $\bar{V}$, that is,

$$
\mathcal{P} u' = 0,
$$

where $\mathcal{P} : V \rightarrow \bar{V}$ is the orthogonal projector on $\bar{V}$. 


The 1D advection-diffusion problem

\begin{align*}
\begin{cases}
\bar{u} \in \bar{V}, \\
u' \in V, \\
\mathcal{L}(\bar{u} + u') = f
\end{cases} \iff u' \in u + \bar{V} \quad \leadsto \delta u' \in \bar{V}.
\end{align*}

Then, under suitable condition on \( \Phi(\cdot) \):

\[ \Phi(u') = \min_{v' \in u + \bar{V}} \Phi(v') \iff D\Phi(u'; \delta u') = 0, \quad \forall \delta u' \in \bar{V}. \]

In case of \( \Phi(\cdot) = \| \cdot \|^2 \), then \( D\Phi(u'; \delta u') = (u', \delta u') = 0 \), for all \( \delta u' \in \bar{V} \), that is, \( u' \) is orthogonal to \( \bar{V} \), that is,

\[ \mathcal{P} u' = 0. \]

where \( \mathcal{P} : V \rightarrow \bar{V} \) is the orthogonal projector on \( \bar{V} \).
\begin{align*}
\begin{cases}
\tilde{u} \in \tilde{V}, \\
u' \in V, \\
\mathcal{L}(\tilde{u} + u') = f
\end{cases}
\Leftrightarrow u' \in u + \tilde{V} \quad \leadsto \delta u' \in \tilde{V}.
\end{align*}

Then, under suitable condition on \( \Phi(\cdot) \):

\[
\Phi(u') = \min_{v' \in u + \tilde{V}} \Phi(v') \quad \Leftrightarrow \quad D\Phi(u'; \delta u') = 0, \quad \forall \delta u' \in \tilde{V}.
\]

In case of \( \Phi(\cdot) = \| \cdot \|^2 \), then \( D\Phi(u'; \delta u') = (u', \delta u') = 0 \), for all \( \delta u' \in \tilde{V} \), that is, \( u' \) is orthogonal to \( \tilde{V} \), that is,

\[
\mathcal{P}u' = 0,
\]

where \( \mathcal{P} : V \to \tilde{V} \) is the orthogonal projector on \( \tilde{V} \).
\begin{equation}
\left\{
\begin{align*}
\bar{u} & \in \bar{V}, \\
\upsilon' & \in V, \\
\mathcal{L}(\bar{u} + \upsilon') & = f
\end{align*}
\right. \Leftrightarrow \upsilon' \in \upsilon + \bar{V} \sim \delta\upsilon' \in \bar{V}.
\end{equation}

Then, under suitable condition on $\Phi(\cdot)$:

$$
\Phi(\upsilon') = \min_{\upsilon' \in \upsilon + \bar{V}} \Phi(\upsilon') \Leftrightarrow D\Phi(\upsilon'; \delta\upsilon') = 0, \quad \forall \delta\upsilon' \in \bar{V}.
$$

In case of $\Phi(\cdot) = \| \cdot \|^2$, then $D\Phi(\upsilon'; \delta\upsilon') = (\upsilon', \delta\upsilon') = 0$, for all $\delta\upsilon' \in \bar{V}$, that is, $\upsilon'$ is orthogonal to $\bar{V}$, that is,

$$
P\upsilon' = 0,
$$

where $P : V \rightarrow \bar{V}$ is the orthogonal projector on $\bar{V}$. 
Scales splitting + optimization

Find $\bar{u}$ and $u'$ such that:

\[
\begin{cases}
\bar{u} \in \tilde{V}, \\
u' \in V, \text{ with } P u' = 0, \\
\mathcal{L}(\bar{u} + u') = f
\end{cases}
\]

- the fine scale space is implicitly defined by the optimality condition:
  \[V' = \{ v \in V : P v = 0 \};\]

- we have $\tilde{V} \oplus V'$;
- when $\Phi(\cdot) = \| \cdot \|^2$, i.e., $P$ is an orthogonal projector, $V'$ is the orthogonal complement of $\tilde{V}$ (in $V$) and $\bar{u} = Pu$;
- how do we eliminate $u'$?
Scales splitting + optimization

Find \( \bar{u} \) and \( u' \) such that:

\[
\begin{cases}
\bar{u} \in \bar{V}, \\
u' \in V, \text{ with } \mathcal{P}u' = 0, \\
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Scales splitting + optimization

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Scales splitting + optimization

Find \( \bar{u} \) and \( u' \) such that:

\[
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\bar{u} & \in \bar{V}, \\
u' & \in V, \text{ with } \mathcal{P} u' = 0, \\
\mathcal{L}(\bar{u} + u') & = f
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Scales splitting + optimization

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## Scales splitting + optimization

Find $\bar{u}$ and $u'$ such that:

$$
\begin{cases}
\bar{u} \in \bar{V}, \\
u' \in V, \text{ with } P u' = 0, \\
\mathcal{L}(\bar{u} + u') = f
\end{cases}
$$

- the fine scale space is implicitly defined by the optimality condition:
  $$V' = \{ v \in V : P v = 0 \};$$

- we have $\bar{V} \oplus V'$;

  - when $\Phi(\cdot) = \| \cdot \|_2^2$, i.e., $P$ is an orthogonal projector, $V'$ is the orthogonal complement of $\bar{V}$ (in $V$) and $\bar{u} = Pu$;

- how do we eliminate $u'$? \(\rightsquigarrow\)
Scales splitting + optimization

Find \( \tilde{u} \) and \( u' \) such that:

\[
\begin{align*}
    \tilde{u} & \in \bar{V}, \\
    u' & \in V, \text{ with } \mathcal{P}u' = 0, \\
    \mathcal{L}(\tilde{u} + u') &= f
\end{align*}
\]

- the fine scale space is implicitly defined by the optimality condition:
  \[
  V' = \{ v \in V : \mathcal{P}v = 0 \};
  \]

- we have \( \bar{V} \oplus V' \);

- when \( \Phi(\cdot) = \| \cdot \|^2 \), i.e., \( \mathcal{P} \) is an orthogonal projector, \( V' \) is the orthogonal complement of \( \bar{V} \) (in \( V \)) and \( \tilde{u} = \mathcal{P}u \);

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Scales splitting + optimization

Find $\bar{u}$ and $u'$ such that:

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\bar{u} & \in \bar{V}, \\
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- the fine scale space is implicitly defined by the optimality condition:
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- we have $\bar{V} \oplus V'$;
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- how do we eliminate $u'$?
The fine-scale problem reads: find \( u' \) such that \( P u' = 0 \) and

\[
V^* \langle L u', v' \rangle_V = V^* \langle r, v' \rangle_V, \quad \forall v' \in V \text{ with } P v' = 0,
\]

where \( r := f - L \bar{u} \). It can be written in unconstrained form introducing a Lagrange multiplier: find \( u' \in V \), and \( \lambda \in \bar{V}^* \) s.t.

\[
L u' + P^T \lambda = r, \quad (\text{in } V^*)
\]
\[
P u' = 0, \quad (\text{in } \bar{V})
\]

where \( P^T : \bar{V}^* \to V^* \). We want \( G' \) such that \( u' = G' r \).

**Theorem**

Let \( G = L^{-1} \) be the Green’s operator. Then,

\[
G' = G - G P^T (P G P^T)^{-1} P G.
\]

Moreover: \( G' P^T = 0, P G' = 0 \).
The fine-scale problem reads: find $u'$ such that $\mathcal{P}u' = 0$ and

$$v^* \langle \mathcal{L}u', v' \rangle_V = v^* \langle r, v' \rangle_V, \quad \forall v' \in V \text{ with } \mathcal{P}v' = 0,$$

where $r := f - \mathcal{L}\bar{u}$. It can be written in unconstrained form introducing a Lagrange multiplier: find $u' \in V$, and $\bar{\lambda} \in \bar{V}^*$ s.t.

$$\mathcal{L}u' + \mathcal{P}^T\bar{\lambda} = r, \quad (\text{in } V^*)$$
$$\mathcal{P}u' = 0, \quad (\text{in } \bar{V}),$$

where $\mathcal{P}^T : \bar{V}^* \rightarrow V^*$. We want $g'$ such that $u' = g'r$.

**Theorem**

Let $g = \mathcal{L}^{-1}$ be the Green’s operator. Then,

$$g' = g - g\mathcal{P}^T(\mathcal{P}g\mathcal{P}^T)^{-1}\mathcal{P}g.$$  

Moreover: $g'\mathcal{P}^T = 0, \mathcal{P}g' = 0.$
The fine-scale problem reads: find \( u' \) such that \( Pu' = 0 \) and

\[
V^* \langle L u', v' \rangle_V = V^* \langle r, v' \rangle_V, \quad \forall v' \in V \text{ with } P v' = 0,
\]

where \( r := f - L \bar{u} \). It can be written in unconstrained form introducing a Lagrange multiplier: find \( u' \in V \), and \( \bar{\lambda} \in \bar{V}^* \) s.t.

\[
Lu' + P^T \bar{\lambda} = r, \quad \text{ (in } V^*)
\]
\[
P u' = 0, \quad \text{ (in } \bar{V}),
\]

where \( P^T : \bar{V}^* \rightarrow V^* \). We want \( G' \) such that \( u' = G'r \).

**Theorem**

Let \( G \equiv L^{-1} \) be the Green’s operator. Then,

\[
G' = G - GP^T (PGP^T)^{-1} PG.
\]

Moreover: \( G'P^T = 0, PG' = 0 \).
The fine-scale problem reads: find $u'$ such that $\mathcal{P}u' = 0$ and

$$V^* \langle \mathcal{L}u', v' \rangle_V = V^* \langle r, v' \rangle_V, \quad \forall v' \in V \text{ with } \mathcal{P}v' = 0,$$

where $r := f - \mathcal{L}\bar{u}$. It can be written in unconstrained form introducing a Lagrange multiplier: find $u' \in V$, and $\bar{\lambda} \in \bar{V}^*$ s.t.

$$\mathcal{L}u' + \mathcal{P}^T\bar{\lambda} = r, \quad (\text{in } V^*)$$
$$\mathcal{P}u' = 0, \quad (\text{in } \bar{V}),$$

where $\mathcal{P}^T : \bar{V}^* \to V^*$. We want $\mathcal{G}'$ such that $u' = \mathcal{G}'r$.

**Theorem**

Let $\mathcal{G} \equiv \mathcal{L}^{-1}$ be the Green's operator. Then,

$$\mathcal{G}' = \mathcal{G} - \mathcal{G}\mathcal{P}^T(\mathcal{P}\mathcal{G}\mathcal{P}^T)^{-1}\mathcal{P}\mathcal{G}.$$

Moreover: $\mathcal{G}'\mathcal{P}^T = 0$, $\mathcal{P}\mathcal{G}' = 0$. 


The fine-scale problem reads: find $u'$ such that $Pu' = 0$ and

$$V^* \langle \mathcal{L}u', v' \rangle_V = V^* \langle r, v' \rangle_V, \quad \forall v' \in V \text{ with } P v' = 0,$$

where $r := f - \mathcal{L}\bar{u}$. It can be written in unconstrained form introducing a Lagrange multiplier: find $u' \in V$, and $\bar{\lambda} \in \bar{V}^*$ s.t.

\[
\mathcal{L}u' + P^T \bar{\lambda} = r, \quad \text{(in } V^*)
\]
\[
P u' = 0, \quad \text{(in } \bar{V}),
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where $P^T : \bar{V}^* \to V^*$. We want $G'$ such that $u' = G' r$.

**Theorem**

Let $G = L^{-1}$ be the Green’s operator. Then,

$$G' = G - G P^T (P G P^T)^{-1} P G.$$

Moreover: $G' P^T = 0$, $P G' = 0$. 
First part:

\[ \mathcal{L}u' + \mathcal{P}^T\bar{\lambda} = r \quad \text{(in } V^*) \],
\[ \mathcal{P}u' = 0 \quad \text{(in } \bar{V}) \],

where \( r := f - \mathcal{L}\bar{u} \). From (1) we get

\[ u' = g(r - \mathcal{P}^T\bar{\lambda}); \] (3)

substituting in (2) gives \( \mathcal{P}g r - \mathcal{P}g \mathcal{P}^T\bar{\lambda} = 0 \), whence
\[ \bar{\lambda} = (\mathcal{P}g \mathcal{P}^T)^{-1}\mathcal{P}g r. \]

Finally, using this in (3) yield

\[ u' = (g - g \mathcal{P}^T(\mathcal{P}g \mathcal{P}^T)^{-1}\mathcal{P}g) r. \] (4)

Second part:

\[ g'\mathcal{P}^T = g\mathcal{P}^T - g\mathcal{P}^T(\mathcal{P}g \mathcal{P}^T)^{-1}(\mathcal{P}g \mathcal{P}^T) = g\mathcal{P}^T - g\mathcal{P}^T = 0, \]
\[ \mathcal{P}g' = \mathcal{P}g - (\mathcal{P}g \mathcal{P}^T)(\mathcal{P}g \mathcal{P}^T)^{-1}\mathcal{P}g = \mathcal{P}g - \mathcal{P}g = 0. \]
First part:

\[ \mathcal{L}u' + P^T\bar{\lambda} = r \] (in \( V^* \)), \hfill (1)

\[ Pu' = 0 \] (in \( \bar{V} \)), \hfill (2)

where \( r := f - \mathcal{L}\bar{u} \). From (1) we get

\[ u' = G(r - P^T\bar{\lambda}); \] \hfill (3)

substituting in (2) gives \( PGr - PGP^T\bar{\lambda} = 0 \), whence

\[ \bar{\lambda} = (PGP^T)^{-1}PGr. \]

Finally, using this in (3) yield

\[ u' = (G - GP^T(PGP^T)^{-1}PG)r. \] \hfill (4)

Second part:

\[ G'P^T = GP^T - GP^T(PGP^T)^{-1}(PGP^T) = GP^T - GP^T = 0, \]

\[ PG' = PG - (PGP^T)(PGP^T)^{-1}PG = PG - PG = 0. \]
First part:

\[ \mathcal{L}u' + \mathcal{P}^T \bar{\lambda} = r \]  
\[ \text{(in } V^*) , \]  
\[ \mathcal{P}u' = 0 , \]  
\[ \text{(in } \bar{V} ) , \]  

where \( r := f - \mathcal{L}\bar{u} \). From (1) we get

\[ u' = \mathcal{G}(r - \mathcal{P}^T \bar{\lambda}); \]  
\[ \text{(3)} \]

substituting in (2) gives \( \mathcal{P}\mathcal{G}r - \mathcal{P}\mathcal{G}\mathcal{P}^T \bar{\lambda} = 0 \), whence

\[ \bar{\lambda} = (\mathcal{P}\mathcal{G}\mathcal{P}^T)^{-1} \mathcal{P}\mathcal{G}r . \]

Finally, using this in (3) yield

\[ u' = (\mathcal{G} - \mathcal{G}\mathcal{P}^T (\mathcal{P}\mathcal{G}\mathcal{P}^T)^{-1} \mathcal{P}\mathcal{G})r . \]  
\[ \text{(4)} \]

Second part:

\[ \mathcal{G}'\mathcal{P}^T = \mathcal{G}\mathcal{P}^T - \mathcal{G}\mathcal{P}^T (\mathcal{P}\mathcal{G}\mathcal{P}^T)^{-1} (\mathcal{P}\mathcal{G}\mathcal{P}^T) = \mathcal{G}\mathcal{P}^T - \mathcal{G}\mathcal{P}^T = 0 , \]

\[ \mathcal{P}\mathcal{G}' = \mathcal{P}\mathcal{G} - (\mathcal{P}\mathcal{G}\mathcal{P}^T)(\mathcal{P}\mathcal{G}\mathcal{P}^T)^{-1} \mathcal{P}\mathcal{G} = \mathcal{P}\mathcal{G} - \mathcal{P}\mathcal{G} = 0 . \]
First part:

\[ \mathcal{L} u' + \mathcal{P}^T \bar{\lambda} = r \quad \text{(in } V^*) \],
\[ \mathcal{P} u' = 0, \quad \text{(in } \bar{V}) \],

where \( r := f - \mathcal{L} \bar{u} \). From (1) we get

\[ u' = G(r - \mathcal{P}^T \bar{\lambda}); \quad (3) \]

substituting in (2) gives \( \mathcal{P} G r - \mathcal{P} G \mathcal{P}^T \bar{\lambda} = 0 \), whence

\[ \bar{\lambda} = (\mathcal{P} G \mathcal{P}^T)^{-1} \mathcal{P} G r. \]

Finally, using this in (3) yield

\[ u' = \left( G - G \mathcal{P}^T (\mathcal{P} G \mathcal{P}^T)^{-1} \mathcal{P} G \right) r. \quad (4) \]

Second part:

\[ G' \mathcal{P}^T = G \mathcal{P}^T - G \mathcal{P}^T (\mathcal{P} G \mathcal{P}^T)^{-1} (\mathcal{P} G \mathcal{P}^T) = G \mathcal{P}^T - G \mathcal{P}^T = 0, \]
\[ \mathcal{P} G' = \mathcal{P} G - (\mathcal{P} G \mathcal{P}^T)(\mathcal{P} G \mathcal{P}^T)^{-1} \mathcal{P} G = \mathcal{P} G - \mathcal{P} G = 0. \]
First part:

\[ \mathcal{L}u' + P^T \bar{\lambda} = r \quad \text{(in } V^*), \]  
\[ P u' = 0, \quad \text{(in } \bar{V}), \]  

where \( r := f - \mathcal{L} \bar{u}. \) From (1) we get

\[ u' = G(r - P^T \bar{\lambda}); \]  

substituting in (2) gives \( P G r - P G P^T \bar{\lambda} = 0, \) whence \( \bar{\lambda} = (P G P^T)^{-1} P G r. \) Finally, using this in (3) yield

\[ u' = (G - G P^T (P G P^T)^{-1} P G) r. \]  

Second part:

\[ G' P^T = G P^T - G P^T (P G P^T)^{-1} (P G P^T) = G P^T - G P^T = 0, \]  
\[ P G' = P G - (P G P^T) (P G P^T)^{-1} P G = P G - P G = 0. \]
1D advection-diffusion problem and $H^1_0$-optimality

\[ \mathcal{L} u := -\kappa \frac{d^2}{dx^2} u + \beta \frac{d}{dx} u = f \quad \text{in } (0, L), \quad u(0) = u(L) = 0. \]

The Green’s operator is represented by the Green’s function:

\[ u(y) = \int_0^L g(x, y) f(x) \, dx, \]

We set $V = H^1_0 \equiv H^1_0(0, L)$, $\bar{V} =$finite elements;

\[ \Phi(v) = \| v \|_{H^1_0}^2 := \int_0^L \left( \frac{d}{dx} v(x) \right)^2 \, dx, \]

\[ \int_0^L \frac{d}{dx} (P v - v) \frac{d}{dx} \bar{v} = 0, \quad \forall \bar{v} \in \bar{V}. \]
\begin{align*}
\int_0^L \mathcal{L} \bar{u}(x) \bar{v}(x) \, dx + \int_0^L \mathcal{L} u'(x) \bar{v}(x) \, dx &= \int_0^L f(x) \bar{v}(x) \, dx, \quad \forall \bar{v} \in \bar{V} \\
\int_0^L \mathcal{L} u'(x) v'(x) \, dx &= \int_0^L (f(x) - \mathcal{L} \bar{u}(x)) \bar{v}(x) \, dx, \quad \forall v' \in V'.
\end{align*}

\begin{align*}
u'(y) &= \int_0^L g'(x, y)(f(x) - \mathcal{L} \bar{u}(x)) \, dx = \int_0^L g'(x, y) r(x) \, dx,
\end{align*}

\implies \text{VMS for 1D advection-diffusion equation:}

\begin{align*}
\int_0^L \mathcal{L} \bar{u}(x) \bar{v}(x) \, dx + \int_0^L \int_0^L \mathcal{L}^* \bar{v}(y) g'(x, y) r(x) \, dx \, dy &= \int_0^L f(x) \bar{v}(x) \, dx, \quad \forall \bar{v} \in \bar{V}.
\end{align*}
\[
\int_0^L \mathcal{L} \bar{u}(x) \bar{v}(x) \, dx + \int_0^L \mathcal{L} u'(x) \bar{v}(x) \, dx = \int_0^L f(x) \bar{v}(x) \, dx, \quad \forall \bar{v} \in \bar{V}
\]

\[
\int_0^L \mathcal{L} u'(x) \bar{v}'(x) \, dx = \int_0^L (f(x) - \mathcal{L} \bar{u}(x)) \bar{v}(x) \, dx, \quad \forall \bar{v}' \in V'.
\]

\[
u'(y) = \int_0^L g'(x, y) (f(x) - \mathcal{L} \bar{u}(x)) \, dx = \int_0^L g'(x, y) r(x) \, dx,
\]

\[\rightsquigarrow \text{VMS for 1D advection-diffusion equation:}\]

\[
\int_0^L \mathcal{L} \bar{u}(x) \bar{v}(x) \, dx + \int_0^L \int_0^L \mathcal{L}^* \bar{v}(y) g'(x, y) r(x) \, dx \, dy
\]

\[= \int_0^L f(x) \bar{v}(x) \, dx, \quad \forall \bar{v} \in \bar{V}.
\]
Consider a grid of nodes $0 = x_0 < x_1 < \ldots < x_{n_{el} - 1} < x_{n_{el}} = L$, that subdivides $(0, L)$ into $n_{el}$ elements $(x_{i-1}, x_i)$ ($i = 1, \ldots, n_{el}$), and take piecewise affine $\bar{V} \subset H^1_0$, with $N := \dim(\bar{V}) \equiv n_{el} - 1$.

Then the abstract formula gives:

$$g'(x, y) = g(x, y) - \begin{bmatrix} g(x_1, x_1) & \cdots & g(x_N, x_1) \\ \vdots & \ddots & \vdots \\ g(x_1, x_N) & \cdots & g(x_N, x_N) \end{bmatrix}^{-1} \begin{bmatrix} g(x, x_1) \\ \vdots \\ g(x, x_N) \end{bmatrix}$$
Structure of $g'(\cdot, \cdot)$ for 1D, linear element, $H^1_0$-optimality

- $g'(x, y) \neq 0$ only if $x$ and $y$ belong to the same element
- $g'$ is the element Green’s function $g^e$ on each $(x_{i-1}, x_i) \times (x_{i-1}, x_i)$

The structure of $g'$ for this case is known (RFB-FEM), indeed

$$V' = \bigoplus_{i=1, \ldots, n_{el}} H^1_0(x_{i-1}, x_i),$$

($u'$ is a bubble, $\bar{u}$ is nodally exact), whence

$$\mathcal{L} u' = f - \mathcal{L} \bar{u}, \text{ on } (x_{i-1}, x_i), \quad u'(x_{i-1}) = u'(x_i) = 0.$$
Structure of $g'(\cdot, \cdot)$ for 1D, linear element, $H^1_0$-optimality

- $g'(x, y) \neq 0$ only if $x$ and $y$ belong to the same element
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$$\mathcal{L}u' = f - \mathcal{L}\bar{u}, \text{ on } (x_{i-1}, x_i), \quad u'(x_{i-1}) = u'(x_i) = 0.$$
Comparison between the Green’s function $g$ (left) and the fine scale Green’s function $g'$ (right) for linear elements, $\kappa = 10^{-1}$, $\beta = 1$, $L = 4$ and a grid of $n_{el} = 4$ uniform elements.
Effect of the fine scale on the coarse scale equation:

\[
\int_0^L \int_0^L \mathcal{L}^* \bar{v}(y) g'(x, y) r(x) \, dx \, dy \\
= \sum_{i=1}^{n_{el}} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \mathcal{L}^* \bar{v}(y) g'(x, y) r(x) \, dx \, dy \\
= \sum_{i=1}^{n_{el}} \frac{\int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} g'(x, y) \, dx \, dy}{|x_i - x_{i-1}|} \int_{x_{i-1}}^{x_i} r(x) \mathcal{L}^* \bar{v}(x) \, dx \\
= \sum_{i=1}^{n_{el}} \tau_1 \int_{x_{i-1}}^{x_i} r(x) \mathcal{L}^* \bar{v}(x) \, dx \\
= \sum_{i=1}^{n_{el}} \tau_1 \int_{x_{i-1}}^{x_i} \left( \beta \frac{d}{dx} \bar{u}(x) - f(x) \right) \left( \beta \frac{d}{dx} \bar{v}(x) \right) \, dx
\]
Fine scale Green’s functions $g'$ for linears in the diffusive ($\alpha = 10^{-2}$, left) and in the advective ($\alpha = 10^2$, right) regime; $\alpha := \frac{\beta h}{2\kappa}$ is the mesh Peclét number.
Higher-order coarse scales

\[ \bar{V} = \left\{ \bar{v} \in H^1_0(0, L) \text{ such that } \bar{v}|_{(x_{i-1}, x_i)} \in \mathbb{P}_k, 0 \leq i \leq n_{el} \right \}; \]

unlike the linear case, \( \bar{V} \) now contains bubbles, and then:

\[ V' \subset \bigoplus_{i=1,\ldots,n_{el}} H^1_0(x_{i-1}, x_i). \]

Then

\[ \int_{x_{i-1}}^{x_i} \mathcal{L}u'(x)v'(x) \, dx = \int_{x_{i-1}}^{x_i} (f(x) - \mathcal{L}\bar{u}(x))v'(x) \, dx, \quad \forall v' \in V' \]

\[ \downarrow \]

\[ \mathcal{L}u' = f - \mathcal{L}\bar{u}, \text{ on } (x_{i-1}, x_i), \quad u'(x_{i-1}) = u'(x_i) = 0. \]
Higher-order coarse scales

\[ \bar{V} = \left\{ \bar{v} \in H^1_0(0, L) \text{ such that } \bar{v}|_{(x_{i-1}, x_i)} \in P_k, \ 0 \leq i \leq n_{el} \right\}; \]

unlike the linear case, \( \bar{V} \) now contains bubbles, and then:

\[ V' \subsetneq \bigoplus_{i=1,...,n_{el}} H^1_0(x_{i-1}, x_i). \]

Then

\[ \int_{x_{i-1}}^{x_i} \mathcal{L}u'(x)v'(x) \, dx = \int_{x_{i-1}}^{x_i} (f(x) - \mathcal{L}\bar{u}(x))v'(x) \, dx, \quad \forall v' \in V' \]

\[ \Downarrow \]

\[ \mathcal{L}u' = f - \mathcal{L}\bar{u}, \text{ on } (x_{i-1}, x_i), \quad u'(x_{i-1}) = u'(x_i) = 0. \]
If \( k = 2 \) then we obtain for \( 0 \leq x, y \leq h \)

\[
g'(x, y) = g^{el}(x, y) - \frac{\int_0^h g^{el}(s, y)ds \int_0^h g^{el}(x, t)dt}{\int_0^h \int_0^h g^{el}(s, t)dsdt} = I + II.
\]

Term \( I \) is the element Green’s function, and term \( II \) is:

\[
II = 2 \left( ye^{\frac{\beta h}{\kappa}} - he^{\frac{\beta y}{\kappa}} + h - y \right) \left( -x - he^{\frac{\beta h}{\kappa}} + e^{-\frac{\beta(-h+x)}{\kappa}} h + e^{\frac{\beta h}{\kappa}} x \right).
\]

\[
h \left( e^{\frac{\beta h}{\kappa}} - 1 \right) \left( he^{\frac{\beta h}{\kappa}} \beta - 2 e^{\frac{\beta h}{\kappa}} \kappa + \beta h + 2 \kappa \right).
\]
Fine scale Green's functions $g'$ for quadratics in the diffusive ($\alpha = 10^{-2}$, left) and in the advective ($\alpha = 10^2$, right) regime; $\alpha := \frac{\beta h}{2\kappa}$ is the mesh Peclét number.
For $k = 3$, for $0 \leq x \leq h$ and $0 \leq y \leq h$ we have

$$g'(x, y) = g^{el}(x, y) - \left[ \int_0^h g^{el}(s, y) ds \int_0^h s g^{el}(s, y) ds \right]$$

$$\times \left[ \int_0^h \int_0^h g^{el}(s, t) ds dt \int_0^h \int_0^h s g^{el}(s, t) ds dt \right]^{-1}$$

$$\times \left[ \int_0^h \int_0^h t g^{el}(s, t) ds dt \int_0^h \int_0^h s t g^{el}(s, t) ds dt \right]$$

$$\times \left[ \int_0^h g^{el}(x, t) dt \int_0^h t g^{el}(x, t) dt \right].$$
Fine scale Green’s functions $g'$ for cubics in the diffusive ($\alpha = 10^{-2}$, left) and in the advective ($\alpha = 10^2$, right) regime; $\alpha := \frac{\beta h}{2\kappa}$ is the mesh Peclét number.
The fine-scale effect on the coarse-scale equation is now:

\[
\int_0^L \int_0^L \mathcal{L}^* \vec{v}(y) g'(x, y) r(x) \, dx \, dy
\]

\[
= \sum_{i=1}^{n_{el}} \left( \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \mathcal{L}^* \vec{v}(y) g'(x, y) r(x) \, dx \, dy \right)
\]

\[
= \sum_{i=1}^{n_{el}} \tau_k \int_{x_{i-1}}^{x_i} \frac{d^{k-1} r(x)}{dx^{k-1}} \left( \frac{d^{k-1} \mathcal{L}^* \vec{v}(x)}{dx^{k-1}} \right) \, dx.
\]

- still local (at the element level)
- since \( g' \mathcal{P}^T = 0 \) and \( \mathcal{P} g' = 0 \), then \( g' \) is \( L^2 \)-orthogonal to \( \mathcal{P}_{k-2} \) in both \( x \) and \( y \), on each \((x_{i-1}, x_i) \times (x_{i-1}, x_i)\).
The fine-scale effect on the coarse-scale equation is now:

\[
\int_0^L \int_0^L \mathcal{L}^* \bar{v}(y) g'(x, y) r(x) \, dx \, dy
\]

\[
= \sum_{i=1}^{n_{el}} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \mathcal{L}^* \bar{v}(y) g'(x, y) r(x) \, dx \, dy
\]

\[
= \sum_{i=1}^{n_{el}} \tau_k \int_{x_{i-1}}^{x_i} \left( \frac{d^{k-1}}{dx^{k-1}} r(x) \right) \left( \frac{d^{k-1}}{dx^{k-1}} \mathcal{L}^* \bar{v}(x) \right) \, dx.
\]

- still local (at the element level)
- since \( g' \mathcal{P}^T = 0 \) and \( \mathcal{P} g' = 0 \), then \( g' \) is \( L^2 \)-orthogonal to \( \mathbb{P}_{k-2} \) in both \( x \) and \( y \), on each \((x_{i-1}, x_i) \times (x_{i-1}, x_i)\).
The $\tau_k$ are positive and of order $h^{2k-1}/\beta$ and $\alpha h^{2k-1}/\beta = h^{2k}/\kappa$ in the advective and in the diffusive regimes, respectively.
$H^1_0$ vs. $L^2$-optimality in 1D: localization of $g'$

$g'$ for the 1D problem and P1 coarse scales, $\kappa = 10^{-3}$, $\beta = 1$, $L = 1$, 16 elem.; $\mathcal{P} = H^1_0$–proj. (left) and $\mathcal{P} = L^2$–proj. (right): in the latter case, $g'$ is global and unattenuated.
Consider:

\[
\begin{cases}
-\kappa \Delta u + \beta \cdot \nabla u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

\[
\kappa = 2 \cdot 10^{-3}, \|\beta\| = 1/\sqrt{2}.
\]
domain for plotting $x \mapsto g'(x, y)$, with $y = 1/8 \cdot [4.5 \ 5.75]$:

$g$ and $g'$ are computed numerically.
domain for plotting \( x \mapsto g'(x, y) \), with \( y = 1/8 \cdot [4.5 \ 5.75] \):

\[ g \text{ and } g' \text{ are computed numerically.} \]
Plot of: $x \mapsto g(x, y)$
plot of $x \mapsto g'(x, y)$, with $\mathcal{P} \equiv H^1_0$-proj.
plot of $x \mapsto g'(x, y)$, with $\mathcal{P} \equiv L^2$-proj.
Coarse-scale component $\bar{u}$ for the model problem.
$\mathcal{P} = H^1_0$-proj. (left) and $\mathcal{P} = L^2$-proj. (right).
Coarse-scale component $\bar{u}$ for the model problem (different mesh). $\mathcal{P} = H^1_0$-proj. (left) and $\mathcal{P} = L^2$-proj. (right).
\[ \mathcal{P} = H^1_0\text{-proj. (left) and } \mathcal{P} = \text{“nodal”-proj. (right)}. \]
Comparison of $x \mapsto g(x, y)$ (left), $x \mapsto g'(x, y)$ for $P = H^1_0$-proj. (middle) and $P =$ “nodal”-proj. (right).
$H^1_0$-optimal method and SUPG.

In the case $\mathcal{P} = H^1_0$-proj., because of $PG' = 0$, we have:

$$\int_\Omega g'(x, y) \Delta \bar{v}(y) \, dy = 0, \quad \forall \bar{v} \in \bar{V}.$$ 

Therefore, the fine-scale effect on the coarse-scale eq. is:

$$\int_\Omega \int_\Omega (f(x) - \mathcal{L}\bar{u}(x)) g'(x, y) \mathcal{L}^* \bar{v}(y) \, dxdy$$

$$= -\int_\Omega \int_\Omega (f(x) - \mathcal{L}\bar{u}(x)) g'(x, y) \beta \cdot \nabla \bar{v}(y) \, dxdy.$$ 

- In 1D, where $g'$ is fully localized, this is the classical SUPG stabilization [A. N. Brooks and T. J. R. Hughes, '82]
- In 2D, $g'$ is not fully localized, and SUPG is obtained replacing $g'$ by the element-wise constant $\tau$. 
Summary

- We have derived an expression for the fine-scale Green’s function $g'$ arising in VMS: the specification of a functional $\Phi(\cdot)$, and then of a projector $P$ defining the decomposition into coarse and fine-scales, renders the problem well-posed.

- For the adv.-diff. 1D problem, we have explicitly calculated $g'$: for higher order-elements, we have obtained a new higher-order residual-based stabilization.

- For the 2D problem, we have numerically computed $g'$: it is found that the projector induced by the $H^1_0$-seminorm is associated to a $g'$ with dominantly local support, whereas the projector induced by the $L^2$-norm is not; $g'$ is only attenuated for the “nodal” interpolation.

- Further extension: non-linear fine-scale optimization.
References

