Γ-convergence for high order phase field fracture: continuum and isogeometric formulations

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Abstract. We consider second order phase field functionals, in the continuum setting, and their discretization with isogeometric tensor product B-splines. We prove that these functionals, continuum and discrete, Γ-converge to a brittle fracture energy, defined in the space $GSBD^2$. In particular, in the isogeometric setting, since the projection operator is not Lagrangian (i.e., interpolatory) a special construction is neeeded in order to guarantee that recovery sequences take values in $[0,1]$; convergence holds, as expected, if $h = o(\varepsilon)$, being h the size of the physical mesh and ε the internal length in the phase field energy.

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1 Introduction

For $\varepsilon > 0$ and $\eta_\varepsilon > 0$ we consider the phase field functionals [8, 28]

$$\mathcal{F}_\varepsilon(u,v) = \int_\Omega (v^2+\eta_\varepsilon)\, W(\boldsymbol{\epsilon}(u))\, dx + \tfrac{1}{4}\, G_c \int_\Omega \varepsilon^{-1}|v-1|^2 + 2\varepsilon|\nabla v|^2 + \varepsilon^3|\Delta v|^2\, dx, \tag{1}$$

where $u \in \mathcal{U} = H^1(\Omega,\mathbb{R}^2)$ and $v \in \mathcal{V} = H^2(\Omega;[0,1])$. Here $W(\boldsymbol{\epsilon})$ is the linear elastic energy density (non-necessarily isotropic) while $G_c > 0$ is toughness. As a first result, we show that for $\eta_\varepsilon = o(\varepsilon)$ the Γ-limit [20, 12] of $\mathcal{F}_\varepsilon$ (as $\varepsilon \to 0$ and with respect to the strong L^2-topology) is the brittle fracture energy

$$F(u) = \int_{\Omega\setminus J(u)} W(\boldsymbol{\epsilon}(u))\, dx + G_c\mathcal{H}^1(J(u)), \quad \text{for } u \in GSBD^2(\Omega). \tag{2}$$

This result, in the continuum Sobolev space setting, fits into a prolific line of research, tracing back to [2] with the approximation, in the sense of Γ-convergence, of the Mumford-Shah functional

$$MS(u) = \tfrac{1}{2}\int_\Omega \mu|\nabla u|^2\, dx + G_c\mathcal{H}^1(J(u)), \quad \text{for } u \in GSBV^2(\Omega), \tag{3}$$

by means of the (first order Ambrosio-Tortorelli) functional

$$\mathcal{AT}_\varepsilon(u,v) = \tfrac{1}{2}\int_\Omega \mu(v^2+\eta_\varepsilon)|\nabla u|^2\, dx + \tfrac{1}{4}G_c\int_\Omega \varepsilon^{-1}|v-1|^2 + \varepsilon|\nabla v|^2\, dx, \tag{4}$$

where $u \in H^1(\Omega)$ and $v \in H^1(\Omega,[0,1])$. Note that here u is a scalar. After [2] several advances have been obtained through the years; among the most recent we mention [18], dealing with cohesive energies in the anti-plane setting, and [13], dealing with the approximation of the Mumford-Shah functional by means of second order energies; technically [18] is set in the spaces $GSVB$ and GBV, while [13] in $GSBV^2$.

From the technical point of view, switching from the scalar Mumford-Shah functional (3) to its vectorial counterpart (2) is not as simple as it may seem. Indeed, a complete Γ-convergence proof for first order vectorial phase field energies of the form

$$\int_\Omega (v^2+\eta_\varepsilon)\, W(\boldsymbol{\epsilon}(u))\, dx + \tfrac{1}{4}\, G_c \int_\Omega \varepsilon^{-1}|v-1|^2 + \varepsilon|\nabla v|^2\, dx, \tag{5}$$

was obtained several years after [2], first by [15] in the framework of the space SBD^2 (by means of a density result) and later by [22] in the framework of the larger space $GSBD^2$ (once $GSBD$ was introduced in [21]). Using $GSBD$ spaces, instead of SBD spaces, allows to employ displacement fields in L^1 rather than in L^∞.

Here, we employ the general framework of $GSBD^2$ spaces; as a first step, toward applications, we provide a rigorous convergence result for second order phase field approximations (1) of the Griffith energy (2). Our proof employs a classical approach for Γ-convergence, here applied in $GSBD^2$, in which the Γ-liminf inequality is obtained by slicing [21], together with a one dimensional liminf estimate, while the Γ-limsup inequality is obtained by density [15, 23], together with a regularization of the one dimensional optimal profile. We remark that the slicing technique is made possible by the definition itself of $GSBD$ fields. Since the structure of the convergence result is classical and based on previous results, we spent some effort in providing short and simple proofs.

It should be noted that in real life application for fracture mechanics it is often necessary to distinguish between traction and compression regimes; this is usually implemented by means of an additive decomposition of the elastic phase field energy energy, of the form

$$\int_\Omega (v^2+\eta_\varepsilon)W_+(\boldsymbol{\epsilon}(u)) + W_-(\boldsymbol{\epsilon}(u))\,dx,$$

where W_+ takes into account traction and W_- compression. In the literature there are different choices for W_- and W_+ either in terms of principal strains or in terms of volumetric and deviatoric parts of the strain, see e.g. [3, 30] or [1]. A characterization of the Γ-limit with this type of elastic bulk energies, in terms of incompenetration on the crack, has been recently proved [16] in the two dimensional setting for first order functionals and bounded SBD^2 fields.

Our second result is instead inspired by applications to fracture simulation, and above all by [8, 28]. We consider the discretizations

$$\mathcal{F}_{\varepsilon,h}(u_h,v_h) = \int_\Omega (v_h^2+\eta)\,W(\boldsymbol{\epsilon}(u_h))\,dx + \int_\Omega \varepsilon^{-1}|v_h-1|^2 + 2\varepsilon|\nabla v_h|^2 + \varepsilon^3|\Delta v_h|^2\,dx \tag{6}$$

obtained by restriction of the functionals $\mathcal{F}_\varepsilon$ to discrete spaces $\mathcal{U}_h \subset H^1(\Omega,\mathbb{R}^2)$ and $\mathcal{V}_h \subset H^2(\Omega;[0,1])$ (corresponding to $\mathcal{U}$ and $\mathcal{V}$ respectively) of isogeometric tensorial B-splines, which are very natural and efficient for high order problems. In the discrete setting, we show that for $\eta_\varepsilon = o(\varepsilon)$ and $h = o(\varepsilon)$ (the element size) the Γ-limit of $\mathcal{F}_{\varepsilon,h}$ is again Griffith's functional F in $GSBD^2$, i.e., (2). Comparing with the continuum setting, the discrete Γ-limsup inequality requires to take into account the fact that "interpolation" in the space of tensor product B-splines and C^1 elements does not preserve L^∞-bounds; as a consequence the projection v_h of the continuum phase-field profile v, which is a natural candidate for the recovery sequence, may not take value in $[0,1]$. This technical issue is by-passed using an *ad hoc* local modification of v_h, at the price of introducing an additional approximation error, vanishing in the limit for $\varepsilon \to 0$. We stress the fact that the condition $h = o(\varepsilon)$ is necessary and natural in order to guarantee a good enough approximation v_h of the field function v in the transition layer, which is indeed of order ε. In the applications this condition is often guaranteed by h-adaptive mesh refinement in a neighbourhood of the crack tip. Theoretically, it appears also in [7], where it was proven that the Mumford-Shah functional can be approximated, again in the sense of Γ-convergence, by a family of Ambrosio-Tortorelli functionals defined in spaces of C^0 finite elements, for both u and v. Our result, with minor modifications, holds also for C^1 finite element spaces.

We conclude this introduction with some comments about applications and evolutions based on phase-field functionals. Initially, phase-field energies like (4) have been used in image segmentation problems, e.g. [29], and later, after [10], they spread in fracture mechanics, see e.g., [30, 26, 34, 8, 28, 31, 24], the book [11] and the review [1]. Γ-convergence provides the mathematical framework to prove that phase-field energies are indeed regularized approximation of free discontinuity energies. On the other hand, applications in fracture mechanics require, beside energy, an evolution law which governs the propagation of the crack. In the applications, this is usually given by incremental (in time) problems. For phase field fracture a very efficient way to implementing such incremental problems is the alternate minimization, or staggered, scheme [10]: at each time step, the updated configuration is obtained by a

sequence of configurations. Ideally, in the limit, the sequence converge to an equilibrium configuration of the energy, in practice, iterations are arrested by some (suitable) criterion. In this respect, second order functionals proved to be very efficient, indeed they converge to an approximated equlibrium point much faster than first order problems (see e.g., [8, Figure 10 and Tables 4, 5] and similarly [13, Table 1]). The time-continuous limit of these evolutions, obtained by time discretization and alternate minimization schemes, have been studied for first order phase-field functionals in [27] (for the dynamic case) and in [32, 25] (for the quasi-static case).

Contents

2 Setting and statement of the Γ-convergence results

2.1 Continuum setting

We assume that the reference domain $\Omega \subset \mathbb{R}^2$ is open, bounded and connected. The space of admissible *continuum displacements* is given by $\mathcal{U} = H^1(\Omega, \mathbb{R}^2)$ while the space of admissible *phase-field functions* is $\mathcal{V} = H^2(\Omega, [0,1])$. The space of admissible *discontinuous displacements* is instead provided by $GSBD^2(\Omega)$ (see Appendix A for the definition and the basic properties of this space and to [21] for the original work).

For technical reasons, natural in Γ-convergence, we will employ the "extended" functionals $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ defined in $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$ and given by

$$\mathcal{F}_\varepsilon(u,v) = \begin{cases} \displaystyle\int_\Omega (v^2 + \eta_\varepsilon)\, W(\boldsymbol{\epsilon}(u))\, dx + \int_\Omega \varepsilon^{-1}|v-1|^2 + 2\varepsilon|\nabla v|^2 + \varepsilon^3|\Delta v|^2\, dx & (u,v) \in \mathcal{U} \times \mathcal{V}, \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

$$\mathcal{F}(u,v) = \begin{cases} \displaystyle\int_{\Omega \setminus J(u)} W(\boldsymbol{\epsilon}(u))\, dx + 4\mathcal{H}^1(J(u)) & u \in GSBD^2(\Omega) \text{ and } v = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{8}$$

We will assume that W is coercive and continuous in $\mathbb{R}^{2\times 2}_{sym}$, i.e that $c_1|E|^2 \le W(E) \le c_2|E|^2$ for $c_i > 0$ and every $E \in \mathbb{R}^{2\times 2}_{sym}$.

Remark 2.1 The choice of L^2 in the definition of $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ is due to the fact that Γ-convergence will be proven with respect to the L^2-norm, which seems general enough for our applications. More general choices could be of theoretical interest: for instance, taking full advantage of the generality of $GSBD$ spaces, the functionals $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ could be defined in the metric space of measurable vector fields endowed with convergence in measure.

Our main result in the continuum setting is stated in the following Theorem.

Theorem 2.2 *For $\eta_\varepsilon = o(\varepsilon)$ the functionals $\mathcal{F}_\varepsilon$ Γ-converge to $\mathcal{F}$ (as $\varepsilon \to 0$) with respect to the strong topology of $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$.*

Remark 2.3 Analogous converce results hold with volume loads in L^2 and with Dirichlet boundary conditions for the displacement field. As a standard byproduct of Γ-convergence we have, upon compactness, the strong convergence of minimizer.

2.2 Isogeometric quadratic tensor product B-splines

We follow the assumptions and notation of [5] (see also [6]). Let $(0,1)^2$ be the (parametrizing) patch and let $\mathcal{Q}_h = \{Q\}$ be a family of uniformly shape regular meshes of elements Q with diameter $h_Q \le h$. Shape regularity means that the ratio between the length of the edges and the diameter is bounded (from below) uniformly with respect to Q and $\mathcal{Q}_h$. Let $\mathbf{F} : (0,1)^2 \to \Omega$ be the parametrization map for the physical domain Ω and denote by $K = \mathbf{F}(Q)$ the elements of the physical mesh $\mathcal{K}_h = \{K\}$. We assume that globally (from $(0,1)^2$ to Ω) the map $\mathbf{F}$ is invertible and that locally (from Q to $K = \mathbf{F}(Q)$) it is a diffeomorphism of class $W^{2,\infty}$. As a consequence the family $\mathcal{K}_h$ is still shape regular and $h_K \le Ch$ uniformly with respect to K and $\mathcal{K}_h$.

We will not enter into the details about the generation of the spaces of quadratic (tensor product) B-splines on $\mathcal{K}_h$ since it is not crucial for our analysis, the reader will find a brief description in [5] and a comprehensive treatise in [33]. We will denote by $\mathcal{U}_h$ and $\mathcal{V}_h$ (on the physical meshes $\mathcal{K}_h$) the discrete spaces of B-splines for the displacement field and the phase-field function respectively. It is important to remark that, in general, functions $v_h \in \mathcal{V}_h$ are allowed to take any real value and thus they may not satisfy the constraint $0 \le v_h \le 1$.

We denote by $\tilde{K} \subset \Omega$ the extended support of $K \in \mathcal{K}_h$, i.e. the union of the supports of the basis functions (of both $\mathcal{U}_h$ and $\mathcal{V}_h$) whose support intersects K. We remark that $\tilde{K} \subset \Omega$ and that $\tilde{K} \subset \{\mathrm{dist}(x, K) \le \tilde{C}h\}$ for $\tilde{C} > 0$ independent of K and $\mathcal{K}_h$. By [5, Theorem 3.1] we know that there exists a linear approximation operator $\Pi_{\mathcal{U}_h} : H^2(\Omega, \mathbb{R}^2) \to \mathcal{U}_h$ such that for every $0 \le k \le l \le 2$ and every element K of $\mathcal{K}_h$ it holds

$$|u - \Pi_{\mathcal{U}_h} u|_{H^k(K,\mathbb{R}^2)} \le Ch^{l-k} \|u\|_{H^l(\tilde{K},\mathbb{R}^2)}. \tag{9}$$

Similarly, there exists a linear approximation operator $\Pi_{\mathcal{V}_h} : H^3(\Omega) \to \mathcal{V}_h$ such that for every $0 \le k \le l \le 3$ and every element K of $\mathcal{K}_h$ it holds

$$|v - \Pi_{\mathcal{V}_h} v|_{H^k(K)} \le Ch^{l-k} \|v\|_{H^l(\tilde{K})}. \tag{10}$$

Note that in the previous estimate the norm in the right hand side is evaluated in the extended element $\tilde{K}$. Clearly, from local estimates we get also the global ones:

$$|u - \Pi_{\mathcal{U}_h} u|_{H^k(\Omega,\mathbb{R}^2)} \le Ch^{l-k} \|u\|_{H^l(\Omega,\mathbb{R}^2)}. \qquad |v - \Pi_{\mathcal{V}_h} v|_{H^k(\Omega)} \le Ch^{l-k} \|v\|_{H^l(\Omega)}. \tag{11}$$

Remark 2.4 Note that, even if v takes values in $[0,1]$, in general $\Pi_{\mathcal{V}_h} v$ does not take values in $[0,1]$ even if the basis functions do. Indeed, high order "interpolation" in spline or polynomial spaces is not Lagrangian (i.e., interpolatory), it is rather a projection operator which does not preserve ordering and L^∞-bounds (see for instance [33, §12]). A similar issue occurs also for C^1 finite elements. In §6 we will provide an "ad hoc" local modification of the projection $\Pi_{\mathcal{V}_h} v$ (for a special function v) taking values in $[0,1]$.

Since the elements are (uniformly) affine equivalent by a simple change of variable and by Sobolev embedding it is immediate to see that there exists a constant $C > 0$ (independent of $h > 0$) such that

$$\|z\|_{L^\infty(K)} \le C\big(h^{-2}\|z\|^2_{L^2(K)} + |z|^2_{H^1(K)} + h^2|z|^2_{H^2(K)}\big)^{1/2} \tag{12}$$

for every $K \in \mathcal{K}_h$ and every $z \in H^2(K)$. Note that this estimate holds for every function in $H^2(\Omega)$ and not only for B-splines.

At this point we can introduce the discrete functionals $\mathcal{F}_{\varepsilon,h}$ given by

$$\mathcal{F}_{\varepsilon,h}(u_h, v_h) = \int_\Omega (v_h^2 + \eta)\, W(\boldsymbol{\epsilon}(u_h))\, dx + \int_\Omega \varepsilon^{-1}|v_h - 1|^2 + 2\varepsilon|\nabla v_h|^2 + \varepsilon^3|\Delta v_h|^2\, dx \tag{13}$$

if $(u_h, v_h) \in \mathcal{U}_h \times \mathcal{V}_h$ with $0 \le v_h \le 1$ and by $\mathcal{F}_{\varepsilon,h}(u_h, v_h) = +\infty$ otherwise in $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$. Note that $\mathcal{F}_{\varepsilon,h}$ is just the restriction of the functional $\mathcal{F}_\varepsilon$ to $\mathcal{U}_h \times \mathcal{V}_h$. The convergence result is the following.

Theorem 2.5 *If $\eta = o(\varepsilon)$ and $h = o(\varepsilon)$ the functionals $\mathcal{F}_{\varepsilon,h}$ Γ-converge to $\mathcal{F}$ (as $\varepsilon \to 0$) with respect to the strong topology of $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$.*

The proof of the previous Theorem will follow from Proposition 4.1 and Proposition 5.2.

Remark 2.6 The condition $h = o(\varepsilon)$, which appears also in [7], allows to have an accurate approximation of the transition of the phase-field variable; in practice it should be satisfied only in a neighborhood on the discontinuity set and often is obtained by local h-refinement, e.g. [4, 9, 14].

2.3 Finite Elements

The proofs contained in § 6 have been written in the context of isogeometric tensor product B-splines, because this is the setting of [8] and because it requires some special care when dealing with extended supports $\tilde{K}$. Actually, a convergence result like Theorem 2.5 holds, as a byproduct, also for finite element spaces (roughly speaking, by replacing $\tilde{K}$ with K). More precisely, let $\mathcal{K}_h = \{K\}$ be a regular family of (triangular or quadrilateral) affine equivalent finite elements in the physical domain Ω. Denote again by $\mathcal{U}_h \subset H^1(\Omega, \mathbb{R}^2)$ and by $\mathcal{V}_h \subset H^2(\Omega)$ the finite element spaces for the displacement fields and phase field functions respectively. We assume also that there exists a linear approximation operator $\Pi_{\mathcal{U}_h} : H^2(\Omega, \mathbb{R}^2) \to \mathcal{U}_h$ such that for every $0 \le k \le l \le 2$ and every element K of $\mathcal{K}_h$ it holds

$$|u - \Pi_{\mathcal{U}_h} u|_{H^k(K,\mathbb{R}^2)} \le C h^{l-k} \|u\|_{H^l(K,\mathbb{R}^2)} \tag{14}$$

and that there exists a linear approximation operator $\Pi_{\mathcal{V}_h} : H^3(\Omega) \to \mathcal{V}_h$ such that for every $0 \le k \le l \le 3$ and every element K of $\mathcal{K}_h$ it holds

$$|v - \Pi_{\mathcal{V}_h} v|_{H^k(K)} \le C h^{l-k} \|v\|_{H^l(K)}. \tag{15}$$

We remark that the condition $\mathcal{V}_h \subset H^2(\Omega)$ requires continuity of the gradient across element boundaries, i.e. C^1 finite elements; we refer to the classic book [17] for several examples of elements, for forth order problems, enjoying this property together with the previous interpolation estimates. Once again, these elements are not Lagrangian and thus interpolation does not preserve, in general, L^∞-bounds.

The discrete functionals $\mathcal{F}_{\varepsilon,h}$ is then defined as above by

$$\mathcal{F}_{\varepsilon,h}(u_h, v_h) = \int_\Omega (v_h^2 + \eta)\, W(\boldsymbol{\epsilon}(u_h))\, dx + \int_\Omega \varepsilon^{-1}|v_h - 1|^2 + 2\varepsilon|\nabla v_h|^2 + \varepsilon^3|\Delta v_h|^2\, dx$$

if $(u_h, v_h) \in \mathcal{U}_h \times \mathcal{V}_h$ with $0 \le v_h \le 1$ and by $\mathcal{F}_{\varepsilon,h}(u_h, v_h) = +\infty$ otherwise in $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$. Note that $\mathcal{F}_{\varepsilon,h}$ is just the restriction of the functional $\mathcal{F}_\varepsilon$ to $\mathcal{U}_h \times \mathcal{V}_h$.

Theorem 2.7 *If $\eta = o(\varepsilon)$ and $h = o(\varepsilon)$ the functionals $\mathcal{F}_{\varepsilon,h}$ Γ-converge to $\mathcal{F}$ (as $\varepsilon \to 0$) with respect to the strong topology of $L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega)$.*

3 Preliminary one dimensional estimates

For $R \in (0, +\infty]$ consider the functionals $\mathcal{J}_R : H^2(0, R) \to [0, +\infty)$ given by

$$\mathcal{J}_R(w) = \int_{(0,R)} w^2 + 2|w'|^2 + |w''|^2\, dr. \tag{16}$$

Lemma 3.1 *Let $w_\infty(r) = e^{-r}(1 + r)$. Then*

$$w_\infty \in \operatorname{argmin}\{\mathcal{J}_\infty(w) : w(0) = 1,\ w'(0) = 0\} \quad \textit{and} \quad \mathcal{J}_\infty(w_\infty) = 2. \tag{17}$$

Proof. The Euler-Lagrange equation for $\mathcal{J}_\infty$ reads $w^{(4)} + 2w'' + w = 0$ whose solutions are of the form $w(r) = e^r(C_1 + C_2 r) + e^{-r}(C_3 + C_4 r)$. Considering the boundary conditions, the unique solution in $H^2(0, +\infty)$ is given by w_∞. An explicit computation gives $\mathcal{J}_\infty(w_\infty) = 2$. ■

Note that w_∞ belongs to $W^{m,\infty}(0, +\infty) \cap C^\infty(0, +\infty)$, for m arbitrarily large, and that w_∞ is monotone decreasing with $\lim_{r\to+\infty} w_\infty(r) = 0$, in particular $0 \le w_\infty \le 1$.

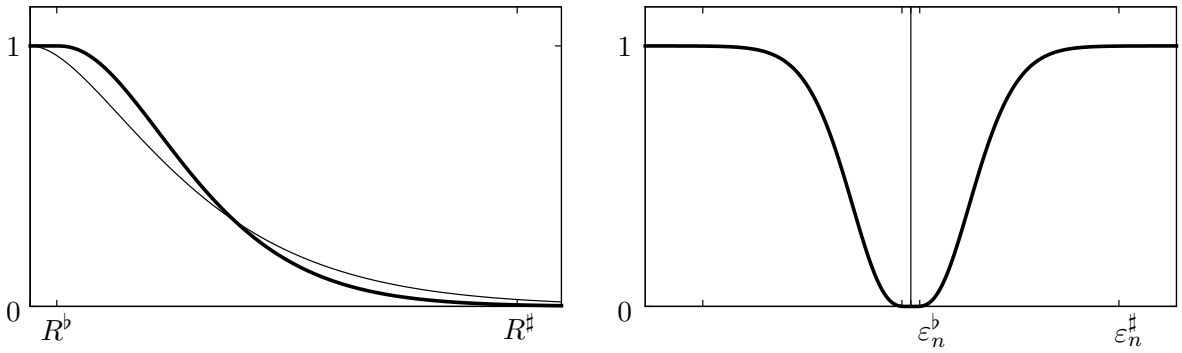

Figure 1: Left: profile of the functions w_∞ from Lemma 3.1 and w (solid) from Lemma 3.2. Right: profile of a function z_n from Lemma 3.4.

Lemma 3.2 *For $\delta > 0$ there exists $w \in W^{m,\infty}(0, +\infty) \cap C^\infty(0, +\infty)$, for m arbitrarily large, with $0 \le w \le 1$, $w = 1$ in $(0, R^\flat)$ and $w = 0$ in $(R^\sharp, +\infty)$, for $0 < R^\flat < 1 < R^\sharp$, and such that $\mathcal{J}_{R^\sharp}(w) = \mathcal{J}_\infty(w) < 2 + \delta$.*

Proof. Let ϕ be a smooth function in the real line with $\phi(r) = 1$ for $r < -1$, $\phi(r) = 0$ for $r > 0$ and $0 \le \phi \le 1$. For $0 < r_k \to 0^+$ and $R_k \to +\infty$ let

$$w_k(r) = \begin{cases} 1 & r \le r_k \\ w_\infty(r - r_k)\phi(r - R_k) & \text{otherwise.} \end{cases} \tag{18}$$

Note that $w_k \in H^2(0, +\infty) \cap C^1(0, +\infty)$ because $w_\infty(r) \approx 1 - r^2$ for $r \approx 0$. Since $w_k = 0$ in $(R_k, +\infty)$ it is an admissible competitor in (17), thus we have $\mathcal{J}_{R_k}(w_k) = \mathcal{J}_\infty(w_k) \ge \mathcal{J}_\infty(w_\infty)$. It is easy to check that $w_k \to w_\infty$ strongly in $H^2(0, +\infty)$ and thus $\mathcal{J}_\infty(w_k) \to \mathcal{J}_\infty(w_\infty)$, by continuity of $\mathcal{J}_\infty$.

For $0 < s_n \to 0^+$ let $\rho_n(r) = \hat{\rho}(r/s_n)/s_n$ be a sequence of smooth mollifiers. Denote $w_{k,n} = w_k * \rho_n$. Clearly $w_{k,n} \to w_k$ in $H^2(0, +\infty)$ and thus $\mathcal{J}_\infty(w_{k,n}) \to \mathcal{J}_\infty(w_k)$. Moreover, $w'_{k,n} = w'_k * \rho_n$ is continuous with compact support. As a consequence $w_{k,n} \in W^{1,\infty}(0, +\infty)$. The same argument holds for the derivative of any order, hence $w_{n,k} \in W^{m,\infty}(0, +\infty) \cap C^\infty(0, +\infty)$, for m arbitrarily large.

It is then sufficient to choose $w = w_{k,n}$ for k and n sufficiently large. ■

Lemma 3.3 *For $R_n \to +\infty$ let $w_n \in H^2(0, R_n)$ such that*

$$\lim_{n\to+\infty} w_n(0) = 1, \quad w'_n(0) = 0, \quad \lim_{n\to+\infty} w_n(R_n) = 0, \quad \lim_{n\to+\infty} w'_n(R_n) = 0.$$

Then $\liminf_{n\to+\infty} \mathcal{J}_{R_n}(w_n) \ge \mathcal{J}_\infty(w_\infty) = 2$.

Proof. By classical results on Sobolev function, there exists $C > 0$ and a lifting $z_n \in H^2(0, +\infty)$ with $z_n(0) = w_n(R_n)$, $z'_n(0) = w'_n(R_n)$ and $\|z_n\|_{H^2(0,+\infty)} \le C(|w_n(R_n)| + |w'_n(R_n)|)$. Hence

$$Z_n = \int_{(0,+\infty)} z_n^2 + 2|z'_n|^2 + |z''_n|^2 \, dr \to 0.$$

Let $\tilde{w}_n \in H^2(0, +\infty)$ be the extension of w_n given by $\tilde{w}_n(r) = z_n(r - R_n)$ for $r \in (R_n, +\infty)$. Denote $\lambda_n = 1/w_n(0)$. Clearly $\lambda_n \tilde{w}_n(0) = 1$ and $\lambda_n \tilde{w}'_n(0) = 0$, moreover

$$\mathcal{J}_\infty(\lambda_n \tilde{w}_n) = \lambda_n^2 \mathcal{J}_\infty(\tilde{w}_n) = \lambda_n^2(\mathcal{J}_{R_n}(w_n) + Z_n).$$

As $\lambda_n \to 1$ and $Z_n \to 0$ we have, by minimality of w_∞,

$$\liminf_{n\to+\infty} \mathcal{J}_{R_n}(w_n) = \liminf_{n\to+\infty} \lambda_n^2(\mathcal{J}_{R_n}(w_n) + Z_n) \geq \liminf_{n\to+\infty} \mathcal{J}_\infty(\lambda_n \tilde{w}_n) \geq \mathcal{J}_\infty(w_\infty),$$

which concludes the proof. ■

3.1 An approximate limsup inequality

Lemma 3.4 *For $\delta > 0$ let w be the function provided by Lemma 3.2. Define $z_n(s) = 1 - w(|s|/\varepsilon_n)$. Then $z_n \in C^\infty(\mathbb{R})$, $z_n \to 1$ in $L^2_{loc}(\mathbb{R})$ and*

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \varepsilon_n^{-1}|z_n - 1|^2 + 2\varepsilon_n|z_n'|^2 + \varepsilon_n^3|z_n''|^2\, ds < 4 + 2\delta.$$

Moreover, there exists $C > 0$ such that

$$\int_{\mathbb{R}} \varepsilon_n^5|z_n^{(3)}|^2\, ds \leq C \quad \text{for every } n \in \mathbb{N}.$$

Note that $0 \leq z_n \leq 1$ and that $z_n(s) = 0$ for $|s| \leq \varepsilon_n^\flat$ and $z_n(s) = 1$ for $|s| \geq \varepsilon_n^\sharp$, where $\varepsilon_n^\flat = \varepsilon_n R^\flat$ and $\varepsilon_n^\sharp = \varepsilon_n R^\sharp$.

Proof. Since $w = 0$ in $(R^\sharp, +\infty)$ it follows that $z_n(s) = 1$ for $|s| > \varepsilon_n R^\sharp = \varepsilon_n^\sharp$. Hence $z_n \to 1$ in $L^2_{loc}(\mathbb{R})$. Since $w = 1$ in $(0, R^\flat)$ and $w \in C^\infty(0, +\infty)$ we have $z_n(s) = 0$ for $|s| \leq \varepsilon_n R^\flat = \varepsilon_n^\flat$ and $z_n \in C^\infty(\mathbb{R})$. The change of variable $s = \varepsilon_n r$ yields

$$\int_{(0,+\infty)} \varepsilon_n^{-1}|z_n - 1|^2 + 2\varepsilon_n|z_n'|^2 + \varepsilon_n^3|z_n''|^2\, ds = \int_{(0,+\infty)} |w|^2 + 2|w'|^2 + |w''|^2\, dr = \mathcal{J}_\infty(w) < 2 + \delta$$

which provides the first estimate. The estimate for the third derivative can be derived in the same way by a change of variable since $w^{(3)} \in L^\infty(0, +\infty)$ and it has compact support. ■

3.2 A liminf inequality

Let $I = (a, b)$, with $a, b \in \mathbb{R}$, and let $\mathcal{I}_\varepsilon : L^2(I) \times L^2(I) \to [0, +\infty]$ be defined by

$$\mathcal{I}_\varepsilon(u, z) = \begin{cases} \frac{1}{2}\displaystyle\int_I (z^2 + \eta_\varepsilon)|u'|^2\, ds + \int_I \varepsilon^{-1}|z - 1|^2 + 2\varepsilon|z'|^2 + \varepsilon^3|z''|^2\, ds & \text{if } (u, z) \in H^1(I) \times H^2(I) \\ +\infty & \text{otherwise.} \end{cases}$$

Considering a sequence $\varepsilon_n \to 0$ we will denote $\mathcal{I}_n = \mathcal{I}_{\varepsilon_n}$.

Lemma 3.5 *If $(u_n, z_n) \to (u, z)$ in $L^2(I) \times L^2(I)$ and $\liminf_{n\to+\infty} \mathcal{I}_n(u_n, z_n) < +\infty$ then $u \in SBV^2(I)$ and*

$$4\,\#(J(u)) \leq \liminf_{n\to+\infty} \int_I \varepsilon_n^{-1}|z_n - 1|^2 + 2\varepsilon_n|z_n'|^2 + \varepsilon_n^3|z_n''|^2\, ds. \tag{19}$$

Proof. Neglecting the term $|z_n''|^2$ we get

$$\mathcal{I}_n(u_n, z_n) \geq \tfrac{1}{2}\int_I (z_n^2 + \eta_n)|u_n'|^2\, ds + \int_I \varepsilon_n^{-1}|z_n - 1|^2 + \varepsilon_n|z_n'|^2\, ds = AT_n(u_n, z_n),$$

where the right hand side is a one dimensional Ambrosio-Tortorelli [2] functional. Invoking for instance [12, Theorem 3.15] we get that $z = 1$ a.e. in I and that $u \in SBV^2(I)$ with $\#(J(u)) < +\infty$. Let $J(u) = \{s_j\}$. For $\delta \ll 1$ consider the disjoint intervals $I_j^\delta = (s_j - \delta, s_j + \delta) \subset (a, b)$. Writing

$$\mathcal{I}_n(u_n, z_n) \geq \sum_j \int_{I_j^\delta} \varepsilon_n^{-1}|z_n - 1|^2 + 2\varepsilon_n|z_n'|^2 + \varepsilon_n^3|z_n''|^2\, ds$$

we will check that

$$\liminf_{n\to+\infty}\int_{I_j^\delta}\varepsilon_n^{-1}|z_n-1|^2+2\varepsilon_n|z_n'|^2+\varepsilon_n^3|z_n''|^2\,ds\geq 4\quad\text{for every } j, \tag{20}$$

from which (19) follows. As a preliminary step, we extract a subsequence (not relabelled) such that $z_n\to 1$ a.e. in I and such that each liminf in (20) is actually a limit.

Fix an interval $I_j^\delta=(s_j-\delta,s_j+\delta)$. Assume, without loss of generality, that $s_j=0$ and denote $I_j^\delta=[-\delta,\delta]$. Fix $\delta^\flat$ and $\delta^\sharp$ (independent of n) with $0<\delta^\flat<\delta^\sharp<\delta$ such that $z_n(\pm\delta^\flat)\to 1$ and $z_n(\pm\delta^\sharp)\to 1$. First, we show that there exist a subsequence (not relabelled) and a couple of points, $s_n^\flat\in(-\delta^\flat,\delta^\flat)$ and $s_n^\sharp\in(\delta^\sharp,\delta)$ such that

$$z_n(s_n^\flat)\to 0,\quad z_n'(s_n^\flat)=0,\quad z_n(s_n^\sharp)\to 1,\quad |z_n'(s_n^\sharp)|\leq 1. \tag{21}$$

Let $s_n^\flat\in\operatorname{argmin}\{z_n(s):s\in[-\delta^\flat,\delta^\flat]\}$. Let us see that $z_n(s_n^\flat)\to 0$. Assume (by contradiction) that there exists a subsequence (not relabelled) such that $\min\{z_n(s):s\in[-\delta^\flat,\delta^\flat]\}\geq C>0$ for every $n\in\mathbb{N}$. Then

$$\mathcal{I}_n(u_n,z_n)\geq\tfrac12\int_{(-\delta^\flat,\delta^\flat)}(z_n^2+\eta_n)|u_n'|^2\,ds\geq C\int_{(-\delta^\flat,\delta^\flat)}|u_n'|^2\,ds.$$

Since $\mathcal{I}_n(u_n,z_n)$ is bounded it follows that u_n is bounded in $H^1(-\delta^\flat,\delta^\flat)$. As consequence its limit u belongs to $H^1(-\delta^\flat,\delta^\flat)$, which contradicts the fact that $0=s_j\in S_u$. Since $z_n(\pm\delta^\flat)\to 1$ the minimizer $s_n^\flat$ belongs to $(-\delta^\flat,\delta^\flat)$, thus $z_n'(s_n^\flat)=0$.

For $\tau>0$ consider the open set $E_n^\tau=\{s\in[\delta^\sharp,\delta]:1-\tau<z_n(s)\}$. Since $z_n\to 1$ in measure we have $|[\delta^\sharp,\delta]\setminus E_n^\tau|\to 0$. Assume (by contradiction) that $z_n'>1$ in E_n^τ. Then, if $s^*\in E_n^\tau$ we have $z_n(s)\geq z_n(s^*)+(s-s^*)$ for $s\geq s^*$ and thus the upper bound $z_n\leq 1$ would be violated. An analogous argument applies for $z_n'<-1$. In all the other cases, by the continuity of z_n', there exists a point $s_n^\sharp$ in E_n^τ with $|z_n'(s_n^\sharp)|\leq 1$. Choosing $\tau_n\to 0^+$ provides the required sequence.

Define the rescaled functions $w_n(r)=1-z_n(\varepsilon_n r+s_n^\flat)$ and let $R_n=(s_n^\sharp-s_n^\flat)/\varepsilon_n\geq(\delta^\sharp-\delta^\flat)/\varepsilon_n$. Then $R_n\to+\infty$ and by (21)

$$\begin{aligned}w_n(0)&=(1-z_n(s_n^\flat))\to 1,\quad w_n'(0)=-\varepsilon_n z_n'(s_n^\flat)=0,\\ w_n(R_n)&=(1-z_n(s_n^\sharp))\to 0,\quad w_n'(R_n)=-\varepsilon_n z_n'(s_n^\sharp)\to 0.\end{aligned}$$

By the change of variable $s=\varepsilon_n r+s_n^\flat$ we have

$$\int_{(s_n^\flat,\,s_n^\sharp)}\varepsilon_n^{-1}|z_n-1|^2+2\varepsilon_n|z_n'|^2+\varepsilon_n^3|z_n''|^2\,ds=\int_{(0,\,R_n)}|w_n|^2+2|w_n'|^2+|w_n''|^2\,dr=\mathcal{J}_{R_n}(w_n).$$

Then by Lemma 3.3 we get $\liminf_{n\to+\infty}\mathcal{J}_{R_n}(w_n)\geq 2$.

By symmetry, we can get the same estimate in the interval $(-\delta,s_n^\flat)$ and (20) is proved. ■

4 Γ-liminf inequality

Let $\varepsilon_n\to 0^+$ and $\eta_n=o(\varepsilon_n)$. For simplicity we will employ the notation $\mathcal{F}_n$ for $\mathcal{F}_{\varepsilon_n}$. The Γ-liminf inequality is based on slicing and on the following standard property, employed also in [13]: if $v\in H_0^2(\Omega)$ then

$$\int_\Omega|\Delta v|^2\,dx=\int_\Omega|D^2v|^2\,dx, \tag{22}$$

where $|\cdot|$ in the right hand side denotes Frobenius norm.

Proposition 4.1 *Let $(u_n,v_n)\in\mathcal{U}\times\mathcal{V}$ such that $u_n\to u$ in $L^2(\Omega,\mathbb{R}^2)$ and $v_n\to v$ in $L^2(\Omega)$. If $\mathcal{F}_n(u_n,v_n)$ is uniformly bounded then $v=1$ a.e. in Ω, $u\in GSBD^2(\Omega)$ and*

$$\liminf_{n\to+\infty}\mathcal{F}_n(u_n,v_n)\geq\mathcal{F}(u,v). \tag{23}$$

Proof. Using the first order bound

$$\mathcal{F}_n(u_n, v_n) \geq \int_\Omega (v_n^2 + \eta_n) W(\boldsymbol{\epsilon}(u_n))\, dx + \int_\Omega \varepsilon_n^{-1}|v_n - 1|^2 + \varepsilon_n |\nabla v_n|^2\, dx$$

and then arguing as in [23, Theorem 4.3] we get that $u \in GSBD^2(\Omega)$ and that $(v_n^2 + \eta_n)^{1/2}\boldsymbol{\epsilon}(u_n) \rightharpoonup \boldsymbol{\epsilon}(u)$ in $L^2(\Omega, \mathbb{R}^{d\times d}_{sym})$; thus

$$\liminf_{n\to+\infty} \int_\Omega (v_n^2 + \eta_n)\, W(\boldsymbol{\epsilon}(u_n))\, dx = \liminf_{n\to+\infty} \int_\Omega W\big((v_n^2 + \eta_n)^{1/2}\boldsymbol{\epsilon}(u_n)\big)\, dx \geq \int_{\Omega\setminus J(u)} W(\boldsymbol{\epsilon}(u))\, dx. \tag{24}$$

To get the right bound for the jump we need also the second derivatives. To this end, first we replace (locally) the Laplacian with the norm of Hessian, introducing a small error, vanishing in the limit. Given an open set $A \subset\subset \Omega$ let $\phi \in C_c^\infty(\Omega)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ on A. Using Young's inequality $(a+b)^2 \leq (1+\delta)\, a^2 + (1+\delta^{-1})\, b^2$ for $\delta > 0$, we can write

$$|\Delta(v\phi)|^2 = |(\Delta v)\phi + 2\nabla v \cdot \nabla\phi + v(\Delta\phi)|^2 \leq (1+\delta)|\Delta v|^2\phi^2 + C_\phi(1+\delta^{-1})(|\nabla v|^2 + |v|^2),$$

where C_ϕ depends only on ϕ. Hence, for $C_{\delta,\phi} = C_\phi(1+\delta)^{-1}(1+\delta^{-1})$, being $0 \leq v \leq 1$ we have

$$\int_\Omega |\Delta v|^2\, dx \geq \int_\Omega |\Delta v|^2\phi^2\, dx \geq (1+\delta)^{-1}\int_\Omega |\Delta(v\phi)|^2\, dx - C_{\delta,\phi}\int_\Omega |\nabla v|^2 + 1\, dx.$$

Moreover, by (22)

$$\int_\Omega |\Delta(v\phi)|^2\, dx = \int_\Omega |D^2(v\phi)|^2\, dx \geq \int_A |D^2(v\phi)|^2\, dx = \int_A |D^2 v|^2\, dx.$$

Hence

$$\int_\Omega |\Delta v|^2\, dx \geq (1+\delta)^{-1}\int_A |D^2 v|^2\, dx - C_{\delta,\phi}\int_\Omega |\nabla v|^2 + 1\, dx.$$

Thus, for every $\delta > 0$, we can write

$$\begin{aligned}\mathcal{F}_n(u_n, v_n) &= \int_\Omega (v_n^2 + \eta_n) W(\boldsymbol{\epsilon}(u_n))\, dx + \int_\Omega \varepsilon_n^{-1}|v_n - 1|^2 + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\Delta v_n|^2\, dx \\ &\geq \int_\Omega (v_n^2 + \eta_n) W(\boldsymbol{\epsilon}(u_n))\, dx + \\ &\quad + (1+\delta)^{-1}\int_A \varepsilon_n^{-1}|v_n - 1|^2 + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|D^2 v_n|^2\, dx - \varepsilon_n^3 C_{\delta,\phi}\int_\Omega |\nabla v_n|^2 + 1\, dx.\end{aligned}$$

Being

$$0 \leq \varepsilon_n^2 \int_\Omega \varepsilon_n|\nabla v_n|^2 + \varepsilon_n\, dx \leq \varepsilon_n^2(\mathcal{F}_n(u_n, v_n) + \varepsilon_n|\Omega|) \to 0$$

for every open set $A \subset\subset \Omega$ and every $\delta > 0$ we have

$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{F}_n(u_n, v_n) \geq \liminf_{n\to+\infty} &\int_\Omega (v_n^2 + \eta_n) W(\boldsymbol{\epsilon}(u_n))\, dx + \\ &+ (1+\delta)^{-1}\liminf_{n\to+\infty}\int_A \varepsilon_n^{-1}|v_n - 1|^2 + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|D^2 v_n|^2\, dx.\end{aligned} \tag{25}$$

By (25) it is clear that $v_n \to 1$ in $L^2(\Omega)$. Let us check that, for every $\xi \in \mathbb{S}^1$ we have

$$\liminf_{n\to+\infty}\int_A \varepsilon_n^{-1}|v_n - 1|^2 + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|D^2 v_n|^2\, dx \geq 4\int_{J^\xi(u)\cap A} |\nu\cdot\xi|\, d\mathcal{H}^1.$$

We will use the slicing technique, see §A. For $\xi \in \mathbb{S}^1$ and $y \in \xi^\perp$ we denote $A_y^\xi = \{s \in \mathbb{R} : y + s\xi \in A\}$. Accordingly, let $v_y^\xi(s) = v(y + s\xi)$ and $u_y^\xi(s) = u(y + s\xi)\cdot\xi$. Since $u_n \to u$ in $L^2(\Omega, \mathbb{R}^2)$ and $v_n \to 1$ in

$L^2(\Omega)$ then for every $\xi \in \mathbb{S}^1$ we have $(u_n)^\xi_y \to u^\xi_y$ and $(v_n)^\xi_y \to 1$ in $L^2(\Omega^\xi_y)$ for a.e. $y \in \xi^\perp$. Note also that u^ξ_y belongs to $SBV(A^\xi_y)$, by Definition A.3, and that, for a.e. $y \in \xi^\perp$, we have (a.e. in A^ξ_y)

$$|\nabla v_n| \geq D_\xi v_n = D(v_n)^\xi_y\,, \qquad |D^2 v_n| \geq D^2_{\xi\xi} v_n = D^2(v_n)^\xi_y.$$

We remark that replacing the Laplacian with the full Hessian allows to get the previous bound on the second derivative of the slice. Then Fubini's Theorem yields

$$\int_A \varepsilon_n^{-1}|v_n - 1|^2 + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|D^2 v_n|^2\,dx \geq$$
$$+ \int_{\xi^\perp} \Big(\int_{A^\xi_y} \varepsilon_n^{-1}|(v_n)^\xi_y - 1|^2 + \varepsilon_n|D(v_n)^\xi_y|^2 + \varepsilon_n^3|D^2(v_n)^\xi_y|^2\,ds\Big) d\mathcal{H}^1(y).$$

By Lemma 3.5 and Theorem A.2 we get

$$\liminf_{n\to+\infty} \int_{A^\xi_y} \varepsilon_n^{-1}|(v_n)^\xi_y - 1|^2 + \varepsilon_n|D(v_n)^\xi_y|^2 + \varepsilon_n^3|D^2(v_n)^\xi_y|^2\,ds \geq 4\,\#\big(J(u^\xi_y) \cap A^\xi_y\big) = 4\,\#\big((J^\xi(u) \cap A)^\xi_y\big).$$

Therefore, Fatou's Lemma and Theorem A.2 give

$$\begin{aligned}\liminf_{n\to+\infty} \int_A \varepsilon_n^{-1}|v_n - 1|^2 + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|D^2 v_n|^2\,dx &\geq 4\int_{\xi^\perp} \#\big((J^\xi(u) \cap A)^\xi_y\big)\,d\mathcal{H}^1(y)\\ &= 4\int_{J^\xi(u)\cap A} |\nu\cdot\xi|\,d\mathcal{H}^1. \end{aligned} \tag{26}$$

Using (25) and (26) in (24) and taking the supremum with respect to $A \subset\subset \Omega$ we get

$$\liminf_{n\to+\infty} \mathcal{F}_n(u_n, v_n) \geq \int_\Omega W(\boldsymbol{\epsilon}(u))\,dx + 4(1+\delta)^{-1}\int_{J^\xi(u)} |\nu\cdot\xi|\,d\mathcal{H}^1 \qquad \text{for every } \xi \in \mathbb{S}^1.$$

To conclude we will employ a supremum of measures argument, see [12, Proposition 1.16]. Let $B \subset \Omega$ be an open set. Denote

$$\mathcal{F}_n(u, v\,|B) = \int_B (v_n^2 + \eta_n)\,W(\boldsymbol{\epsilon}(u_n))\,dx + \int_B \varepsilon_n^{-1}|v_n - 1|^2 + 2\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\Delta v_n|^2\,dx.$$

Arguing as above, just replacing Ω with B, we get

$$\liminf_{n\to+\infty} \mathcal{F}_n(u_n, v_n\,|B) \geq \int_B W(\boldsymbol{\epsilon}(u))\,dx + 4(1+\delta)^{-1}\int_{J^\xi(u)\,\cap\,B} |\nu\cdot\xi|\,d\mathcal{H}^1 \qquad \text{for every } \xi \in \mathbb{S}^1.$$

For a.e. $\xi \in \mathbb{S}^1$ we have $\mathcal{H}^1(J(u) \setminus J^\xi(u)) = 0$, again by Theorem A.2, and thus

$$\liminf_{n\to+\infty} \mathcal{F}_n(u_n, v_n\,|B) \geq \int_B W(\boldsymbol{\epsilon}(u))\,dx + 4(1+\delta)^{-1}\int_{J(u)\,\cap\,B} |\nu\cdot\xi|\,d\mathcal{H}^1$$

for a.e. $\xi \in \mathbb{S}^1$ and every $B \subset \Omega$. Note that $\sup_\xi |\nu\cdot\xi| = 1$, even if the supremum is taken with respect to a.e. $\xi \in \mathbb{S}^1$. Therefore, by [12, Proposition 1.16] and (25) we get

$$\liminf_{n\to+\infty} \mathcal{F}_n(u_n, v_n) \geq \int_\Omega W(\boldsymbol{\epsilon}(u))\,dx + 4(1+\delta)^{-1}\mathcal{H}^1(J(u)),$$

which concludes the proof, by arbitrariness of δ. ■

5 Γ-limsup inequality

By a standard diagonal argument in the theory of Γ-convergence together with Theorem A.4 it is enough to prove the limsup estimate stated in the next Proposition. We provide first a "geometrical" Lemma.

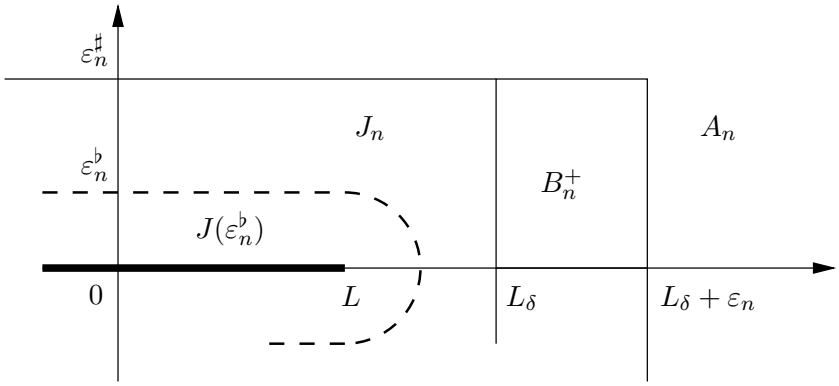

Figure 2: Sets involved in the proof of the Γ-limsup inequality.

Lemma 5.1 *Let $J \subset \mathbb{R}^2$ be a closed line segment, the distance function* $\mathrm{dist}(\cdot, J)$ *is of class $W^{1,\infty}_{loc}(\mathbb{R}^2) \cap W^{2,\infty}_{loc}(\mathbb{R}^2 \setminus J)$ with $\|\nabla \mathrm{dist}(\cdot, J)\|_\infty = 1$.*

Proposition 5.2 *Let $J \subset \Omega$ be a closed line segment and let $u \in W^{2,\infty}(\Omega \setminus J, \mathbb{R}^2)$. There exists $C > 0$ such that for every $\delta > 0$ there exist $u_n \in H^2(\Omega, \mathbb{R}^2)$ and $v_n \in W^{m,\infty}(\Omega)$, for m arbitrarily large, with $0 \le v_n \le 1$, such that $u_n \to u$ in $L^2(\Omega, \mathbb{R}^2)$, $v_n \to 1$ in $L^2(\Omega)$ and*

$$\limsup_{n\to+\infty} \mathcal{F}_n(u_n, v_n) \le \int_{\Omega\setminus J} W(\boldsymbol{\epsilon}(u))\, dx + 4\mathcal{H}^1(J) + C\delta. \tag{27}$$

Proof. Step 1. Consider a system of Cartesian coordinates (x_1, x_2) in which $J = [-L, L] \times \{0\}$.

Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a smooth function with $0 \le \zeta \le 1$, $\zeta = 0$ in $(-\infty, 0)$ and $\zeta = 1$ in $(1, +\infty)$. Denote $L_\delta = L + \delta$ and

$$\zeta_n(r) = \zeta\left(\frac{|r| - L_\delta}{\varepsilon_n}\right).$$

In this way ζ_n is smooth in $\mathbb{R}$ with $\zeta_n(r) = 0$ for $|r| \le L_\delta$ and $\zeta_n(r) = 1$ for $|r| \ge L_\delta + \varepsilon_n$.

Define

$$v_n(x) = \zeta_n(x_1) + (1 - \zeta_n(x_1))\, z_n(x_2),$$

where z_n is the (recovery) sequence provided by Lemma 3.4. Note that $v_n \in C^\infty(\Omega)$. We introduce the disjoint sets (see Figure 2)

$$J_n = [-L_\delta, L_\delta] \times [-\varepsilon_n^\sharp, \varepsilon_n^\sharp], \quad B_n = [-L_\delta - \varepsilon_n, L_\delta + \varepsilon_n] \times [-\varepsilon_n^\sharp, \varepsilon_n^\sharp] \setminus J_n, \quad A_n = \Omega \setminus (J_n \cup B_n).$$

where $\varepsilon_n^\sharp = \varepsilon_n R^\sharp$ has been introduced, together with $\varepsilon_n^\flat$, in Lemma 3.4. Note that $v_n = 0$ in $[-L_\delta, L_\delta] \times [-\varepsilon_n^\flat, \varepsilon_n^\flat]$ and that $v_n = 1$ in A_n. Hence $v_n \in W^{m,\infty}(\Omega)$ (for any m) and

$$\int_{A_n} \varepsilon_n^{-1}|v_n - 1|^2 + 2\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\Delta v_n|^2\, dx = 0.$$

Next, let us see that

$$\limsup_{n\to+\infty} \int_{J_n} \varepsilon_n^{-1}|v_n - 1|^2 + 2\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\Delta v_n|^2\, dx \le 4\mathcal{H}^1(J) + C\delta, \tag{28}$$

for a suitable constant $C > 0$, independent of $\delta \ll 1$. In the set J_n we have $\zeta_n = 0$, hence $v_n(x) = z_n(x_2)$, $|\nabla v_n(x)| = |z_n'(x_2)|$ and $|\Delta v_n(x)| = |z_n''(x_2)|$; by Lemma 3.4 we get

$$\limsup_{n\to+\infty} \int_{J_n} \varepsilon_n^{-1}|v_n - 1|^2 + 2\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\Delta v_n|^2\, dx \le$$
$$\le \limsup_{n\to+\infty} 2L_\delta \int_{\mathbb{R}} \varepsilon_n^{-1}|z_n - 1|^2 + 2\varepsilon_n|z_n'|^2 + \varepsilon_n^3|z_n''|^2\, dx_2 \le 2L_\delta(4 + 2\delta)$$

from which we get (28) since $2L_\delta = \mathcal{H}^1(J) + 2\delta$.

Now, we show that

$$\lim_{n\to+\infty} \int_{B_n} \varepsilon_n^{-1}|v_n - 1|^2 + 2\varepsilon_n|\nabla v_n|^2 + \varepsilon^3|\Delta v_n|^2\,dx = 0.$$

By symmetry it is enough to consider the set $B_n^+ = [L_\delta, L_\delta + \varepsilon_n] \times [0, \varepsilon_n^\sharp]$. For the first integrand it is enough to write

$$\int_{B_n^+} \varepsilon_n^{-1}|v_n - 1|^2\,dx \le |B_n^+|\varepsilon_n^{-1} = \varepsilon_n^\sharp \to 0.$$

For every multi-index (α, β) write $D^{(\alpha,\beta)}v_n = \zeta_n^{(\alpha)}1^{(\beta)} + (1-\zeta_n)^{(\alpha)}z_n^{(\beta)}$ (where $1^{(\beta)}$ is need to make the first term vanish for $\beta \neq 0$). In particular

$$D^{(1,0)}v_n = \zeta_n'(1 - z_n), \qquad D^{(0,1)}v_n = (1-\zeta_n)z_n',$$
$$D^{(2,0)}v_n = \zeta_n''(1 - z_n), \qquad D^{(0,2)}v_n = (1-\zeta_n)z_n''$$

and hence

$$|\nabla v_n|^2 \le |\zeta_n'|^2 + |z_n'|^2, \qquad |\Delta v_n|^2 \le 2|\zeta_n''|^2 + 2|z_n''|^2.$$

As $B_n^+ = [L_\delta, L_\delta + \varepsilon_n] \times [0, \varepsilon_n^\sharp]$ it is convenient to separate the variables, writing

$$\int_{B_n^+} \varepsilon_n|\nabla v_n|^2 + \varepsilon_n^{-3}|\Delta v_n|^2\,dx \le \varepsilon_n^\sharp \int_{(L_\delta, L_\delta+\varepsilon_n)} 2\varepsilon_n|\zeta_n'(x_1)|^2 + 2\varepsilon_n^3|\zeta_n''(x_1)|^2\,dx_1 +$$
$$+ \varepsilon_n \int_{(0,\varepsilon_n^\sharp)} 2\varepsilon_n|z_n'(x_2)|^2 + 2\varepsilon_n^3|z_n''(x_2)|^2\,dx_2.$$

Since $\zeta_n(x_1) = \zeta\big(\frac{x_1 - L_\delta}{\varepsilon_n}\big)$ by the change of variable $s = \frac{x_1 - L_\delta}{\varepsilon_n}$ we get

$$\varepsilon_n^\sharp \int_{(L_\delta, L_\delta+\varepsilon_n)} \varepsilon_n|\zeta_n'(x_1)|^2 + \varepsilon_n^3|\zeta_n''(x_1)|^2\,dx_1 = \varepsilon_n^\sharp \int_{(0,1)} |\zeta'(s)|^2 + |\zeta''(s)|^2\,ds \to 0.$$

For the second term by Lemma 3.4 we have

$$\varepsilon_n \int_{(0,\varepsilon_n^\sharp)} \varepsilon_n|z_n'(x_2)|^2 + \varepsilon_n^3|z_n''(x_2)|^2\,dx \to 0.$$

In conclusion

$$\limsup_{n\to+\infty} \int_\Omega \varepsilon_n^{-1}|v_n - 1|^2 + 2\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\Delta v_n|^2\,dx \le 4\mathcal{H}^1(J) + C\delta. \tag{29}$$

Let $\phi \in C^\infty(0, +\infty)$ with $0 \le \phi \le 1$, $\phi = 0$ in $(0, 1/4)$ and $\phi = 1$ in $(1/2, +\infty)$ (the choice $1/4$ and $1/2$ will be useful in the proof of Proposition 6.1). Let $\phi_n(r) = \phi(r/\varepsilon_n^\flat)$, where $\varepsilon_n^\flat = \varepsilon_n R^\flat$ has been defined in Lemma 3.4. Let $\varphi_n(x) = \phi_n \circ \mathrm{dist}(x, J)$ and denote $J(r) = \{x \in \Omega : \mathrm{dist}(x, J) < r\}$. Note that $\varphi_n = 0$ in $J(\varepsilon_n^\flat/4)$ and $\varphi_n = 1$ in $\Omega \setminus J(\varepsilon_n^\flat/2)$, hence by Lemma 5.1 we know that $\varphi_n \in W^{2,\infty}(\Omega)$.

Step 2. We define $u_n(x) = \varphi_n(x)\,u(x)$. Since $u \in W^{2,\infty}(\Omega \setminus J, \mathbb{R}^2)$ and since $\varphi_n = 0$ in $J(\varepsilon_n^\flat/4)$ the displacement field $u_n(x)$ belongs to $H^2(\Omega, \mathbb{R}^2)$ and $Du_n = \varphi_n\,Du + \nabla\varphi_n \otimes u$ (where Du is the derivative in $\Omega \setminus J$). Since $\|\varphi_n\|_\infty \le 1$ and $\|\nabla\varphi_n\|_\infty \le \|\phi_n'\|_\infty \le C/\varepsilon_n^\flat$ we get $\|Du_n\|_{L^\infty(\Omega,\mathbb{R}^{2\times 2})} \le C/\varepsilon_n^\flat$. Remembering that the phase field variable $v_n = 0$ in $[-L_\delta, L_\delta] \times [-\varepsilon_n^\flat, \varepsilon_n^\flat] \supset J(\varepsilon_n^\flat)$ we get

$$\int_{J(\varepsilon_n^\flat)} (v_n^2 + \eta_n)W(\boldsymbol{\epsilon}(u_n)) \le C\eta_n \int_{J(\varepsilon_n^\flat)} |Du_n|^2\,dx \le C'\eta_n/\varepsilon_n^\flat \to 0,$$

since $\eta_n = o(\varepsilon_n)$ and $\varepsilon_n^\flat = \varepsilon_n R^\flat$. Since $\varphi_n = 1$ in $\Omega \setminus J(\varepsilon_n^\flat)$ we have $u_n = u$ in $\Omega \setminus J(\varepsilon_n^\flat)$. Moreover $0 \le v_n \le 1$ and $v_n \to 1$ a.e. in Ω, thus by dominated convergence

$$\int_{\Omega\setminus J(\varepsilon^\flat)} (v_n^2 + \eta_n)W(\boldsymbol{\epsilon}(u_n)) \to \int_{\Omega\setminus J} W(\boldsymbol{\epsilon}(u))\,dx,$$

which concludes the proof. ■

Remark 5.3 The fact that $u \in H^2(\Omega, \mathbb{R}^2)$ and $v \in W^{m,\infty}(\Omega)$, instead of the more natural $H^1(\Omega, \mathbb{R}^2)$ and $H^2(\Omega)$ which would be enough for the Γ-limsup estimate, will be useful in the discrete approximation, see § 6, together with the next Corollary.

Corollary 5.4 *Let v_n be as in Proposition 5.2. Then, there exists $C > 0$ (independent of n) such that $\|D^{(\alpha,\beta)} v_n\|_{L^\infty(\Omega)} \le C\varepsilon_n^{-l}$ for $l = \alpha + \beta = 0, ..., 3$ and $\|v_n - 1\|^2_{H^3(\Omega)} \le C\varepsilon_n^{-5}$.*

Proof. As in the proof of Proposition 5.2, write $D^{(\alpha,\beta)} v_n = \zeta_n^{(\alpha)} 1^{(\beta)} + (1 - \zeta_n)^{(\alpha)} z_n^{(\beta)}$. Remembering that $\zeta_n(x_1) = \zeta\big(\frac{x_1 - L_\delta}{\varepsilon_n}\big)$ and that $z_n(x_2) = 1 - w(|x_2|/\varepsilon_n)$ it is immediate to check that $\|\zeta_n^{(\alpha)}\|_\infty \le C\varepsilon_n^{-\alpha}$ and $\|z_n^{(\beta)}\|_\infty \le C\varepsilon_n^{-\beta}$; the L^∞-bound follows.

To estimate the H^3-norm it is sufficient to employ the L^∞-bound remembering that $v_n - 1$ is supported in the set $[-L_\delta - \varepsilon_n, L_\delta + \varepsilon_n] \times [-\varepsilon_n^\sharp, \varepsilon_n^\sharp]$, whose measure is of order ε_n. ■

6 Γ-limit of $\mathcal{F}_{\varepsilon,h}$

As $\mathcal{F}_{\varepsilon,h}$ is the restriction of $\mathcal{F}_\varepsilon$ to $\mathcal{U}_h \times \mathcal{V}_h$ the Γ-liminf inequality for $\mathcal{F}_{\varepsilon,h}$ follows directly from Proposition 4.1. Moreover, as in the continuum setting, it is enough to prove the following Γ-limsup inequality.

Proposition 6.1 *Let $\varepsilon_n \to 0^+$, $\eta_n = o(\varepsilon_n)$ and $h_n = o(\varepsilon_n)$. Let $J \subset \Omega$ be a closed line segment and let $u \in W^{2,\infty}(\Omega \setminus J, \mathbb{R}^2)$. There exists $C > 0$ such that for every $\delta > 0$ there exist $u_{h_n} \in \mathcal{U}_{h_n}$ and $v_{h_n} \in \mathcal{V}_{h_n}$ with $0 \le v_{h_n} \le 1$ such that $u_{h_n} \to u$ in $L^2(\Omega, \mathbb{R}^2)$, $v_{h_n} \to 1$ in $L^2(\Omega)$ and*

$$\limsup_{n\to+\infty} \mathcal{F}_{\varepsilon_n,h_n}(u_{h_n}, v_{h_n}) \le (1+\delta) \int_{\Omega\setminus J} W(\boldsymbol{\epsilon}(u))\, dx + 4\mathcal{H}^1(J) + C\delta. \tag{30}$$

Proof. Consider a system of Cartesian coordinates (x_1, x_2) in which $J = [-L, L] \times \{0\}$. Note that, this system in general is not aligned with the elements K of the physical mesh $\mathcal{K}_{h_n}$. Let u_n, v_n be provided by Proposition 5.2.

Step 1. Let $\Pi_{\mathcal{V}_{h_n}}$ be the interpolation operator in $H^3(\Omega)$ and denote $w_{h_n} = \Pi_{\mathcal{V}_{h_n}} v_n$. Note that, in general, the inequality $0 \le w_{h_n} \le 1$ may not hold everywhere in Ω; to fix this point let us start with an estimate of the error $\|v_n - w_{h_n}\|_{L^\infty(\Omega)}$. For every element $K \in \mathcal{K}_{h_n}$ in the physical domain, (12) provides

$$\|v_n - w_{h_n}\|_{L^\infty(K)} \le C\big(h^{-2}\|v_n - w_{h_n}\|^2_{L^2(K)} + |v_n - w_{h_n}|^2_{H^1(K)} + h^2|v_n - w_{h_n}|^2_{H^2(K)}\big)^{1/2}. \tag{31}$$

By Corollary 5.4 we know that $\|D^{(\alpha,\beta)} v_n\|_\infty \le C\varepsilon_n^{-3}$ for $\alpha + \beta = 0, .., 3$ and then, invoking (10), for $0 \le k \le 2$ we get

$$|v_n - w_{h_n}|^2_{H^k(K)} \le C\, h_n^{6-2k} \|v_n\|^2_{H^3(\tilde{K})} \le C'\, h_n^{6-2k} |\tilde{K}|\, \varepsilon_n^{-6} \le C''\, h_n^{2-2k} (h_n/\varepsilon_n)^6. \tag{32}$$

By (31) it follows that $\|v_n - w_{h_n}\|_{L^\infty(K)} \le C(h_n/\varepsilon_n)^3$. As the constant C is independent of the element K the previous estimate becomes

$$\|v_n - w_{h_n}\|_{L^\infty(\Omega)} \le C(h_n/\varepsilon_n)^3 = c_n. \tag{33}$$

Hence $\|v_n - w_{h_n}\|_{L^\infty(\Omega)} \to 0$ and $-c_n \le w_{h_n} \le 1 + c_n$ in Ω.

Now, let us see how to modify w_{h_n} in such a way that it takes values in $[0, 1]$. Define

$$\Omega_n^0 = \{v_n = 0\}, \qquad \Omega_n^1 = \{0 < v_n < 2c_n\}, \qquad \Omega_n^2 = \{2c_n \le v_n \le 1 - 2c_n\},$$
$$\Omega_n^3 = \{1 - 2c_n < v_n < 1\}, \quad \Omega_n^4 = \{v_n = 1\}.$$

Define also

$$\mathcal{K}^i_{h_n} = \{K : \tilde{K} \subset \Omega_n^i\} \quad \text{for } i = 0, 2, 4, \qquad \mathcal{K}^i_{h_n} = \{K : \tilde{K} \cap \Omega_n^i \ne \emptyset\} \quad \text{for } i = 1, 3.$$

Note that the previous definitions depends on the extended elements $\tilde{K}$. First, we check that the families $\mathcal{K}^i_{h_n}$ provide a disjoint partition of $\mathcal{K}_{h_n}$. Let $K \in \mathcal{K}_{h_n}$, if $K \notin \mathcal{K}^i_{h_n}$ for $i = 0, 2, 4$ then $K \in \mathcal{K}^1_{h_n}$ and/or

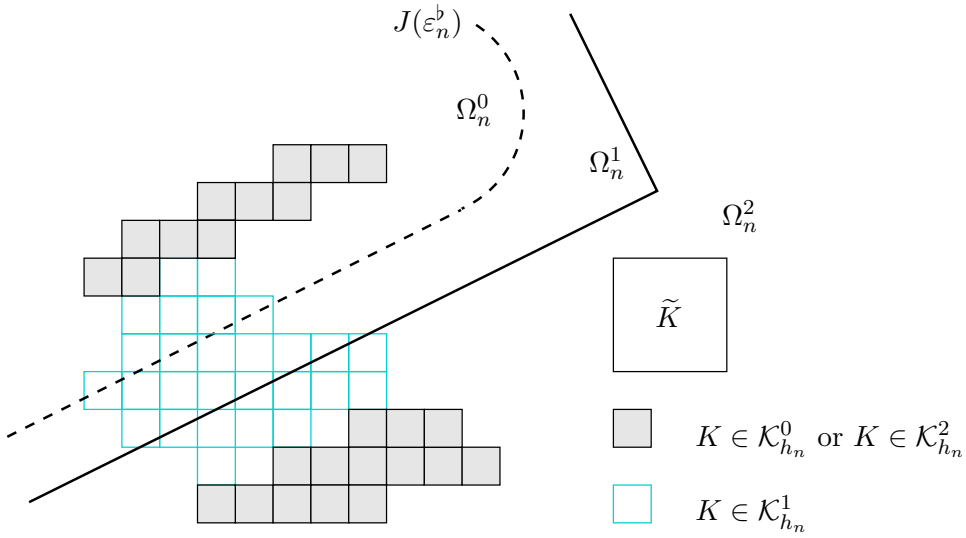

Figure 3: Sets involved in the proof of Proposition 6.1.

$K \in \mathcal{K}_{h_n}^3$; hence, the union of the families $\mathcal{K}_{h_n}^i$ for $i = 0, ..., 4$ is the whole $\mathcal{K}_{h_n}$. Moreover, if $K \in \mathcal{K}_{h_n}^i$ for $i = 0, 2, 4$ then $K \notin (\mathcal{K}_{h_n}^1 \cup \mathcal{K}_{h_n}^3)$, hence the sets $(\mathcal{K}_{h_n}^0 \cup \mathcal{K}_{h_n}^2 \cup \mathcal{K}_{h_n}^4)$ and $(\mathcal{K}_{h_n}^1 \cup \mathcal{K}_{h_n}^3)$ are disjoint. It is clear, from the definition, that the families $\mathcal{K}_{h_n}^i$ are pairwise disjoint for $i = 0, 2, 4$ because the corresponding sets $\Omega_{h_n}^i$ are pairwise disjoint. It remains to check that $\mathcal{K}_{h_n}^1$ and $\mathcal{K}_{h_n}^3$ are disjoint, at least for $n \gg 1$. Remember that $\|\nabla v_n\|_{L^\infty(\Omega)} \leq C/\varepsilon_n$, that $\mathrm{diam}(\tilde{K}) \leq \tilde{C} h_n$ and that $h_n = o(\varepsilon_n)$, then for $n \gg 1$ we have

$$\begin{cases} v_n < 2c_n + \bar{C} h_n/\varepsilon_n < 1/3 \ \text{ in } \tilde{K} & \text{if } K \in \mathcal{K}_{h_n}^1, \\ v_n \geq (1 - 2c_n) - \bar{C} h_n/\varepsilon_n \geq 2/3 \ \text{ in } \tilde{K} & \text{if } K \in \mathcal{K}_{h_n}^3. \end{cases} \tag{34}$$

It follows that $\mathcal{K}_{h_n}^1$ and $\mathcal{K}_{h_n}^3$ are disjoint.

Next, denote by $A_{h_n}^i$ the union of the elements $K \in \mathcal{K}_{h_n}^i$ and by $\tilde{A}_{h_n}^i$ the corresponding union of the extended elements (see Figure 3). We claim that for $n \gg 1$ the sets $A_{h_n}^i$ provide a disjoint partition of Ω and that

$$w_{h_n} = 0 \text{ in } A_{h_n}^0, \qquad c_n \leq w_{h_n} \leq 1 - c_n \text{ in } A_{h_n}^2, \qquad w_{h_n} = 1 \text{ in } A_{h_n}^4, \tag{35}$$

$$-c_n \leq w_{h_n} \leq 1/3 + c_n \text{ in } A_{h_n}^1, \qquad 2/3 - c_n \leq w_{h_n} \leq 1 + c_n \text{ in } A_{h_n}^3 \tag{36}$$

$$0 \leq w_{h_n} \leq 1/3 + c_n \text{ in } \tilde{A}_{h_n}^1 \setminus A_{h_n}^1, \qquad 2/3 - c_n \leq w_{h_n} \leq 1 \text{ in } \tilde{A}_{h_n}^3 \setminus A_{h_n}^3. \tag{37}$$

Since the sets $\mathcal{K}_{h_n}^i$ provide a disjoint partition of $\mathcal{K}_{h_n}$ the corresponding sets $A_{h_n}^i$ give a disjoint partition of Ω. Let us check (35). By definition, if $K \in \mathcal{K}_{h_n}^0$ then $v_n = 0$ on $\tilde{K}$, hence $w_{h_n} = 0$ in K because the projection operator is locally an identity for constant functions (see for instance [5, Lemma 3.2]). In the same way, if $K \in \mathcal{K}_{h_n}^4$ then $v_n = 1$ on $\tilde{K}$, hence $w_{h_n} = 1$ in K. If $K \in \mathcal{K}_{h_n}^2$ then $2c_n \leq v_n \leq 1 - 2c_n$ in K and then by (33) we have $c_n \leq w_{h_n} \leq 1 - c_n$ in K. Let us check (36). If $K \in \mathcal{K}_{h_n}^1$ then, being $v_n \geq 0$, by (34) we have $0 \leq v_n < 1/3$ in $\tilde{K}$ and thus by (33) we get $-c_n \leq w_{h_n} \leq 1/3 + c_n$ in $\tilde{K}$. To get (37) from (36) it is enough to note that $\tilde{A}_{h_n}^1 \setminus A_{h_n}^1$ is contained in the union of the set $A_{h_n}^i$ for $i = 0, 2, 3, 4$ where $w_{h_n} \geq 0$. Similarly for $K \in \mathcal{K}_{h_n}^3$.

We are now ready to modify the function w_{h_n} in the sets $A_{h_n}^i$ for $i = 1, 3$ (where the constraint $0 \leq w_{h_n} \leq 1$ may not be satisfied). Consider all the basis functions $\hat{v}_{h_n}$ whose support intersects an element $K \in \mathcal{K}_{h_n}^1$ and denote by $v_{h_n}^1$ their sum. By definition, basis functions $\hat{v}_{h_n}$ are non-negative, provide locally, on each element, a partition of unity and are supported in the extended elements $\tilde{K}$; hence

$$0 \leq v_{h_n}^1 \leq 1 \text{ in } \Omega, \qquad v_{h_n}^1 = 1 \text{ in } A_{h_n}^1, \qquad \mathrm{supp}(v_{h_n}^1) \subset \tilde{A}_{h_n}^1,$$
$$\|D^{(\alpha,\beta)} v_{h_n}^1\|_{L^\infty(\Omega)} \leq C h_n^{-(\alpha+\beta)} \ \text{ for } (\alpha + \beta) = 1, 2.$$

The L^∞-estimate for the derivatives follows from scaling and from the fact that each bases function intersects a finite number of elements. Note that the support is contained in the enlarged set $\tilde{A}^1_{h_n}$. Similarly we define $v^3_{h_n}$ and finally we can introduce the phase-field functions v_{h_n}, given by

$$v_{h_n} = w_{h_n} + c_n v^1_{h_n} - c_n v^3_{h_n}.$$

Since the supports of $v^1_{h_n}$ and $v^3_{h_n}$ are disjoint we can write v_{h_n} also as

$$v_{h_n} = \begin{cases} w_{h_n} + c_n v^1_{h_n} & \text{in } \tilde{A}^1_{h_n}, \\ w_{h_n} - c_n v^3_{h_n} & \text{in } \tilde{A}^3_{h_n}, \\ w_{h_n} & \text{otherwise.} \end{cases}$$

In the set $A^1_{h_n}$ we have $v^1_{h_n} = 1$ and $-c_n \le w_{h_n} \le 1/3 + c_n$, hence $0 \le v_{h_n} \le 1$. In $\tilde{A}^1_{h_n} \setminus A^1_{h_n}$ we have $0 \le v^1_{h_n} \le 1$ and $0 \le w_{h_n} \le 1 - c_n$, hence $0 \le v_{h_n} \le 1$. We can argue in a similar way for $v^3_{h_n}$. We have checked that that $0 \le v_{h_n} \le 1$ in Ω, for $n \gg 1$. Now, let us provide some error estimates. First, note that

$$\|v_{h_n} - w_{h_n}\|_{L^\infty(\Omega)} \le c_n, \qquad \|D^{(\alpha,\beta)}(v_{h_n} - w_{h_n})\|_{L^\infty(\Omega)} \le C c_n h_n^{-(\alpha+\beta)} \quad \text{for } (\alpha+\beta) = 1, 2.$$

Let us check that the Lebesgue measure of $\mathrm{supp}(v_{h_n} - w_{h_n})$ is of order ε_n. Clearly $\mathrm{supp}(v_{h_n} - w_{h_n}) \subset (\tilde{A}^1_{h_n} \cup \tilde{A}^3_{h_n})$. By Proposition 5.2 (see also Figure 2) $v_n = 1$ in $\Omega \setminus Q_n$ where Q_n is a rectangle of the form $\{|x_1| \le L', |x_2| \le C\varepsilon_n\}$. Thus Ω^1_n and Ω^3_n are contained in Q_n. It follows that $\tilde{A}^1_{h_n}$ and $\tilde{A}^3_{h_n}$ are contained in a rectangle of the form $\{|x_1| \le L' + \tilde{C}h_n, |x_2| \le C\varepsilon_n + \tilde{C}h_n\}$. Since $h_n = o(\varepsilon_n)$ we have the required estimate on the measure of the support. Using $c_n = C(h_n/\varepsilon_n)^3$ and the L^∞-estimates above we get

$$\int_\Omega |v_{h_n} - w_{h_n}|^2 \, dx \le C c_n^2 \varepsilon_n \le C' h_n^6 \varepsilon_n^{-5}, \tag{38}$$

$$\int_\Omega |\nabla v_{h_n} - \nabla w_{h_n}|^2 \le C c_n^2 \varepsilon_n h_n^{-2} \le C' h_n^4 \varepsilon_n^{-5}, \tag{39}$$

$$\int_\Omega |D^2 v_{h_n} - D^2 w_{h_n}|^2 \le C c_n^2 \varepsilon_n h_n^{-4} \le C' h_n^2 \varepsilon_n^{-5}. \tag{40}$$

Before proceeding, let us provide also some global error estimates. We know (see for instance [5, Lemma 3.2]) that $\Pi_{\mathcal{V}_{h_n}} 1 = 1$ and then $v_n - w_{h_n} = (v_n - 1) - \Pi_{\mathcal{V}_{h_n}}(v_n - 1)$. Hence, using (11) for $(v_n - 1)$ and $l = 3$ together with Corollary 5.4 we get, for $k = 0, .., 2$,

$$|v_n - w_{h_n}|^2_{H^k(\Omega)} \le C\, h_n^{6-2k} \|v_n - 1\|^2_{H^3(\Omega)} \le C' h_n^{6-2k} \varepsilon_n^{-5}. \tag{41}$$

Note that this estimate for $k = 0, ..., 2$ is of the same order of (38)-(40). Clearly the right hand side is not infinitesimal, because the sequence v_n is not bounded in $H^3(\Omega)$.

Step 2. Now, let us prove (30). In the sequel we will make frequent use of the following Young's inequality $(a+b)^2 \le (1+\delta^{-1})a^2 + (1+\delta)b^2$ for $\delta > 0$. Let $C'_\delta = (1+\delta)(1+\delta^{-1})$. Using twice Young's inequality, the error estimates (38) and (41) and remembering that $c_n = (h_n/\varepsilon_n)^3$ we get

$$\begin{aligned} \int_\Omega \varepsilon_n^{-1} |v_{h_n} - 1|^2 &\le (1+\delta^{-1}) \int_\Omega \varepsilon_n^{-1} |v_{h_n} - w_{h_n}|^2 \, dx + (1+\delta) \int_\Omega \varepsilon_n^{-1} |w_{h_n} - 1|^2 \, dx \\ &\le (1+\delta^{-1}) \int_\Omega \varepsilon_n^{-1} |v_{h_n} - w_{h_n}|^2 \, dx + C'_\delta \int_\Omega \varepsilon_n^{-1} |w_{h_n} - v_n|^2 \, dx + \\ &\quad + (1+\delta)^2 \int_\Omega \varepsilon_n^{-1} |v_n - 1|^2 \, dx \\ &\le C_\delta\, c_n^2 + (1+\delta)^2 \int_\Omega \varepsilon_n^{-1} |v_n - 1|^2 \, dx. \end{aligned}$$

Similarly,

$$\begin{aligned}\int_\Omega \varepsilon_n|\nabla v_{h_n}|^2\,dx &\le (1+\delta^{-1})\int_\Omega \varepsilon_n|\nabla v_{h_n}-\nabla w_{h_n}|^2\,dx + (1+\delta)\int_\Omega \varepsilon_n|\nabla w_{h_n}|^2\,dx\\ &\le (1+\delta^{-1})\int_\Omega \varepsilon_n|\nabla v_{h_n}-\nabla w_{h_n}|^2\,dx + C'_\delta\int_\Omega \varepsilon_n|\nabla w_{h_n}-\nabla v_n|^2\,dx\\ &\quad + (1+\delta)^2\int_\Omega \varepsilon_n|\nabla v_n|^2\,dx\\ &\le C_\delta\, c_n^{4/3} + (1+\delta)^2\int_\Omega \varepsilon_n|\nabla v_n|^2\,dx.\end{aligned}$$

Finally,

$$\begin{aligned}\int_\Omega \varepsilon_n^3|\Delta v_{h_n}|^2\,dx &\le (1+\delta^{-1})\int_\Omega \varepsilon_n^3|\Delta v_{h_n}-\Delta w_{h_n}|^2\,dx + (1+\delta)\int_\Omega \varepsilon_n^3|\Delta w_{h_n}|^2\,dx\\ &\le (1+\delta^{-1})\int_\Omega \varepsilon_n^3|\Delta v_{h_n}-\Delta w_{h_n}|^2\,dx + C'_\delta\int_\Omega \varepsilon_n^3|\Delta w_{h_n}-\Delta v_n|^2\,dx\\ &\quad + (1+\delta)^2\int_\Omega \varepsilon_n^3|\Delta v_n|^2\,dx\\ &\le C_\delta\, c_n^{2/3} + (1+\delta)^2\int_\Omega \varepsilon_n|\Delta v_n|^2\,dx.\end{aligned}$$

In conclusion,

$$\begin{aligned}&\int_\Omega \varepsilon_n^{-1}|v_{h_n}-1|^2 + 2\varepsilon_n|\nabla v_{h_n}|^2 + \varepsilon_n^3|\Delta v_{h_n}|^2\,dx \le\\ &\qquad\qquad \le (1+\delta)^2\int_\Omega \varepsilon_n^{-1}|v_n-1|^2 + 2\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\Delta v_n|^2\,dx + o(1)\end{aligned}$$

and thus, by Proposition 5.2

$$\limsup_{n\to+\infty}\int_\Omega \varepsilon_n^{-1}|v_{h_n}-1|^2 + 2\varepsilon_n|\nabla v_{h_n}|^2 + \varepsilon_n^3|\Delta v_{h_n}|^2\,dx \le 4\mathcal{H}^1(J) + C\delta$$

for a suitable $C>0$.

Step 3. Let $u_{h_n} = \Pi_{\mathcal{U}_{h_n}} u_n$. Denote $\varepsilon_n^* = \frac{3}{4}\varepsilon_n^\flat$ where $\varepsilon_n^\flat = \varepsilon_n R^\flat$. From the proof of Proposition 5.2 (see also Figure 2) we know that $v_n = 0$ in $J(\varepsilon_n^\flat)$; since $h_n = o(\varepsilon_n^\flat)$ we have $v_{h_n} = 0$ in $J(\varepsilon_n^*)$ (remember that interpolation is non-local). Since W is quadratic, by Young's inequality we can write

$$W(\boldsymbol{\epsilon}(u_{h_n})) \le (1+\delta)W(\boldsymbol{\epsilon}(u)) + C_\delta|Du - Du_{h_n}|^2.$$

Being $v_{h_n}\le 1$ we get

$$\int_{\Omega\setminus J(\varepsilon_n^*)} (v_{h_n}^2+\eta_n)W(\boldsymbol{\epsilon}(u_{h_n})) \le (1+\eta_n)\int_{\Omega\setminus J(\varepsilon_n^*)} (1+\delta)W(\boldsymbol{\epsilon}(u)) + C_\delta|Du-Du_{h_n}|^2\,dx.$$

Clearly, for the first term we have

$$(1+\eta_n)\int_{\Omega\setminus J(\varepsilon_n^*)} W(\boldsymbol{\epsilon}(u))\,dx \to \int_{\Omega\setminus J} W(\boldsymbol{\epsilon}(u))\,dx.$$

From Step 2 in Proposition 5.2 we know that $u_n = \varphi_n u = u$ in $\Omega\setminus J(\varepsilon_n^\flat/2)$ and thus in $\Omega\setminus J(\varepsilon_n^*)$. As $u\in W^{2,\infty}(\Omega\setminus J)$ by the interpolation error estimate (9) the limit of the second term is estimated by

$$\int_{\Omega\setminus J(\varepsilon_n^*)} |Du_n - Du_{h_n}|^2\,dx \le Ch_n^2 \to 0.$$

Since $v_{h_n} = 0$ in $J(\varepsilon_n^*)$ we can write

$$\int_{J(\varepsilon_n^*)} (v_{h_n}^2 + \eta_n) W(\boldsymbol{\epsilon}(u_{h_n}))\, dx \le C\eta_n \int_{J(\varepsilon_n^*)} |Du_{h_n}|^2\, dx$$
$$\le C'\eta_n \int_{J(\varepsilon_n^*)} |Du_n|^2\, dx + C'\eta_n \int_{J(\varepsilon_n^*)} |Du_n - Du_{h_n}|^2\, dx.$$

In Proposition 5.2 we have already shown that the first term is infinitesimal. Since $u_n = \varphi_n u$, we have $\|Du_n\|_{L^\infty(\Omega,\mathbb{R}^{2\times 2})} \le C\varepsilon_n^{-1}$. Thus, using the fact that $\varepsilon_n^* + \tilde{C}h_n \le \varepsilon_n^\flat$ (for $n \gg 1$) the error estimate (9) for $k = l = 1$ provides

$$\eta_n \int_{J(\varepsilon_n^*)} |Du_n - Du_{h_n}|^2\, dx \le C\eta_n \int_{J(\varepsilon_n^\flat)} |Du_n|^2\, dx \le C\eta_n \varepsilon_n^\flat \varepsilon_n^{-2} \le C'\eta_n/\varepsilon_n \to 0,$$

because $\eta_n = o(\varepsilon_n)$. The proof is concluded. ■

A *GSBD* spaces

We provide just the definition and the main properties of vector fields in $GSBD(\Omega)$ and $GSBD^2(\Omega)$ for Ω an open subset of $\mathbb{R}^2$. For a general and detailed work the reader should refer to [21].

For $\xi \in \mathbb{S}^1 = \{\xi \in \mathbb{R}^2 : |\xi| = 1\}$ let $\xi^\perp = \{y \in \mathbb{R}^2 : y \cdot \xi = 0\}$. For $B \subset \Omega$ and $y \in \xi^\perp$ let $B_y^\xi = \{s \in \mathbb{R} : y + s\xi \in B\}$ denote the "slice" of B. If $u : \Omega \to \mathbb{R}^2$ we consider its projected ξ-slice in B, i.e., the function $u_y^\xi : B_y^\xi \to \mathbb{R}$ given by $u_y^\xi(s) = u(y + s\xi) \cdot \xi$. Note that u_y^ξ is scalar valued.

Definition A.1 *A measurable function $u : \Omega \to \mathbb{R}^2$ belongs to $GSBD(\Omega, \mathbb{R}^2)$ if for every $\xi \in \mathbb{S}^1$ and a.e. $y \in \xi^\perp$ the slices u_y^ξ belong to $SBV_{loc}(\Omega_y^\xi)$ and if there exists a finite Radon measure μ such that for every Borel set $B \subset \Omega$ we have*

$$\int_{\xi^\perp} |Du_y^\xi|\big(B_y^\xi \setminus J^{(1)}(u_y^\xi)\big) + \#\big(B_y^\xi \cap J^{(1)}(u_y^\xi)\big)\, dy \le \mu(B) \quad \text{for every } \xi \in \mathbb{S}^1 \text{ and a.e. } y \in \xi^\perp.$$

Here $Du_y^\xi \in \mathcal{M}_{loc}(\Omega_y^\xi)$ is the distributional derivative of u_y^ξ while $J^{(1)}(u_y^\xi) = \{s \in \Omega_y^\xi : |[\![u_y^\xi(s)]\!]| \ge 1\}$.

Theorem A.2 *Let $u \in GSBD(\Omega)$ and $\xi \in \mathbb{S}^1$. For a.e. $y \in \xi^\perp$ we have $(J^\xi(u))_y^\xi = J(u_y^\xi)$ where*

$$J^\xi(u) = \{x \in J(u) : (u^+(x) - u^-(x)) \cdot \xi \ne 0\}.$$

Moreover, for a.e. $\xi \in \mathbb{S}^1$ we have $\mathcal{H}^1(J^\xi(u) \setminus J(u)) = 0$ and

$$\int_{\xi^\perp} \#(J(u_y^\xi))\, dy = \int_{J(u)} |\xi \cdot \nu|\, d\mathcal{H}^1.$$

Definition A.3 *A measurable function $u : \Omega \to \mathbb{R}^2$ belongs to $GSBD^2(\Omega, \mathbb{R}^2)$ if $u \in GSBD(\Omega, \mathbb{R}^2)$, $\boldsymbol{\epsilon}(u) \in L^2(\Omega, \mathbb{R}^{2\times 2})$ and $\mathcal{H}^1(J(u)) < +\infty$.*

Combining [23] and [19] yields the following approximation result.

Theorem A.4 *Let $u \in GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^2)$. Then there exists a sequence $u_k \in SBV^2(\Omega, \mathbb{R}^2)$ such that $u_k \to u$ in $L^2(\Omega, \mathbb{R}^2)$, $\boldsymbol{\epsilon}(u_k) \to \boldsymbol{\epsilon}(u)$ in $L^2(\Omega, \mathbb{R}^{2\times 2})$ and $\mathcal{H}^1(J(u_k)) \to \mathcal{H}^1(J(u))$. Further, u_k can be chosen in such a way that $J(u_k) \subset \Omega$ is the finite union of closed, disjoint line segments and $u_k \in W^{m,\infty}(\Omega \setminus J(u_k), \mathbb{R}^2)$ (for m arbitrarily large).*

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References

[1] M. Ambati, T. Gerasimov, and L. De Lorenzis. A review on phase-field models of brittle fracture and a new fast hybrid formulation. *Comp. Mech.*, 55(2):383–405, 2015.

[2] L. Ambrosio and V.M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Comm. Pure Appl. Math.*, 43(8):999–1036, 1990.

[3] H. Amor, J.-J. Marigo, and C. Maurini. Regularized formulation of the variational brittle fracture with unilateral contact: Numerical experiments. *Journal of the Mechanics and Physics of Solids*, 57(8):1209 – 1229, 2009.

[4] M. Artina, M. Fornasier, S. Micheletti, and S. Perotto. Anisotropic mesh adaptation for crack detection in brittle materials. *SIAM J. Sci. Comput.*, 37(4):B633–B659, 2015.

[5] Y. Bazilevs, L. Beirão da Veiga, J.A. Cottrell, T.J.R. Hughes, and G. Sangalli. Isogeometric analysis: approximation, stability and error estimates for h-refined meshes. *Math. Models Methods Appl. Sci.*, 16(7):1031–1090, 2006.

[6] L. Beirão da Veiga, A. Buffa, G. Sangalli, and R. Vázquez. An introduction to the numerical analysis of isogeometric methods. In *Numerical simulation in physics and engineering*, volume 9 of *SEMA SIMAI Springer Ser.*, pages 3–69. Springer, [Cham], 2016.

[7] G. Bellettini and A. Coscia. Discrete approximation of a free discontinuity problem. *Numer. Funct. Anal. Optim.*, 15(3-4):201–224, 1994.

[8] M.J. Borden, T.J.R. Hughes, C.M. Landis, and C.V. Verhoosel. A higher-order phase-field model for brittle fracture: Formulation and analysis within the isogeometric analysis framework. *Comput. Methods Appl. Mech. Engrg.*, 273:100 – 118, 2014.

[9] B. Bourdin and A. Chambolle. Implementation of an adaptive finite-element approximation of the Mumford-Shah functional. *Numer. Math.*, 85(4):609–646, 2000.

[10] B. Bourdin, G. A. Francfort, and J.-J. Marigo. Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids*, 48(4):797–826, 2000.

[11] B. Bourdin, G.A. Francfort, and J.-J. Marigo. The variational approach to fracture. *J. Elasticity*, 91:5–148, 2008.

[12] A. Braides. *Approximation of free-discontinuity problems.* Springer-Verlag, Berlin, 1998.

[13] M. Burger, T. Esposito, and C.I. Zeppieri. Second-order edge-penalization in the Ambrosio-Tortorelli functional. *Multiscale Model. Simul.*, 13(4):1354–1389, 2015.

[14] S. Burke, C. Ortner, and E. Süli. An adaptive finite element approximation of a variational model of brittle fracture. *SIAM J. Numer. Anal.*, 48(3):980–1012, 2010.

[15] A. Chambolle. An approximation result for special functions with bounded deformation. *J. Math. Pures Appl. (9)*, 83(7):929–954, 2004.

[16] A. Chambolle, S. Conti, and G. A. Francfort. Approximation of a brittle fracture energy with a constraint of non-interpenetration. *https://arxiv.org/pdf/1709.04208*, 2017.

[17] P.G. Ciarlet. *The finite element method for elliptic problems.* Studies in Mathematics and its Applications, Vol. 4. North-Holland Publishing Co., Amsterdam, 1978.

[18] S. Conti, M. Focardi, and F. Iurlano. Phase field approximation of cohesive fracture models. *Ann. I. H. Poincaré*, 33:1033–1067, 2016.

[19] G. Cortesani and R. Toader. A density result in SBV with respect to non-isotropic energies. *Nonlinear Anal.*, 38(5):585–604, 1999.

[20] G. Dal Maso. *An introduction to* Γ*-convergence.* Birkhäuser, Boston, 1993.

[21] G. Dal Maso. Generalised functions of bounded deformation. *J. Eur. Math. Soc.*, 15(5):1943–1997, 2013.

[22] G. Dal Maso and F. Iurlano. Fracture models as Γ-limits of damage models. *Commun. Pure Appl. Anal.*, 12(4):1657–1686, 2013.

[23] F. Iurlano. A density result for $GSBD$ and its application to the approximation of brittle fracture energies. *Calc. Var. Partial Differential Equation*, 51:315–342, 2014.

[24] J. Kiendl, M. Ambati, L. De Lorenzis, H. Gomez, and A. Reali. Phase-field description of brittle fracture in plates and shells. *Computer Methods in Applied Mechanics and Engineering*, 312:374 – 394, 2016. Phase Field Approaches to Fracture.

[25] D. Knees and M. Negri. Convergence of alternate minimization schemes for phase field fracture and damage. *Math. Models Methods Appl. Sci.*, (to appear).

[26] C. Kuhn and R. Müller. A continuum phase field model for fracture. *Engineering Fracture Mechanics*, 77(18):3625 – 3634, 2010.

[27] C.J. Larsen, C. Ortner, and E. Süli. Existence of solutions to a regularized model of dynamic fracture. *Math. Models Methods Appl. Sci.*, 20(7):1021–1048, 2010.

[28] B. Li, C. Peco, D. Millán, I. Arias, and M. Arroyo. Phase-field modeling and simulation of fracture in brittle materials with strongly anisotropic surface energy. *Int. J. Numer. Methods Eng.*, 102(3-4):711–727, 2015.

[29] R. March. Visual reconstruction with discontinuities using variational methods. *Vision Computing*, 10:30–38, 1992.

[30] C. Miehe, F. Welschinger, and M. Hofacker. Thermodynamically consistent phase-field models of fracture: variational principles and multi-field FE implementations. *Internat. J. Numer. Methods Engrg.*, 83(10):1273–1311, 2010.

[31] A. Mikelić, M. F. Wheeler, and T. Wick. A quasi-static phase-field approach to pressurized fractures. *Nonlinearity*, 28(5):1371–1399, 2015.

[32] M. Negri. Quasi-static evolutions in brittle fracture generated by gradient flows: sharp crack and phase-field approaches. In *Innovative Numerical Approaches for Multi-Physics and Multi-Scale Problems*, volume 81 of *Lecture Notes in Applied and Computational Mechanics*, pages 197–216. Springer, 2016.

[33] L.L. Schumaker. *Spline functions: basic theory.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2007.

[34] P. Sicsic, J.-J. Marigo, and C. Maurini. Initiation of a periodic array of cracks in the thermal shock problem: A gradient damage modeling. *J. Mech. Phys. Solids*, 63:256–284, 2014.