Some results on polygon and hyperpolygon spaces

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Abstract

This note briefly illustrates the wall-crossing behavior of the moduli space of polygons and of its hyperkähler analogue. Polygon and hyperpolygon spaces arise respectively as Kähler and hyperkähler reduction and their geometrical structures depend upon the choice of a level set for the (Kähler and hyperkähler respectively) moment map, i.e. depends upon the choice of a vector $r \in \mathbb{R}_+^n$, called *length vector*. For r^+ and r^- in different chambers of the moment polytope, the associated polygon spaces $M(r^+)$ and $M(r^-)$ are related by a birational map, which we characterize explicitly (cf. [12]), while the hyperpolygon spaces $X(r^+)$ and $X(r^-)$ are related by a Mukai transform (cf. [4]).

The polygon space $M(r)$ is the space of closed piece-wise linear paths in $\mathbb{R}^3$ such that the j-th step has norm r_j, modulo rotations and translations. The geometric structure of these spaces has been investigated by several points of view, see for example [1, 4, 6, 7, 8, 11, 12] and references therein.

Polygon spaces can be described by means of a symplectic quotient as follows [7]. Let $\mathcal{S}_r = \prod_{j=1}^n S^2_{r_j}$ be the product of n spheres in $\mathbb{R}^3$ of radii $r_1, \ldots, r_n$ and center the origin. The diagonal $SO(3)$-action on $\mathcal{S}_r$ is Hamiltonian with moment map $\mu : \mathcal{S}_r \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$, $\mu(e_1, \ldots, e_n) = e_1 + \cdots + e_n$. Note that an element $(e_1, \ldots, e_n) \in \mathcal{S}_r$ is in $\mu^{-1}(0)$ if and only if the polygon in $\mathbb{R}^3$ of edges $e_1, \ldots, e_n$ closes. The moduli space of polygons $M(r)$ arises then as the symplectic quotient $\mu^{-1}(0)/SO(3) =: \mathcal{S}_r /\!\!/_0 SO(3)$.

More generally, the polygon space $M(r)$ is the Kähler quiver variety associated to a star-shaped quiver $\mathcal{Q}$ as in Figure 1. A quiver, as introduced by Nakajima, is an oriented graph formed by vertices and arrows between some of the vertices. In particular, a star shaped quiver has vertex set $I \cup \{0\}$, for $I := \{1, \ldots, n\}$, and edge set $E = \{(i, 0) \mid i \in I\}$, meaning that for any $i \in I$ there is an arrow from i to 0.

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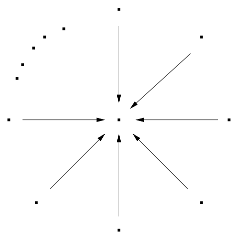

Figure 1: Star-shaped quiver

Consider the representation of $\mathcal{Q}$ obtained by taking vector spaces $V_i = \mathbb{C}$ for $i \in I$, and $V_0 = \mathbb{C}^2$. Then one gets the Kähler quiver variety associated with $\mathcal{Q}$ by performing Kähler reduction on the representation space

$$\bigoplus_{i \in I} \mathrm{Hom}(V_i, V_0) \simeq \mathbb{C}^{2n}$$

with respect to the action by conjugation of the group

$$K := \big(U(2) \times U(1)^n\big)/\Delta = (SU(2) \times U(1)^n)/\mathbb{Z}_2,$$

where $U(1)^n$ is the maximal torus of diagonal matrices in the unitary group $U(n)$ and Δ is the diagonal circle in $U(2) \times U(1)^n$. Hence the polygon space $M(r)$ is the symplectic reduction

$$\mathbb{C}^{2n} /\!\!/_{(0,r)} K$$

cf [6]. Performing reduction in stages, one obtains the product of spheres $\mathcal{S}_r$ as the quotient of $\mathbb{C}^{2n}$ by $U(1)^n$. The residual $U(2)/U_1 \simeq SO(3)$ action is the one described above, and one recovers the description of the polygon space $M(r)$ as the symplectic quotient $\mathcal{S}_r /\!\!/_0 SO(3)$.

Performing the reduction in stages in the opposite order, one first obtains the Grassmanian $Gr(2,n)$ of complex planes in $\mathbb{C}^n$ as the reduction $\mathbb{C}^{2n} /\!\!/_0 U(2)$. Then the quotient by the residual $U(1)^n/U(1)$ action on $Gr(2,n)$ yields the polygons space $M(r)$ (cf. [6, 12] for details).

The moduli space $M(r)$ is non singular if and only if the lengths are chosen so that, for each $I \subset \{1, \dots, n\}$, the quantity

$$\varepsilon_I(r) := \sum_{i \in I} r_i - \sum_{i \in I^c} r_i$$

is non-zero. Equivalently, if and only if no element in $M(r)$ is represented by a polygon contained in a line. In fact, if such a polygon existed, the

$SO(3)$-action would not be free since the stabilizer of this polygon would be the circle of rotations around the corresponding line. A length vector $r \in \mathbb{R}^n_+$ is called *generic* if $\varepsilon_I(r) \neq 0$ for any index set I. If r is generic the polygon space $M(r)$ is a smooth Kähler manifold of complex dimension $n-3$ (when not empty). The set of non-generic r is the union of finitely many walls

$$W_I := \{r \in \mathbb{R}^n_+ \mid \varepsilon_I(r) = 0\}$$

for any $I \subset \{1, \dots, n\}$. Note that an index set I and its complement I^c define the same wall. Moreover, a wall W_I separates two connected components of the set of generic r, called chambers, say Δ^+ and Δ^-, such that $\varepsilon_I(r) > 0$ for every $r \in \Delta^+$ and $\varepsilon_I(r) < 0$ for every $r \in \Delta^-$. The diffeotype of the polygon space $M(r)$ depends upon the length vector $r \in \mathbb{R}^n_+$. In fact, by the Duistermaat–Heckman Theorem [3], $M(r)$ and $M(r')$ are diffeomorphic for r and r' in the same chamber, but the diffeotype of $M(r^\pm)$ is different if r^+ and r^- are in different chambers. In particular, if r^+ and r^- lie in opposite sides of a wall W_S, then $M(r^+)$ and $M(r^-)$ are related by a blow up followed by a blow down. This is a classical result for reduced spaces (see, for example [5, 2]) and has been worked out in detail in the case of polygon spaces in [12], where the submanifolds involved in the birational transformation are characterized in terms of lower dimensional polygon spaces as follows.

For any index set $I \subset \{1, \dots n\}$, let $M_I(r)$ be the (eventually empty) submanifold of $M(r)$ of those polygons such that the edges e_i, for $i \in I$, are positive proportional to each other. Precisely

$$M_I(r) := \widetilde{M}_I(r)/SO(3)$$

where

$$\widetilde{M}_I(r) := \Big\{(e_1, \dots, e_n) \in \mathcal{S}_r \mid \sum_{j=1}^n e_j = 0,\, e_i = \lambda_i e_k, \quad \text{for all } j,k \in I \text{ and } \lambda_j \in \mathbb{R}_+\Big\}.$$

Theorem 1. *[12] Let the length vector r cross a wall W_I from Δ^+ to Δ^- as above, and let r^c be the wall crossing point. The diffeotype of the moduli space of polygons $M(r)$ changes by blowing up the submanifold $M_{I^c}(r^+) \simeq \mathbb{CP}^{|I|-2}$ and blowing down the projectivized normal bundle of $M_I(r^-) \simeq \mathbb{CP}^{n-|I|-2}$.*

The polygon spaces $M_I(r^-)$ and $M_{I^c}(r^+)$ are resolutions of the singularity in $M(r^c)$, and both are dominated by the blow up $\widetilde{M}$ of $M(r^c)$ at the singular point, with exceptional divisor $\mathbb{CP}^{|I|-2} \times \mathbb{CP}^{n-|I|-2}$.

Hyperpolygon spaces have been introduced by Konno [8] as the hyperkähler analogue of polygon spaces, i.e. as the hyperkähler quotient

$$X(r) := T^*\mathbb{C}^{2n} /\!\!/\!\!/\!\!/_{(r.0)} K.$$

For any length vector $r \in \mathbb{R}^n_+$ the space $X(r)$ is a non-compact hyperkähler manifold of complex dimension $2(n-3)$ containing the cotangent bundle of the moduli space of polygons. The wall-crossing behavior of hyperpolygon spaces is quite different from the one of polygon spaces, as studied in [4]. In fact the diffeotype of $X(r)$ does not depend on the value $(r, 0)$ of the hyperkähler moment map as long as r is generic (see [8]). Nevertheless, if r^+ and r^- are in different chambers, the hyperkähler structures on $X(r^\pm)$ are not the same. Precisely, the hyperpolygon spaces $X(r^\pm)$ are related by a Mukai transform (cf. [10] for an introduction to such transformations), which, if restricting our attention to the polygon space $M(r) \subset X(r)$ is nothing but the birational transformation as in Theorem 1. The wall-crossing analysis for hyperpolygons is based on the fact that the space $X(r)$ is isomorphic to the moduli space $\mathcal{H}(\alpha)$ of parabolic Higgs bundles (under suitable restrictions). In fact the wall-crossing problem has been solved by Thaddeus [13] in the case of moduli spaces of parabolic Higgs bundles, where the changes induced by variations of the parabolic weights are described by means of a transformation (called elementary or Mukai transform). The isomorphism between $X(\alpha)$ and $\mathcal{H}(\alpha)$ (cf. [4]) allows us to translate Thaddeus' work to the case of a hyperpolygon space.

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