Non-isothermal cyclic fatigue in oscillating elasto-plastic structures with hysteresis

Michela Eleuteri

Università degli Studi di Milano

Supported by the FP7-IDEAS-ERC-StG Grant “EntroPhase” #256872 (P.I. E. Rocca)

Milano, June 5th, 2013
Plan of the seminar

- Theory of hysteresis operators powerful tool for solving mathematical problems in various applications (solid mechanics, material fatigue, ferromagnetism, phase transitions)

Elasto-plastic oscillations of beams and plates with material fatigue

M. Eleuteri, J. Kopfová and P. Krejčí, submitted.

- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Influence of energy dissipation (due to material softening on the damage increase) taken into account - fatigue accumulation accelerated - a singularity (material failure) may develop in finite time
- Account also for decreasing fatigue rate (phase parameter $\chi$)
- Discuss thermodynamic consistency of the model with a proper choice of the evolution equation for the fatigue parameter $m$
Plan of the seminar

- Theory of hysteresis operators powerful tool for solving mathematical problems in various applications (solid mechanics, material fatigue, ferromagnetism, phase transitions)

Elasto-plastic oscillations of beams and plates with material fatigue

M. Eleuteri, J. Kopfová and P. Krejčí, submitted.

- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Influence of energy dissipation (due to material softening on the damage increase) taken into account - fatigue accumulation accelerated - a singularity (material failure) may develop in finite time
- Account also for decreasing fatigue rate (phase parameter $\chi$)
- Discuss thermodynamic consistency of the model with a proper choice of the evolution equation for the fatigue parameter $m$
Plan of the seminar

- Theory of hysteresis operators powerful tool for solving mathematical problems in various applications (solid mechanics, material fatigue, ferromagnetism, phase transitions)

Elasto-plastic oscillations of beams and plates with material fatigue

M. Eleuteri, J. Kopfová and P. Krejčí, submitted.

- **Main modeling assumption**: proportionality between fatigue rate and dissipation rate
  - Influence of energy dissipation (due to material softening on the damage increase) taken into account - fatigue accumulation accelerated - a singularity (material failure) may develop in finite time
  - Account also for *decreasing fatigue rate* (phase parameter $\chi$)
  - Discuss thermodynamic consistency of the model with a proper choice of the evolution equation for the fatigue parameter $m$
Plan of the seminar

- Theory of hysteresis operators powerful tool for solving mathematical problems in various applications (solid mechanics, material fatigue, ferromagnetism, phase transitions)

**Elasto-plastic oscillations of beams and plates with material fatigue**

M. Eleuteri, J. Kopfová and P. Krejčí, submitted.

- **Main modeling assumption**: proportionality between fatigue rate and dissipation rate

- Influence of energy dissipation (due to material softening on the damage increase) taken into account - fatigue accumulation accelerated - a singularity (material failure) may develop in finite time

- Account also for *decreasing fatigue rate* (phase parameter $\chi$)

- Discuss thermodynamic consistency of the model with a proper choice of the evolution equation for the fatigue parameter $m$
Plan of the seminar

- Theory of hysteresis operators powerful tool for solving mathematical problems in various applications (solid mechanics, material fatigue, ferromagnetism, phase transitions)

Elasto-plastic oscillations of beams and plates with material fatigue

M. Eleuteri, J. Kopfová and P. Krejčí, submitted.

- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Influence of energy dissipation (due to material softening on the damage increase) taken into account - fatigue accumulation accelerated - a singularity (material failure) may develop in finite time
- Account also for decreasing fatigue rate (phase parameter $\chi$)
- Discuss thermodynamic consistency of the model with a proper choice of the evolution equation for the fatigue parameter $m$
Theory of hysteresis operators powerful tool for solving mathematical problems in various applications (solid mechanics, material fatigue, ferromagnetism, phase transitions)

Elasto-plastic oscillations of beams and plates with material fatigue

- M. Eleuteri, J. Kopfová and P. Krejčí, submitted.

- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Influence of energy dissipation (due to material softening on the damage increase) taken into account - fatigue accumulation accelerated - a singularity (material failure) may develop in finite time
- Account also for decreasing fatigue rate (phase parameter $\chi$)
- Discuss thermodynamic consistency of the model with a proper choice of the evolution equation for the fatigue parameter $m$
Plan of the seminar

- Hysteresis in elastoplasticity
  - The material fatigue
  - Main modeling assumption: proportionality between fatigue rate and dissipation rate
  - Thermodynamic consistency
  - Fatigue and phase transitions
  - Numerics in PDEs with hysteresis in elastoplasticity?
Plan of the seminar

- Hysteresis in elastoplasticity
- The material fatigue
  - Main modeling assumption: proportionality between fatigue rate and dissipation rate
  - Thermodynamic consistency
  - Fatigue and phase transitions
  - Numerics in PDEs with hysteresis in elastoplasticity?
Plan of the seminar

- Hysteresis in elastoplasticity
- The material fatigue
- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Thermodynamic consistency
- Fatigue and phase transitions
- Numerics in PDEs with hysteresis in elastoplasticity?
Plan of the seminar

- Hysteresis in elastoplasticity
- The material fatigue
- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Thermodynamic consistency
- Fatigue and phase transitions
- Numerics in PDEs with hysteresis in elastoplasticity?
Plan of the seminar

- Hysteresis in elastoplasticity
- The material fatigue
- Main modeling assumption: proportionality between fatigue rate and dissipation rate
- Thermodynamic consistency
- Fatigue and phase transitions
- Numerics in PDEs with hysteresis in elastoplasticity?
Plan of the seminar

- Hysteresis in elastoplasticity
- The material fatigue
- **Main modeling assumption**: proportionality between fatigue rate and dissipation rate
- Thermodynamic consistency
- Fatigue and phase transitions
- Numerics in PDEs with hysteresis in elastoplasticity?
Hysteresis: a rate-independent memory effect

- **Hysteresis**: a rate-independent memory effect (multidisciplinary character)

![Hysteresis Diagram]

Tipical hysteresis diagram in ferromagnetism ($h$ magnetic field, $m$ magnetization).

- Hysteresis present not only in ferromagnetism, but also in phase transitions, elastoplasticity, shape memory alloys, magnetostrictive and piezoelectric materials, economy, biology...
Hysteresis: a rate-independent memory effect (multidisciplinary character)

- **Hysteresis**: a rate-independent memory effect (multidisciplinary character)

  Typical hysteresis diagram in ferromagnetism ($h$ magnetic field, $m$ magnetization).

  - Hysteresis present not only in ferromagnetism, but also in phase transitions, elastoplasticity, shape memory alloys, magnetostrictive and piezoelectric materials, economy, biology...
The stop model

Figure 1: The stop model.
The stop model

Figure 2: Hysteretic behaviour of the stop model.
Given a parameter $r > 0$, a function $\varepsilon : [0, T] \rightarrow \mathbb{R}$ and an initial condition $\sigma^0 \in [-r, r]$

We look for functions $\sigma, \xi : [0, T] \rightarrow \mathbb{R}$ such that $\sigma(0) = \sigma^0$ and

$$\sigma(t) + \xi(t) = \varepsilon(t)$$

$$|\sigma(t)| \leq r$$

$$\dot{\xi}(t)(\sigma(t) - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r, r].$$

For all $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the previous problem admits a unique solution $\sigma \in W^{1,1}(0, T)$.

The map $s_r : [-r, r] \times W^{1,1}(0, T) \rightarrow W^{1,1}(0, T)$, $s_r[\sigma^0, \varepsilon] = \sigma$ is called stop or elasto-plastic element. $s_r : [-r, r] \times C([0, T]) \rightarrow C([0, T])$.

Multidimensional extension of the stop model.
Given a parameter $r > 0$, a function $\epsilon : [0, T] \rightarrow \mathbb{R}$ and an initial condition $\sigma^0 \in [-r, r]$.

We look for functions $\sigma, \xi : [0, T] \rightarrow \mathbb{R}$ such that $\sigma(0) = \sigma^0$ and

\[
\sigma(t) + \xi(t) = \epsilon(t)
\]

\[
|\sigma(t)| \leq r
\]

\[
\dot{\xi}(t)(\sigma(t) - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r, r].
\]

For all $\epsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the previous problem admits a unique solution $\sigma \in W^{1,1}(0, T)$.

The map $s_r : [-r, r] \times W^{1,1}(0, T) \rightarrow W^{1,1}(0, T)$, $s_r[\sigma^0, \epsilon] = \sigma$ is called stop or elasto-plastic element. $s_r : [-r, r] \times C([0, T]) \rightarrow C([0, T])$.

Multidimensional extension of the stop model.
Given a parameter $r > 0$, a function $\varepsilon : [0, T] \to \mathbb{R}$ and an initial condition $\sigma^0 \in [-r, r]$

We look for functions $\sigma, \xi : [0, T] \to \mathbb{R}$ such that $\sigma(0) = \sigma^0$ and

\[
\sigma(t) + \xi(t) = \varepsilon(t)
\]

\[
|\sigma(t)| \leq r
\]

\[
\dot{\xi}(t)(\sigma(t) - \tilde{\sigma}) \geq 0 \ \forall \tilde{\sigma} \in [-r, r].
\]

For all $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the previous problem admits a unique solution $\sigma \in W^{1,1}(0, T)$

The map $s_r : [-r, r] \times W^{1,1}(0, T) \to W^{1,1}(0, T)$, $s_r[\sigma^0, \varepsilon] = \sigma$ is called stop or elasto-plastic element. $s_r : [-r, r] \times C([0, T]) \to C([0, T])$

Multidimensional extension of the stop model
Given a parameter $r > 0$, a function $\epsilon : [0, T] \to \mathbb{R}$ and an initial condition $\sigma^0 \in [-r, r]$,

- We look for functions $\sigma, \xi : [0, T] \to \mathbb{R}$ such that $\sigma(0) = \sigma^0$ and

$$\sigma(t) + \xi(t) = \epsilon(t)$$

$$|\sigma(t)| \leq r$$

$$\dot{\xi}(t)(\sigma(t) - \bar{\sigma}) \geq 0 \ \forall \bar{\sigma} \in [-r, r].$$

For all $\epsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the previous problem admits a unique solution $\sigma \in W^{1,1}(0, T)$.

- The map $s_r : [-r, r] \times W^{1,1}(0, T) \to W^{1,1}(0, T)$, $s_r[\sigma^0, \epsilon] = \sigma$ is called stop or elasto-plastic element. $s_r : [-r, r] \times C([0, T]) \to C([0, T])$.

- Multidimensional extension of the stop model.
A classical hysteresis-type model for 1D elastoplasticity

- Given a parameter $r > 0$, a function $\varepsilon : [0, T] \rightarrow \mathbb{R}$ and an initial condition $\sigma^0 \in [-r, r]$.
- We look for functions $\sigma, \xi : [0, T] \rightarrow \mathbb{R}$ such that $\sigma(0) = \sigma^0$ and

$$\sigma(t) + \xi(t) = \varepsilon(t),$$

$$|\sigma(t)| \leq r,$$

$$\dot{\xi}(t)(\sigma(t) - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r, r].$$

- For all $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the previous problem admits a unique solution $\sigma \in W^{1,1}(0, T)$.
- The map $s_r : [-r, r] \times W^{1,1}(0, T) \rightarrow W^{1,1}(0, T)$, $s_r[\sigma^0, \varepsilon] = \sigma$ is called stop or elasto-plastic element. $s_r : [-r, r] \times C([0, T]) \rightarrow C([0, T])$.

- Multidimensional extension of the stop model.
Given a parameter $r > 0$, a function $\varepsilon : [0, T] \rightarrow \mathbb{R}$ and an initial condition $\sigma^0 \in [-r, r]$

We look for functions $\sigma, \xi : [0, T] \rightarrow \mathbb{R}$ such that $\sigma(0) = \sigma^0$ and

$$\sigma(t) + \xi(t) = \varepsilon(t)$$

$$|\sigma(t)| \leq r$$

$$\dot{\xi}(t)(\sigma(t) - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r, r].$$

For all $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the previous problem admits a unique solution $\sigma \in W^{1,1}(0, T)$

The map $s_r : [-r, r] \times W^{1,1}(0, T) \rightarrow W^{1,1}(0, T)$, $s_r[\sigma^0, \varepsilon] = \sigma$ is called **stop** or **elasto-plastic element**. $s_r : [-r, r] \times C([0, T]) \rightarrow C([0, T])$

Multidimensional extension of the stop model
The multidimensional stop model

Given a closed convex set $Z$ including the origin in his interior $0$ in a real separable Banach space $X$, a function $\varepsilon : [0,T] \to X$ and an initial condition $\sigma^0 \in Z$, we look for functions $\sigma, \xi : [0,T] \to X$ such that $\sigma(0) = \sigma^0$ and

$$\begin{align*}
\sigma(x,0) &= Q_Z(\varepsilon(x,0)) \\
\sigma(t) &\in Z \\
\left\langle \dot{\xi}(t), \sigma(t) - \tilde{\sigma} \right\rangle &\geq 0 \quad \forall \tilde{\sigma} \in Z.
\end{align*}$$

For all $\varepsilon \in W^{1,1}(0,T;X)$ and $\sigma^0 \in Z$, the previous system admits a unique solution $s_Z[\sigma^0, \varepsilon] = \sigma \in W^{1,1}(0,T;X)$. The map $s_Z : Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ is continuous and admits a continuous extension $s_Z : Z \times C([0,T];X) \to C([0,T];X)$. 
A classical hysteresis-type model for one-dimensional elastoplasticity was introduced by L. Prandtl and A. Yu. Ishlinskii.

In their model, the relation between (one-dimensional) strain $\varepsilon$ and stress $\sigma$ is given in the form of the so-called Prandtl-Ishlinskii operator

$$\sigma = \mathcal{P}[\varepsilon](t) = \int_{0}^{\infty} s_{r}[\varepsilon](t) \varphi(r) \, dr$$

for all $\varepsilon \in W^{1,1}(0, T)$. Here $\varphi > 0$ is a nonnegative weight function not known a priori and $s_{r}$ represents the one-dimensional elastic-ideally plastic element or stop operator, with the threshold $r > 0$. 
A classical hysteresis-type model for one-dimensional elastoplasticity was introduced by L. Prandtl and A. Yu. Ishlinskii.

In their model, the relation between (one-dimensional) strain $\varepsilon$ and stress $\sigma$ is given in the form of the so-called Prandtl-Ishlinskii operator

$$\sigma = \mathcal{P}[\varepsilon](t) = \int_0^{\infty} s_r[\varepsilon](t) \varphi(r) \, dr$$

for all $\varepsilon \in W^{1,1}(0, T)$. Here $\varphi > 0$ is a nonnegative weight function not known a priori and $s_r$ represents the one-dimensional elastic-ideally plastic element or stop operator, with the threshold $r > 0$.
The stop operators and their combinations

Figure 3: the stop operators and their combinations.

\[ \sigma = \varphi_2 \mathcal{S}_{r_2}[\varepsilon] \]

\[ \sigma = \varphi_1 \mathcal{S}_{r_1}[\varepsilon] \]

\[ \sigma = \varphi_1 \mathcal{S}_{r_1}[\varepsilon] + \varphi_2 \mathcal{S}_{r_2}[\varepsilon] \]
Prandtl-Ishlinskii model v.s. Von Mises model

The Prandtl-Ishlinskii model

- very imaginative and easily understood (superposition of many stops having different thresholds)
- multi-yield: describes gradual plasticization process
- the weight function $\phi$ is not known a priori and must be identified

The Von Mises model

- simple (solution of variational inequality)
- single-yield: sharp interface between elastic and plastic regime - not so realistic!
The Prandtl-Ishlinskii model

😊 very imaginative and easily understood (superposition of many stops having different thresholds)

😊 multi-yield: describes gradual plasticization process

😊 the weight function $\varphi$ is not known a priori and must be identified

The Von Mises model

😊 simple (solution of variational inequality)

😊 single-yield: sharp interface between elastic and plastic regime - not so realistic!
The Prandtl-Ishlinskii model

- very imaginative and easily understood (superposition of many stops having different thresholds)
- \textbf{multi-yield}: describes gradual plasticization process
- the weight function $\varphi$ \textbf{is not known a priori} and must be identified

The Von Mises model

- simple (solution of variational inequality)
- single-yield: sharp interface between elastic and plastic regime - not so realistic!
Prandtl-Ishlinskii model v.s. Von Mises model

The Prandtl-Ishlinskii model

- very imaginative and easily understood (superposition of many stops having different thresholds)
- multi-yield: describes gradual plasticization process
- the weight function \( \varphi \) is not known a priori and must be identified

The Von Mises model

- simple (solution of variational inequality)
- single-yield: sharp interface between elastic and plastic regime - not so realistic!
The Prandtl-Ishlinskii model

- very imaginative and easily understood (superposition of many stops having different thresholds)
- multi-yield: describes gradual plasticization process
- the weight function $\varphi$ is not known a priori and must be identified

The Von Mises model

- simple (solution of variational inequality)
- single-yield: sharp interface between elastic and plastic regime - not so realistic!
The Prandtl-Ishlinskii model

- very imaginative and easily understood (superposition of many stops having different thresholds)
- multi-yield: describes gradual plasticization process
- the weight function $\varphi$ is not known a priori and must be identified

The Von Mises model

- simple (solution of variational inequality)
- single-yield: sharp interface between elastic and plastic regime - not so realistic!
Prandtl-Ishlinskii model v.s. Von Mises model

The Prandtl-Ishlinskii model

- very imaginative and easily understood (superposition of many stops having different thresholds)
- **multi-yield**: describes gradual plasticization process
- the weight function $\varphi$ is not known *a priori* and must be identified

The Von Mises model

- simple (solution of variational inequality)
- **single-yield**: sharp interface between elastic and plastic regime - not so realistic!
They demonstrated that the three-dimensional single-yield von Mises criterion leads after a dimensional reduction to a multi-yield Prandtl-Ishlinskii operator where the weight function $\varphi$ can be explicitly determined!

The new theory is based on the idea that a lower-dimensional observer does not see anymore the sharp transition from the purely elastic to the purely plastic regime as in the von Mises model.
New theory of oscillating elastoplastic beams and plates


- They demonstrated that the three-dimensional single-yield von Mises criterion leads after a dimensional reduction to a multi-yield Prandtl-Ishlinskii operator where the weight function $\varphi$ can be explicitly determined!

- The new theory is based on the idea that a lower-dimensional observer does not see anymore the sharp transition from the purely elastic to the purely plastic regime as in the von Mises model.
They demonstrated that the three-dimensional single-yield von Mises criterion leads after a dimensional reduction to a multi-yield Prandtl-Ishlinskii operator where the weight function $\varphi$ can be explicitly determined!

The new theory is based on the idea that a lower-dimensional observer does not see anymore the sharp transition from the purely elastic to the purely plastic regime as in the von Mises model.
Figure 4: A plate section with grey plasticized zone.
Motivation for the material fatigue

- It is well known that plastic deformations lead to energy dissipation and material fatigue.
- Material fatigue is manifested by material softening, heat release, material failure in finite time.
- Very important: take into account the effects of energy exchange between heat and mechanical energy, thermal stresses, and material fatigue.
- In particular great importance for the applications: methods for estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue.
- The existing mathematical literature mainly goes in the direction of the quasistatic approach of elastoplastic processes with hysteresis; few results on the dynamics.
Motivation for the material fatigue

- It is well known that plastic deformations lead to energy dissipation and material fatigue.
- Material fatigue is manifested by material softening, heat release, material failure in finite time.
- Very important: take into account the effects of energy exchange between heat and mechanical energy, thermal stresses, and material fatigue.
- In particular great importance for the applications: methods for estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue.
- The existing mathematical literature mainly goes in the direction of the quasistatic approach of elastoplastic processes with hysteresis; few results on the dynamics.
Motivation for the material fatigue

- It is well known that plastic deformations lead to energy dissipation and material fatigue
- Material fatigue is manifested by material softening, heat release, material failure in finite time
- Very important: take into account the effects of energy exchange between heat and mechanical energy, thermal stresses, and material fatigue
- In particular great importance for the applications: methods for estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue
- The existing mathematical literature mainly goes in the direction of the quasistatic approach of elastoplastic processes with hysteresis; few results on the dynamics

Michela Eleuteri Non-isothermal cyclic fatigue in oscillating elasto-plastic structures with hysteresis
It is well known that plastic deformations lead to energy dissipation and material fatigue.

Material fatigue is manifested by material softening, heat release, material failure in finite time.

Very important: take into account the effects of energy exchange between heat and mechanical energy, thermal stresses, and material fatigue.

In particular great importance for the applications: methods for estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue.

The existing mathematical literature mainly goes in the direction of the quasistatic approach of elastoplastic processes with hysteresis; few results on the dynamics.
Motivation for the material fatigue

- It is well known that plastic deformations lead to energy dissipation and material fatigue.
- Material fatigue is manifested by material softening, heat release, material failure in finite time.
- Very important: take into account the effects of energy exchange between heat and mechanical energy, thermal stresses, and material fatigue.
- In particular great importance for the applications: methods for estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue.
- The existing mathematical literature mainly goes in the direction of the quasistatic approach of elastoplastic processes with hysteresis; few results on the dynamics.
Literature concerning oscillating elastoplastic processes

- **Classical approach**
  
  **G. Duvaut, J.L. Lions** (1972) - dynamical problem of plasticity of Prandtl-Reuss type is solved through the use of variational inequalities

- **Quasistatic approach**
  
  W. Han, B.D. Reddy (1999); yield condition described by a sharp surface of plasticity - M. Brokate, A.M. Khludnev (2000); M. Kuczma, P. Litewka, J. Rakoswki, J.R. Whiteman (2004); O. Millet, A. Cimetiere, A. Hamdouni (2003)

- **Γ-convergence approach**
  
  (it doesn’t extend to the case of oscillating systems) D. Percivale (1990); G. Friesecke, R.D. James, S. Müller (2006); M. Liero, A. Mielke (2010); M. Liero, T. Roche (preprint 2011) - They get the Prandtl-Ishlinskii model at the limit

- “Multiyield” character but without explicit reference to plates
  
Literature concerning oscillating elastoplastic processes

- **Classical approach**
  
  **G. Duvaut, J.L. Lions** (1972) - dynamical problem of plasticity of Prandtl-Reuss type is solved through the use of variational inequalities

- **Quasistatic approach**
  
  **W. Han, B.D. Reddy** (1999); yield condition described by a sharp surface of plasticity - **M. Brokate, A.M. Khludnev** (2000); **M. Kuczma, P. Litewka, J. Rakoswki, J.R. Whiteman** (2004); **O. Millet, A. Cimetiere, A. Hamdouni** (2003)

- **Γ-convergence approach**
  
  (it doesn’t extend to the case of oscillating systems) **D. Percivale** (1990); **G. Friesecke, R.D. James, S. Müller** (2006); **M. Liero, A. Mielke** (2010); **M. Liero, T. Roche** (preprint 2011) - They get the Prandtl-Ishlinskii model at the limit

- **“Multiyield” character but without explicit reference to plates**
  
  **M. Brokate, C. Carstensen, J. Valdman** (2004)
Literature concerning oscillating elastoplastic processes

- **Classical approach**
  
  *G. Duvaut, J.L. Lions* (1972) - dynamical problem of plasticity of Prandtl-Reuss type is solved through the use of variational inequalities

- **Quasistatic approach**
  

- **Γ-convergence approach**
  
  (it doesn’t extend to the case of oscillating systems) *D. Percivale* (1990); *G. Friesecke, R.D. James, S. Müller* (2006); *M. Liero, A. Mielke* (2010); *M. Liero, T. Roche* (preprint 2011) - They get the Prandtl-Ishlinskii model at the limit

- “Multiyield” character but without explicit reference to plates
  
Literature concerning oscillating elastoplastic processes

- **Classical approach**
  G. Duvaut, J.L. Lions (1972) - dynamical problem of plasticity of Prandtl-Reuss type is solved through the use of variational inequalities

- **Quasistatic approach**
  W. Han, B.D. Reddy (1999); yield condition described by a sharp surface of plasticity - M. Brokate, A.M. Khludnev (2000); M. Kuczma, P. Litewka, J. Rakoswki, J.R. Whiteman (2004); O. Millet, A. Cimetiere, A. Hamdouni (2003)

- **$\Gamma$-convergence approach**
  (it doesn’t extend to the case of oscillating systems) D. Percivale (1990); G. Friesecke, R.D. James, S. Müller (2006); M. Liero, A. Mielke (2010); M. Liero, T. Roche (preprint 2011) - They get the Prandtl-Ishlinskii model at the limit

- "Multiyield" character but without explicit reference to plates
The system of PDEs

- The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

\[
\partial_{tt} w - \partial_{tt} \Delta w + D_2^* \sigma = g,
\]

\[
\sigma = B \varepsilon + \int_0^\infty s_{rZ} [\varepsilon](t) \varphi(r) \, dr
\]

\[
\varepsilon = D_2 w
\]

where \(D_2\) is the second derivative operator \((\partial_{xx}, \partial_{yy}, \partial_{xy})\) and \(D_2^*\) is its adjoint.

- We introduce a positive parameter \(\theta > 0\) indicating the absolute temperature and \(m(x,t) \geq 0\) a parameter which represents the material fatigue accumulated in the point \(x\) in the time interval \([0,t]\).

- Basic modeling assumption: replacing the classical elastoplastic constitutive law with a new one, where we account for the material fatigue and where the memory keeps into consideration the temperature.
The system of PDEs

The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

\[
\partial_{tt} w - \partial_{tt} \Delta w + D_2^* \sigma = g, \\
\sigma = B \varepsilon + \int_0^\infty s_{rZ}[\varepsilon](t) \varphi(r) \, dr \\
\varepsilon = D_2 w
\]

where \( D_2 \) is the second derivative operator \((\partial_{xx}, \partial_{yy}, \partial_{xy})\) and \( D_2^* \) is its adjoint.

We introduce a positive parameter \( \theta > 0 \) indicating the absolute temperature and \( m(x,t) \geq 0 \) a parameter which represents the material fatigue accumulated in the point \( x \) in the time interval \([0,t]\).

Basic modeling assumption: replacing the classical elastoplastic constitutive law with a new one, where we account for the material fatigue and where the memory keeps into consideration the temperature.
The system of PDEs

- The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

\[
\partial_{tt} w - \partial_{tt} \Delta w + D^*_2 \sigma = g,
\]

\[
\sigma = B\varepsilon + \int_0^\infty s_r Z[\varepsilon](t) \varphi(r) \, dr
\]

\[
\varepsilon = D_2 w
\]

where \( D_2 \) is the second derivative operator \((\partial_{xx}, \partial_{yy}, \partial_{xy})\) and \( D^*_2 \) is its adjoint.

- We introduce a positive parameter \( \theta > 0 \) indicating the absolute temperature and \( m(x, t) \geq 0 \) a parameter which represents the material fatigue accumulated in the point \( x \) in the time interval \([0, t]\).

- Basic modeling assumption: replacing the classical elastoplastic constitutive law with a new one, where we account for the material fatigue and where the memory keeps into consideration the temperature.
The system of PDEs

The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

\[ \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial t^2} \Delta w + D_2^* \sigma = g, \]

\[ \sigma = B\varepsilon + \int_0^\infty s_r Z[\varepsilon](t) \varphi(r) \, dr \]

\[ \varepsilon = D_2 w \]

where \( D_2 \) is the second derivative operator \((\partial_{xx}, \partial_{yy}, \partial_{xy})\) and \( D_2^* \) is its adjoint.

- We introduce a positive parameter \( \theta > 0 \) indicating the absolute temperature and \( m(x, t) \geq 0 \) a parameter which represents the material fatigue accumulated in the point \( x \) in the time interval \([0, t]\).

- **Basic modeling assumption:** replacing the classical elastoplastic constitutive law with a new one, where we account for the material fatigue and where the memory keeps into consideration the temperature.
The system of PDEs

- The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

  \[ \partial_{tt} w - \partial_{tt} \Delta w + D_2^* \sigma = g , \]

  \[ \sigma = B(m) \varepsilon + \int_0^\infty s_{rZ} [\varepsilon](t) \varphi(\theta, r) \, dr - \beta (\theta - \theta_c) 1 \]

  \[ \varepsilon = D_2 w \]

  where \( D_2 \) is the second derivative operator \((\partial_{xx}, \partial_{yy}, \partial_{xy})\) and \( D_2^* \) is its adjoint.

- We introduce a positive parameter \( \theta > 0 \) indicating the absolute temperature and \( m(x, t) \geq 0 \) a parameter which represents the material fatigue accumulated in the point \( x \) in the time interval \([0, t]\).

- Basic modeling assumption: replacing the classical elastoplastic constitutive law with a new one, where we account for the material fatigue and where the memory keeps into consideration the temperature.
The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

\[ \partial_{tt}w - \partial_{tt}\Delta w + D^*_2\sigma = g, \]

\[ \sigma = \mathbf{B}(m)\varepsilon + \int_0^\infty s_{rZ}[\varepsilon](t) \varphi(\theta, r) \, dr - \beta(\theta - \theta_c)\mathbf{1} \]

\[ \varepsilon = D_2w, \]

where \( \beta \) is the thermal dilation coefficient, \( \mathbf{1} \) is the constant vector \((1, 1, 0)\), \( \theta_c \) is a fixed reference temperature and the kinematic hardening matrix \( \mathbf{B} \) depends on \( m \).

The aim is now the following: to obtain an evolution equation for \( m \) which is consistent from the thermodynamic point of view.
The system of PDEs

- The resulting system from the theory developed by Krejčí and coworkers (starting point of our new project) is the following:

\[ \partial_{tt}w - \partial_{tt}\Delta w + D^*_2\sigma = g, \]

\[ \sigma = B(m)\varepsilon + \int_0^\infty s_rZ[\varepsilon](t)\varphi(\theta, r)dr - \beta(\theta - \theta_c)1, \]

\[ \varepsilon = D_2w, \]

where \( \beta \) is the thermal dilation coefficient, \( 1 \) is the constant vector \( (1, 1, 0) \), \( \theta_c \) is a fixed reference temperature and the kinematic hardening matrix \( B \) depends on \( m \).

- The aim is now the following: to obtain an evolution equation for \( m \) which is consistent from the thermodynamic point of view.
Coupling with the energy balance laws

- To the previous system we associate the **specific free energy**

\[
\mathcal{F}[\theta, \varepsilon] = c_V \theta (1 - \log(\theta / \theta_c)) + \frac{1}{2} \langle \mathbf{B}(m) \varepsilon, \varepsilon \rangle \\
+ \frac{1}{2} \int_0^\infty \langle s_r Z[\varepsilon], s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr - \beta (\theta - \theta_c) \langle \varepsilon, 1 \rangle,
\]

with a constant specific heat \( c_V > 0 \)

- The **specific entropy** has the following form

\[
\mathcal{S}[\theta, \varepsilon] = c_V \log(\theta / \theta_c) - \frac{1}{2} \int_0^\infty \langle s_r Z[\varepsilon], s_r Z[\varepsilon] \rangle \partial_\theta \varphi(\theta, r) \, dr \\
+ \beta \langle \varepsilon, 1 \rangle,
\]

- from which, exploiting the well know relation \( \mathcal{F} = \mathcal{U} - \theta \mathcal{S} \) we get the following form of the **internal energy**

\[
\mathcal{U}[\theta, \varepsilon] = c_V \theta + \frac{1}{2} \langle \mathbf{B}(m) \varepsilon, \varepsilon \rangle \\
+ \frac{1}{2} \int_0^\infty \langle s_r Z[\varepsilon], s_r Z[\varepsilon] \rangle (\varphi(\theta, r) - \theta \partial_\theta \varphi(\theta, r)) \, dr + \beta \theta_c \langle \varepsilon, 1 \rangle.
\]
Coupling with the energy balance laws

- To the previous system we associate the specific free energy

\[ \mathcal{F}[\theta, \varepsilon] = c_V \theta (1 - \log(\theta/\theta_c)) + \frac{1}{2} \langle \mathbf{B}(m) \varepsilon, \varepsilon \rangle + \frac{1}{2} \int_0^\infty \langle s_rZ[\varepsilon], s_rZ[\varepsilon] \rangle \varphi(\theta, r) \, dr - \beta (\theta - \theta_c) \langle \varepsilon, 1 \rangle, \]

with a constant specific heat \( c_V > 0 \)

- The specific entropy has the following form

\[ \mathcal{S}[\theta, \varepsilon] = c_V \log(\theta/\theta_c) - \frac{1}{2} \int_0^\infty \langle s_rZ[\varepsilon], s_rZ[\varepsilon] \rangle \partial_\theta \varphi(\theta, r) \, dr + \beta \langle \varepsilon, 1 \rangle, \]

from which, exploiting the well know relation \( \mathcal{F} = \mathcal{U} - \theta \mathcal{S} \) we get the following form of the internal energy

\[ \mathcal{U}[\theta, \varepsilon] = c_V \theta + \frac{1}{2} \langle \mathbf{B}(m) \varepsilon, \varepsilon \rangle + \frac{1}{2} \int_0^\infty \langle s_rZ[\varepsilon], s_rZ[\varepsilon] \rangle (\varphi(\theta, r) - \theta \partial_\theta \varphi(\theta, r)) \, dr + \beta \theta_c \langle \varepsilon, 1 \rangle. \]
Coupling with the energy balance laws

- To the previous system we associate the **specific free energy**

\[
\mathcal{F}[\theta, \varepsilon] = c_V \theta (1 - \log(\theta/\theta_c)) + \frac{1}{2} \langle B(m) \varepsilon, \varepsilon \rangle + \frac{1}{2} \int_0^\infty \langle s_{rZ}[\varepsilon], s_{rZ}[\varepsilon] \rangle \varphi(\theta, r) \, dr - \beta (\theta - \theta_c) \langle \varepsilon, 1 \rangle,
\]

with a constant specific heat \(c_V > 0\)

- The **specific entropy** has the following form

\[
\mathcal{S}[\theta, \varepsilon] = c_V \log(\theta/\theta_c) - \frac{1}{2} \int_0^\infty \langle s_{rZ}[\varepsilon], s_{rZ}[\varepsilon] \rangle \partial_\theta \varphi(\theta, r) \, dr + \beta \langle \varepsilon, 1 \rangle,
\]

- from which, exploiting the well know relation \(\mathcal{F} = \mathcal{U} - \theta \mathcal{S}\) we get the following form of the **internal energy**

\[
\mathcal{U}[\theta, \varepsilon] = c_V \theta + \frac{1}{2} \langle B(m) \varepsilon, \varepsilon \rangle + \frac{1}{2} \int_0^\infty \langle s_{rZ}[\varepsilon], s_{rZ}[\varepsilon] \rangle (\varphi(\theta, r) - \theta \partial_\theta \varphi(\theta, r)) \, dr + \beta \theta_c \langle \varepsilon, 1 \rangle.
\]
The energy balance can be written as
\[ \partial_t \mathcal{U}[\theta, \varepsilon] + \text{div} \mathbf{q} = \langle \sigma, \partial_t \varepsilon \rangle, \]
where \( \mathbf{q} \) is the heat flux vector.

We derive the evolution law for the fatigue parameter which has to be compatible with the Second Principle of Thermodynamics, which we state in the form of the Clausius-Duhem inequality
\[ \psi := \partial_t \mathcal{S}[\theta, \varepsilon] + \text{div} \left( \frac{\mathbf{q}}{\theta} \right) \geq 0, \]
where \( \psi \) is the entropy production.

This implies that the dissipation rate
\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{S}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] 
= -\frac{1}{2} \langle \mathcal{B}'(m)\varepsilon, \varepsilon \rangle \partial_t m + \int_0^{\infty} \langle \partial_t (\varepsilon - \mathcal{S}_Z[\varepsilon]), \mathcal{S}_Z[\varepsilon] \rangle \varphi(\theta, r) \, dr \]
has to be non-negative.
Coupling with the energy balance laws

- The **energy balance** can be written as
  \[
  \partial_t \mathcal{U}[\theta, \varepsilon] + \text{div} \mathbf{q} = \langle \sigma, \partial_t \varepsilon \rangle,
  \]
  where \( \mathbf{q} \) is the heat flux vector.

- We derive the evolution law for the fatigue parameter which has to be compatible with the **Second Principle of Thermodynamics**, which we state in the form of the **Clausius-Duhem inequality**
  \[
  \psi := \partial_t \mathcal{S}[\theta, \varepsilon] + \text{div} \left( \frac{\mathbf{q}}{\theta} \right) \geq 0,
  \]
  where \( \psi \) is the entropy production.

- This implies that the **dissipation rate**
  \[
  \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{S}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon]
  = -\frac{1}{2} \langle \mathbf{B}^\prime(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r \mathcal{Z}[\varepsilon]), s_r \mathcal{Z}[\varepsilon] \rangle \varphi(\theta, r) \, dr
  \]
  has to be non-negative.
The **energy balance** can be written as

$$\partial_t \mathcal{U} [\theta, \varepsilon] + \text{div} \, \mathbf{q} = \langle \sigma, \partial_t \varepsilon \rangle,$$

where $\mathbf{q}$ is the heat flux vector.

We derive the evolution law for the fatigue parameter which has to be compatible with the **Second Principle of Thermodynamics**, which we state in the form of the **Clausius-Duhem inequality**

$$\psi := \partial_t \mathcal{S} [\theta, \varepsilon] + \text{div} \left( \frac{\mathbf{q}}{\theta} \right) \geq 0,$$

where $\psi$ is the entropy production.

This implies that the **dissipation rate**

$$\mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{F} [\theta, \varepsilon] - \partial_t \mathcal{F} [\theta, \varepsilon]$$

$$= - \frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z [\varepsilon]), s_r Z [\varepsilon] \rangle \varphi(\theta, r) \, dr$$

has to be non-negative.
Evolution equation for the fatigue

\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{I} [\theta, \varepsilon] - \partial_t \mathcal{F} [\theta, \varepsilon] \]

\[ = -\frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z [\varepsilon]), s_r Z [\varepsilon] \rangle \varphi (\theta, r) dr \]

- The integral is non-negative by virtue of the variational inequality which defines the stop operator.
- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( \mathbf{B}'(m) \) is a negative semidefinite matrix (softening!)
- The system will be complete by assuming the linear Fourier law between the heat flux and the temperature gradient

\[ q = -\kappa \nabla \theta , \]

- Fundamental assumption: proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \).
Evolution equation for the fatigue 

\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{I}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] \]

\[ = -\frac{1}{2} \langle \mathbf{B}'(m)\varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr \]

- The integral is non negative by virtue of the variational inequality which defines the stop operator

- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( \mathbf{B}'(m) \) is a negative semidefinite matrix (softening!)

- The system will be complete by assuming the linear Fourier law between the heat flux and the temperature gradient

\[ \mathbf{q} = -\kappa \nabla \theta , \]

- Fundamental assumption: proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \)
Evolution equation for the fatigue

\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{I} [\theta, \varepsilon] - \partial_t \mathcal{F} [\theta, \varepsilon] \]

\[ = - \frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^{\infty} \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr \]

- The integral is non negative by virtue of the variational inequality which defines the stop operator.
- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( \mathbf{B}'(m) \) is a negative semidefinite matrix (softening!)
- The system will be complete by assuming the linear Fourier law between the heat flux and the temperature gradient

\[ q = -\kappa \nabla \theta, \]

- **Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \).
Evolution equation for the fatigue

\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{I}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] \]

\[ = -\frac{1}{2} \langle B'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr \]

- The integral is non-negative by virtue of the variational inequality which defines the stop operator.

- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( B'(m) \) is a negative semidefinite matrix (softening!)

- The system will be complete by assuming the linear Fourier law between the heat flux and the temperature gradient

\[ q = -\kappa \nabla \theta , \]

**Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \).
Evolution equation for the fatigue

\[
\mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{J}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon]
\]

\[
= -\frac{1}{2} \langle B'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \phi(\theta, r) \, dr
\]

- The integral is non negative by virtue of the variational inequality which defines the stop operator.

- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( B'(m) \) is a negative semidefinite matrix (softening!).

- The system will be complete by assuming the **linear Fourier law** between the heat flux and the temperature gradient

\[
q = -\kappa \nabla \theta
\]

- **Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \).
The rainflow algorithm

- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)
- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy.
- With an undamaged material, we associate fatigue value 0, and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue.
- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude $b$ lead to total failure.
- With each closed cycle of amplitude $b$ we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue.
- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöhler line.
The rainflow algorithm

- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)

- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy.

- With an undamaged material, we associate fatigue value 0, and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue.

- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude $b$ lead to total failure.

- With each closed cycle of amplitude $b$ we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue.

- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöhler line.
The rainflow algorithm

- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)

- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy.

- With an undamaged material, we associate fatigue value 0, and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue.

- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude $b$ lead to total failure.

- With each closed cycle of amplitude $b$ we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue.

- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöler line.
The rainflow algorithm

- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)

- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy.

- With an undamaged material, we associate fatigue value 0, and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue.

- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude $b$ lead to total failure.

- With each closed cycle of amplitude $b$ we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue.

- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöler line.
**The rainflow algorithm**

- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)
- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy.
- With an undamaged material, we associate fatigue value 0, and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue.
- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude $b$ lead to total failure.
- With each closed cycle of amplitude $b$ we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue.
- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöler line.
The rainflow algorithm

- **Rainflow method for cyclic fatigue accumulation** - M. Endo (1968)
- The analysis of the rainflow algorithm in uniaxial processes has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy.
- With an undamaged material, we associate fatigue value 0, and 1 corresponds to total failure. Values between 0 and 1 quantify the degree of fatigue.
- Experimental (decreasing!) curve $n(b)$ (the so-called Wöhler line) determines how many closed cycles of amplitude $b$ lead to total failure.
- With each closed cycle of amplitude $b$ we associate the contribution $d(b) = \frac{1}{n(b)}$ of the individual cycle to total fatigue.
- Therefore, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude (the contribution of the loop to total damage) taken from the Wöler line.
The rainflow algorithm

- This corresponds to the mechanism of energy dissipation: the number associated with a closed loop is its area in this case.
- Let a real loading process consist of a sequence of cycles with amplitudes $b_j$, $j = 1, \ldots, n$. The Palmgren-Miner additivity rule states that the total fatigue can be computed as the sum of all individual contributions $D_n = \Sigma_{j=1}^{n} d(b_j)$.
- When the contribution of a closed cycle to total fatigue is evaluated, the cycle is removed from the history. At the end of the process, the remaining residual of the input signal contains no more closed cycles.
- For a numerical treatment of large data sets it is important to assume that oscillations of very small amplitude cause no damage.
- The rainflow method is then stable with respect to small measurement errors independently of the number of cycles.
The rainflow algorithm

- This corresponds to the mechanism of energy dissipation: the number associated with a closed loop is its area in this case.

- Let a real loading process consist of a sequence of cycles with amplitudes $b_j$, $j = 1, \ldots, n$. The Palmgren-Miner additivity rule states that the total fatigue can be computed as the sum of all individual contributions $D_n = \sum_{j=1}^{n} d(b_j)$.

- When the contribution of a closed cycle to total fatigue is evaluated, the cycle is removed from the history. At the end of the process, the remaining residual of the input signal contains no more closed cycles.

- For a numerical treatment of large data sets it is important to assume that oscillations of very small amplitude cause no damage.

- The rainflow method is then stable with respect to small measurement errors independently of the number of cycles.

The rainflow algorithm

- This corresponds to the mechanism of energy dissipation: the number associated with a closed loop is its area in this case.
- Let a real loading process consist of a sequence of cycles with amplitudes $b_j$, $j = 1, \ldots, n$. The Palmgren-Miner additivity rule states that the total fatigue can be computed as the sum of all individual contributions $D_n = \sum_{j=1}^{n} d(b_j)$.
- When the contribution of a closed cycle to total fatigue is evaluated, the cycle is removed from the history. At the end of the process, the remaining residual of the input signal contains no more closed cycles.
- For a numerical treatment of large data sets it is important to assume that oscillations of very small amplitude cause no damage.
- The rainflow method is then stable with respect to small measurement errors independently of the number of cycles.
The rainflow algorithm

- This corresponds to the mechanism of energy dissipation: the number associated with a closed loop is its area in this case.

- Let a real loading process consist of a sequence of cycles with amplitudes \( b_j, j = 1, \ldots, n \). The Palmgren-Miner additivity rule states that the total fatigue can be computed as the sum of all individual contributions \( D_n = \sum_{j=1}^{n} d(b_j) \).

- When the contribution of a closed cycle to total fatigue is evaluated, the cycle is removed from the history. At the end of the process, the remaining residual of the input signal contains no more closed cycles.

- For a numerical treatment of large data sets it is important to assume that oscillations of very small amplitude cause no damage.

- The rainflow method is then stable with respect to small measurement errors independently of the number of cycles.

The rainflow algorithm

- This corresponds to the mechanism of energy dissipation: the number associated with a closed loop is its area in this case.
- Let a real loading process consist of a sequence of cycles with amplitudes $b_j$, $j = 1, \ldots, n$. The Palmgren-Miner additivity rule states that the total fatigue can be computed as the sum of all individual contributions $D_n = \sum_{j=1}^{n} d(b_j)$.
- When the contribution of a closed cycle to total fatigue is evaluated, the cycle is removed from the history. At the end of the process, the remaining residual of the input signal contains no more closed cycles.
- For a numerical treatment of large data sets it is important to assume that oscillations of very small amplitude cause no damage.
- The rainflow method is then stable with respect to small measurement errors independently of the number of cycles.

The rainflow algorithm

- This corresponds to the mechanism of energy dissipation: the number associated with a closed loop is its area in this case.

- Let a real loading process consist of a sequence of cycles with amplitudes $b_j$, $j = 1, \ldots, n$. The Palmgren-Miner additivity rule states that the total fatigue can be computed as the sum of all individual contributions $D_n = \sum_{j=1}^{n} d(b_j)$.

- When the contribution of a closed cycle to total fatigue is evaluated, the cycle is removed from the history. At the end of the process, the remaining residual of the input signal contains no more closed cycles.

- For a numerical treatment of large data sets it is important to assume that oscillations of very small amplitude cause no damage.

- The rainflow method is then stable with respect to small measurement errors independently of the number of cycles.

In multiassial processes?

- **Drawback:** the rainflow method is exclusively uniaxial - no closed cycles in the vector case
- In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known
- On the other hand, the notion of energy dissipation is independent of the experimental setting: experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak
- In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry).

**Fundamental assumption**

Assuming that the fatigue rate is proportional to the dissipation rate, and that the material parameters depend on the fatigue parameter

- Assumption realistic: plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid
In multiassial processes?

- **Drawback**: the rainflow method is exclusively uniaxial - no closed cycles in the vector case
- In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known
- On the other hand, the notion of energy dissipation is independent of the experimental setting: experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak
- In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry).

**Fundamental assumption**

Assuming that the fatigue rate is proportional to the dissipation rate, and that the material parameters depend on the fatigue parameter

- Assumption realistic: plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid
In multiassial processes?

- **Drawback:** the rainflow method is exclusively uniaxial - no closed cycles in the vector case
- In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known
- On the other hand, the notion of energy dissipation is independent of the experimental setting: experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak
- In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry).

**Fundamental assumption**

Assuming that the fatigue rate is proportional to the dissipation rate, and that the material parameters depend on the fatigue parameter

- Assumption realistic: plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid
In multiassial processes?

- **Drawback**: the rainflow method is exclusively uniaxial - no closed cycles in the vector case
- In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known
- On the other hand, the notion of energy dissipation is independent of the experimental setting: experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak
- In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry).

**Fundamental assumption**

Assuming that the fatigue rate is proportional to the dissipation rate, and that the material parameters depend on the fatigue parameter

- Assumption realistic: plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid
In multiassial processes?

- **Drawback:** the rainflow method is exclusively uniaxial - no closed cycles in the vector case.

- In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known.

- On the other hand, the notion of energy dissipation is independent of the experimental setting: experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak.

- In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry).

**Fundamental assumption**

Assuming that the fatigue rate is proportional to the dissipation rate, and that the material parameters depend on the fatigue parameter.

- Assumption realistic: plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid.
In multiassial processes?

- Drawback: the rainflow method is exclusively uniaxial - no closed cycles in the vector case.
- In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known.
- On the other hand, the notion of energy dissipation is independent of the experimental setting: experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak.
- In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry).

Fundamental assumption

Assuming that the fatigue rate is proportional to the dissipation rate, and that the material parameters depend on the fatigue parameter.

- Assumption realistic: plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid.
Evolution equation for the fatigue

\[ D = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{S}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] \]

\[ = -\frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \phi(\theta, r) \, dr \]

- The integral is non negative by virtue of the variational inequality which defines the stop operator.

- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( \mathbf{B}'(m) \) is a negative semidefinite matrix (softening!)

- **Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( D \)

\[ \left( \frac{1}{C(\theta)} + \frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \right) \partial_t m = \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \phi(\theta, r) \, dr \]

- \( \mathbf{B}'(m) \leq 0 \) softening \( \Rightarrow \) singularity! Material failure in finite time!

Michela Eleuteri

Non-isothermal cyclic fatigue in oscillating elasto-plastic structures with hysteresis
Evolution equation for the fatigue

\[ D = \langle \sigma, \partial_t \epsilon \rangle - \partial_t \theta \mathcal{S}[\theta, \epsilon] - \partial_t \mathcal{F}[\theta, \epsilon] \]

\[ = -\frac{1}{2} \langle B'(m) \epsilon, \epsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\epsilon - s_r Z[\epsilon]), s_r Z[\epsilon] \rangle \varphi(\theta, r) dr \]

- The integral is non negative by virtue of the variational inequality which defines the stop operator
- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( B'(m) \) is a negative semidefinite matrix (softening!)
- **Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( D \)

\[ \left( \frac{1}{C(\theta)} + \frac{1}{2} \langle B'(m) \epsilon, \epsilon \rangle \right) \partial_t m = \int_0^\infty \langle \partial_t (\epsilon - s_r Z[\epsilon]), s_r Z[\epsilon] \rangle \varphi(\theta, r) dr \]

- \( B'(m) \leq 0 \) softening \( \Rightarrow \) singularity! Material failure in finite time!
Evolution equation for the fatigue

\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{H}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] \]

\[ = -\frac{1}{2} \langle B'(m)\varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \phi(\theta, r) \, dr \]

- The integral is non negative by virtue of the variational inequality which defines the stop operator.

- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( B'(m) \) is a negative semidefinite matrix (softening!)

- **Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \)

\[ \left( \frac{1}{C(\theta)} + \frac{1}{2} \langle B'(m)\varepsilon, \varepsilon \rangle \right) \partial_t m = \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \phi(\theta, r) \, dr \]

- \( B'(m) \leq 0 \) softening \( \Rightarrow \) singularity! Material failure in finite time!
Evolution equation for the fatigue

\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{S}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] \]

\[ = -\frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_{rZ}[\varepsilon]), s_{rZ}[\varepsilon] \rangle \varphi(\theta, r) \, dr \]

- The integral is non negative by virtue of the variational inequality which defines the stop operator.
- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( \mathbf{B}'(m) \) is a negative semidefinite matrix (softening!)

- **Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \)

\[ \left( \frac{1}{C(\theta)} + \frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \right) \partial_t m = \int_0^\infty \langle \partial_t (\varepsilon - s_{rZ}[\varepsilon]), s_{rZ}[\varepsilon] \rangle \varphi(\theta, r) \, dr \]

- \( \mathbf{B}'(m) \leq 0 \) softening \( \Rightarrow \) singularity! Material failure in finite time!
Evolution equation for the fatigue

\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{S}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] \]

\[ = -\frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_{rZ}[\varepsilon]), s_{rZ}[\varepsilon] \rangle \phi(\theta, r) \, dr \]

- The integral is non-negative by virtue of the variational inequality which defines the stop operator.

- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( \mathbf{B}'(m) \) is a negative semidefinite matrix (softening!).

- **Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \)

\[ \left( \frac{1}{C(\theta)} + \frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \right) \partial_t m = \int_0^\infty \langle \partial_t (\varepsilon - s_{rZ}[\varepsilon]), s_{rZ}[\varepsilon] \rangle \phi(\theta, r) \, dr \]

- \( \mathbf{B}'(m) \leq 0 \) softening \( \Rightarrow \) singularity! Material failure in finite time!
\[ \mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{S}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon] \]
\[ = -\frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr \]

- The integral is non-negative by virtue of the variational inequality which defines the stop operator.

- The fatigue accumulation rate \( \partial_t m \) should be nonnegative. Hence, it suffices to assume that \( \mathbf{B}'(m) \) is a negative semidefinite matrix (softening!)

**Fundamental assumption:** proportionality between the rate of fatigue \( \partial_t m \) and the dissipation rate \( \mathcal{D} \)

\[ \left( \frac{1}{C(\theta)} + \frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle \right) \partial_t m = \int_0^\infty \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr \]

- \( \mathbf{B}'(m) \leq 0 \) softening \( \Rightarrow \) **singularity!** Material failure in finite time!
The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**

**Phase transition equation in the form of melting-solidification law**

\[ \alpha \chi_t \in - \nabla \chi \mathcal{F} [\varepsilon, \theta, \chi] \quad \chi \in [0, 1] \]

\[ \chi_0 \in [0, 1] \text{ some initial condition, } A(x,t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right)(x, \tau) \, d\tau \]

\[ (\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1] \]
The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**
- account for phase transition in the model
- material fatigue and \( \chi \) degree of melting
- the time of failure of the material can be shifted
- possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

**Phase transition equation in the form of melting-solidification law**

\[
\alpha \chi_t \in - \partial_\chi \mathcal{F}[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]
\]

\( \chi_0 \in [0, 1] \) some initial condition,

\[
A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{d}{d\tau} (\theta - \theta_c) \right) (x, \tau) d\tau
\]

\[
(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]
\]
The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**
- account for phase transition in the model
- material fatigue and $\chi$ degree of melting
- the time of failure of the material can be shifted
- possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

**Phase transition equation in the form of melting-solidification law**

$$\alpha \chi_t \in - \partial_\chi \mathcal{F}[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]$$

$\chi_0 \in [0, 1]$ some initial condition, $A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau$

$(\chi_t - A_t)(z - \chi) \geq 0$ for all $z \in [0, 1]$. 
The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**
- account for phase transition in the model
- $m$ material fatigue and $\chi$ degree of melting
- the time of failure of the material can be shifted
- possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

Phase transition equation in the form of melting-solidification law

$$\alpha \chi_t \in -\partial_\chi F[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]$$

$\chi_0 \in [0, 1]$ some initial condition, $A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau$

$$(\chi_t - A_t) (z - \chi) \geq 0 \text{ for all } z \in [0, 1]$$
The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**
- account for phase transition in the model
  - \( m \) material fatigue and \( \chi \) degree of melting
  - the time of failure of the material can be shifted
  - possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

**Phase transition equation in the form of melting-solidification law**

\[
\alpha \chi_t \in - \partial_\chi \mathcal{F} [\varepsilon, \theta, \chi] \\
\chi \in [0, 1]
\]

\( \chi_0 \in [0, 1] \) some initial condition, \( A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau \)

\( (\chi_t - A_t)(z - \chi) \geq 0 \) for all \( z \in [0, 1] \)
The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**
- account for phase transition in the model
- $m$ material fatigue and $\chi$ degree of melting
  - the time of failure of the material can be shifted
  - possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

Phase transition equation in the form of melting-solidification law

$$
\alpha \chi_t \in -\partial \chi \mathcal{F}[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]
$$

$$
\chi_0 \in [0, 1] \text{ some initial condition, } A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L_c}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau
$$

$$
(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]
$$
The model with phase transition

Motivation:
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

How to achieve this goal:
- account for phase transition in the model
- $m$ material fatigue and $\chi$ degree of melting
- the time of failure of the material can be shifted
- possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

Phase transition equation in the form of melting-solidification law

$$\alpha \chi_t \in -\partial \chi \mathcal{F} [\varepsilon, \theta, \chi] \quad \chi \in [0, 1]$$

$$\chi_0 \in [0, 1] \text{ some initial condition, } A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) \, d\tau$$

$$(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]$$
The model with phase transition

Motivation:
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

How to achieve this goal:
- account for phase transition in the model
- $m$ material fatigue and $\chi$ degree of melting
- the time of failure of the material can be shifted
- possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

Phase transition equation in the form of melting-solidification law

$$\alpha \chi_t \in -\partial_{\chi} \mathcal{F}[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]$$

$$\chi_0 \in [0, 1] \text{ some initial condition, } A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) \, d\tau$$

$$(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]$$
The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**
- account for phase transition in the model
- $m$ material fatigue and $\chi$ degree of melting
- the time of failure of the material can be shifted
- possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

**Phase transition equation in the form of melting-solidification law**

$$\alpha \chi_t \in -\partial \chi \mathcal{F}[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]$$

$$\chi_0 \in [0, 1] \text{ some initial condition, } A(x, t) : = \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau$$

$$(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]$$
The model with phase transition

- **Motivation:**
  - possibility to account also for decreasing fatigue rate (in view of engineering applications)
  - the material can be partially repaired by local melting

- **How to achieve this goal:**
  - account for phase transition in the model
  - $m$ material fatigue and $\chi$ degree of melting
  - the time of failure of the material can be shifted
  - possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

- Phase transition equation in the form of melting-solidification law

\[
\alpha \chi_t \in -\partial \chi F[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]
\]

$\chi_0 \in [0, 1]$ some initial condition, $A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau$

\[(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]\]
The model with phase transition

- **Motivation:**
  - possibility to account also for decreasing fatigue rate (in view of engineering applications)
  - the material can be partially repaired by local melting

- **How to achieve this goal:**
  - account for phase transition in the model
  - $m$ material fatigue and $\chi$ degree of melting
  - the time of failure of the material can be shifted
  - possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

- **Phase transition equation in the form of melting-solidification law**

  \[
  \alpha \chi_t \in -\partial \chi \mathcal{F} [\varepsilon, \theta, \chi] \quad \chi \in [0, 1]
  \]

  \[
  \chi_0 \in [0, 1] \text{ some initial condition, } A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau
  \]

  \[
  (\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]
  \]
The model with phase transition

- **Motivation:**
  - possibility to account also for decreasing fatigue rate (in view of engineering applications)
  - the material can be partially repaired by local melting

- **How to achieve this goal:**
  - account for phase transition in the model
  - $m$ material fatigue and $\chi$ degree of melting
  - the time of failure of the material can be shifted
  - possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

- Phase transition equation in the form of melting-solidification law

  \[ \alpha \chi_t \in -\partial \chi F[\varepsilon, \theta, \chi] \quad \chi \in [0, 1] \]

  \[ \chi_0 \in [0, 1] \text{ some initial condition, } A(x, t) := \int_0^t \frac{1}{\alpha} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right)(x, \tau) \, d\tau \]

  $\chi \in s[0,1][\chi_0,A]$ $s[0,1]$ is a shifted stop
Thermodynamical consistency

- If we introduce $\mathcal{F}[\varepsilon, \theta, \chi]$ specific free energy, $\mathcal{I}[\varepsilon, \theta, \chi]$ specific entropy and $\mathcal{U}[\varepsilon, \theta, \chi]$ internal energy we are able to show that the first and second principles of thermodynamics are satisfied

\[
\frac{\partial}{\partial t} \mathcal{U}[\varepsilon, \theta, \chi] + \text{div} q = \langle \sigma, \varepsilon_t \rangle \quad \text{(energy conservation)}
\]
\[
\frac{\partial}{\partial t} \mathcal{I}[\varepsilon, \theta, \chi] + \text{div} \frac{q}{\theta} \geq 0, \quad \text{(Clausius-Duhem inequality)}
\]

- Evolution equation for $m$:

\[
(C + \mathcal{K}[m, \theta, \varepsilon]) m_t = -h(\chi_t) + \mathcal{D}[m, \theta, \varepsilon]
\]

- where

\[
\mathcal{D}[m, \theta, \varepsilon] := \int_0^\infty \varphi(m, \theta, r) \langle \mathbf{K}_{s_{rZ}}[\varepsilon], (\varepsilon - s_{rZ}[\varepsilon])_t \rangle \, dr
\]
\[
\mathcal{K}[m, \theta, \varepsilon] := -\frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle - \frac{1}{2} \int_0^\infty \varphi_m(m, \theta, r) \langle \mathbf{K}_{s_{rZ}}[\varepsilon], s_{rZ}[\varepsilon] \rangle \, dr.
\]

- allow the possibility of decreasing rate (i.e. $m_t < 0$) but only in the case if $\chi$ grows faster than the plastic dissipation rate (strong melting).
Thermodynamical consistency

- If we introduce $\mathcal{F}[\varepsilon, \theta, \chi]$ specific free energy, $\mathcal{S}[\varepsilon, \theta, \chi]$ specific entropy and $\mathcal{U}[\varepsilon, \theta, \chi]$ internal energy we are able to show that the first and second principles of thermodynamics are satisfied

\[
\frac{\partial}{\partial t} \mathcal{U}[\varepsilon, \theta, \chi] + \text{div} q = \langle \sigma, \varepsilon_t \rangle \quad \text{(energy conservation)}
\]

\[
\frac{\partial}{\partial t} \mathcal{S}[\varepsilon, \theta, \chi] + \text{div} \frac{q}{\theta} \geq 0, \quad \text{(Clausius-Duhem inequality)}
\]

- Evolution equation for $m$:

\[
(C + \mathcal{K}[m, \theta, \varepsilon]) m_t = -h(\chi_t) + \mathcal{D}[m, \theta, \varepsilon]
\]

- where

\[
\mathcal{D}[m, \theta, \varepsilon] := \int_0^\infty \varphi(m, \theta, r) \langle Ks_{rZ}[\varepsilon], (\varepsilon - s_{rZ}[\varepsilon])_t \rangle \, dr
\]

\[
\mathcal{K}[m, \theta, \varepsilon] := -\frac{1}{2} \langle B'(m) \varepsilon, \varepsilon \rangle - \frac{1}{2} \int_0^\infty \varphi_m(m, \theta, r) \langle Ks_{rZ}[\varepsilon], s_{rZ}[\varepsilon] \rangle \, dr.
\]

- allow the possibility of decreasing rate (i.e. $m_t < 0$) but only in the case if $\chi$ grows faster than the plastic dissipation rate (strong melting)
Thermodynamical consistency

- If we introduce $\mathcal{F}[\varepsilon, \theta, \chi]$ specific free energy, $\mathcal{S}[\varepsilon, \theta, \chi]$ specific entropy and $\mathcal{U}[\varepsilon, \theta, \chi]$ internal energy we are able to show that the first and second principles of thermodynamics are satisfied

$$\frac{\partial}{\partial t} \mathcal{U}[\varepsilon, \theta, \chi] + \text{div} \mathbf{q} = \langle \sigma, \varepsilon_t \rangle \quad \text{(energy conservation)}$$

$$\frac{\partial}{\partial t} \mathcal{S}[\varepsilon, \theta, \chi] + \text{div} \frac{\mathbf{q}}{\theta} \geq 0, \quad \text{(Clausius-Duhem inequality)}$$

- **Evolution equation for $m$:**

$$\left( C + \mathcal{K}[m, \theta, \varepsilon] \right) m_t = -h(\chi_t) + \mathcal{D}[m, \theta, \varepsilon]$$

where

$$\mathcal{D}[m, \theta, \varepsilon] := \int_0^{\infty} \varphi(m, \theta, r) \langle \mathbf{K}_r Z[\varepsilon], (\varepsilon - s_r Z[\varepsilon])_t \rangle \, dr$$

$$\mathcal{K}[m, \theta, \varepsilon] := -\frac{1}{2} \langle \mathbf{B}'(m) \varepsilon, \varepsilon \rangle - \frac{1}{2} \int_0^{\infty} \varphi_m(m, \theta, r) \langle \mathbf{K}_r Z[\varepsilon], s_r Z[\varepsilon] \rangle \, dr.$$

- allow the possibility of decreasing rate (i.e. $m_t < 0$) but only in the case if $\chi$ grows faster than the plastic dissipation rate (strong melting)
Thermodynamical consistency

- If we introduce $\mathcal{F}[\varepsilon, \theta, \chi]$ specific free energy, $\mathcal{S}[\varepsilon, \theta, \chi]$ specific entropy and $\mathcal{U}[\varepsilon, \theta, \chi]$ internal energy we are able to show that the first and second principles of thermodynamics are satisfied

\[
\frac{\partial}{\partial t} \mathcal{U}[\varepsilon, \theta, \chi] + \text{div} q = \langle \sigma, \varepsilon_t \rangle \quad \text{(energy conservation)}
\]

\[
\frac{\partial}{\partial t} \mathcal{S}[\varepsilon, \theta, \chi] + \text{div} \frac{q}{\theta} \geq 0, \quad \text{(Clausius-Duhem inequality)}
\]

- Evolution equation for $m$:

\[
(C + \mathcal{H}[m, \theta, \varepsilon]) m_t = -h(\chi_t) + \mathcal{D}[m, \theta, \varepsilon]
\]

- where

\[
\mathcal{D}[m, \theta, \varepsilon] := \int_0^\infty \phi(m, \theta, r) \langle \mathbf{K}s_rZ[\varepsilon], (\varepsilon - s_rZ[\varepsilon])_t \rangle \, dr
\]

\[
\mathcal{H}[m, \theta, \varepsilon] := -\frac{1}{2} \langle \mathbf{B}'(m)\varepsilon, \varepsilon \rangle - \frac{1}{2} \int_0^\infty \phi_m(m, \theta, r) \langle \mathbf{K}s_rZ[\varepsilon], s_rZ[\varepsilon] \rangle \, dr.
\]

- allow the possibility of decreasing rate (i.e. $m_t < 0$) but only in the case if $\chi$ grows faster than the plastic dissipation rate (strong melting)
Thermodynamical consistency

- If we introduce \( \mathcal{F}[\varepsilon, \theta, \chi] \) specific free energy, \( \mathcal{S}[\varepsilon, \theta, \chi] \) specific entropy and \( \mathcal{U}[\varepsilon, \theta, \chi] \) internal energy we are able to show that the first and second principles of thermodynamics are satisfied

\[
\frac{\partial}{\partial t} \mathcal{U}[\varepsilon, \theta, \chi] + \text{div} q = \langle \sigma, \varepsilon_t \rangle \quad \text{(energy conservation)}
\]

\[
\frac{\partial}{\partial t} \mathcal{S}[\varepsilon, \theta, \chi] + \text{div} \frac{q}{\theta} \geq 0, \quad \text{(Clausius-Duhem inequality)}
\]

- Evolution equation for \( m \):

\[
(C + \mathcal{K}[m, \theta, \varepsilon]) m_t = -h(\chi_t) + \mathcal{D}[m, \theta, \varepsilon]
\]

where

\[
\mathcal{D}[m, \theta, \varepsilon] := \int_0^\infty \varphi(m, \theta, r) \langle \mathbf{K}s_r Z[\varepsilon], (\varepsilon - s_r Z[\varepsilon])_t \rangle \, dr
\]

\[
\mathcal{K}[m, \theta, \varepsilon] := -\frac{1}{2} \langle \mathbf{B}'(m)\varepsilon, \varepsilon \rangle - \frac{1}{2} \int_0^\infty \varphi_m(m, \theta, r) \langle \mathbf{K}s_r Z[\varepsilon], s_r Z[\varepsilon] \rangle \, dr.
\]

- allow the possibility of decreasing rate (i.e. \( m_t < 0 \)) but only in the case if \( \chi \) grows faster than the plastic dissipation rate (strong melting)
Numerical approximation for PDEs with hysteresis in elastoplasticity?

- Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem
- M. Siegfanz Ph.D. Thesis
  - 1D wave equation with hysteresis
    - not only numerical scheme proposed but also convergence results and error estimates
- Future work:
Numerical approximation for PDEs with hysteresis in elastoplasticity?

- Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem


- **M. Siegfanz** Ph.D. Thesis
  - 1D wave equation with hysteresis
  - not only numerical scheme proposed but also convergence results and error estimates

- Future work:
Numerical approximation for PDEs with hysteresis in elastoplasticity?

- Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem


- **M. Siegfanz** Ph.D. Thesis
  - 1D wave equation with hysteresis
  - not only numerical scheme proposed but also convergence results and error estimates

- Future work:
Numerical approximation for PDEs with hysteresis in elastoplasticity?

- Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem


- **M. Siegfanz** Ph.D. Thesis
  - 1D wave equation with hysteresis
    - not only numerical scheme proposed but also convergence results and error estimates

- **Future work:**
  - extend the work of M. Siegfanz to the beam equation (1D)
    - propose a numerical scheme and (possibly) prove convergence results and error estimates
Numerical approximation for PDEs with hysteresis in elastoplasticity?

- Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem


- **M. Siegfanz** Ph.D. Thesis
  - 1D wave equation with hysteresis
  - not only numerical scheme proposed but also convergence results and error estimates

- **Future work:**
  - extend the work of M. Siegfanz to the beam equation (1D)
  - propose a numerical scheme and (possibly!) prove convergence results and error estimates
Numerical approximation for PDEs with hysteresis in elastoplasticity?

- Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem


- **M. Siegfanz** Ph.D. Thesis
  - 1D wave equation with hysteresis
  - not only numerical scheme proposed but also convergence results and error estimates

- **Future work:**
  - extend the work of M. Siegfanz to the beam equation (1D)
  - propose a numerical scheme and (possibly!) prove convergence results and error estimates
Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem.


**M. Siegfanz** Ph.D. Thesis
- 1D wave equation with hysteresis
- not only numerical scheme proposed but also convergence results and error estimates

**Future work:**
- extend the work of M. Siegfanz to the beam equation (1D)
- propose a numerical scheme and (possibly!) prove convergence results and error estimates
Numerical approximation for PDEs with hysteresis in elastoplasticity?

- Numerical approximation for PDEs with hysteresis is a challenging (and difficult) problem


- **M. Siegfanz** Ph.D. Thesis
  - 1D wave equation with hysteresis
  - not only numerical scheme proposed but also convergence results and error estimates

- **Future work:**
  - extend the work of M. Siegfanz to the beam equation (1D)
  - propose a numerical scheme and (possibly!) prove convergence results and error estimates
Rainflow method for fatigue evaluation in elastoplastic materials (uniaxial cyclic loading) allow to consider dissipated energy as a measure for fatigue.

The solution cannot be expected to exist globally: singularities (thermal shocks) occur in finite time.

Phase transition in the model accounts also for decreasing fatigue rate.

The time of failure can be shifted and considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found.

The resulting full system of energy and momentum balance equations is consistent with the first and the second principles of thermodynamics; mathematical analysis of the model is work in progress.
Rainflow method for fatigue evaluation in elastoplastic materials (uniaxial cyclic loading) allow to consider dissipated energy as a measure for fatigue.

The solution cannot be expected to exist globally: singularities (thermal shocks) occur in finite time.

Phase transition in the model accounts also for decreasing fatigue rate.

The time of failure can be shifted and considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found.

The resulting full system of energy and momentum balance equations is consistent with the first and the second principles of thermodynamics; mathematical analysis of the model is work in progress.
Rainflow method for fatigue evaluation in elastoplastic materials (uniaxial cyclic loading) allow to consider dissipated energy as a measure for fatigue.

The solution cannot be expected to exist globally: singularities (thermal shocks) occur in finite time.

Phase transition in the model accounts also for decreasing fatigue rate.

The time of failure can be shifted and considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found.

The resulting full system of energy and momentum balance equations is consistent with the first and the second principles of thermodynamics; mathematical analysis of the model is work in progress.
Rainflow method for fatigue evaluation in elastoplastic materials (uniaxial cyclic loading) allow to consider dissipated energy as a measure for fatigue.

The solution cannot be expected to exist globally: singularities (thermal shocks) occur in finite time.

Phase transition in the model accounts also for decreasing fatigue rate.

The time of failure can be shifted and considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found.

The resulting full system of energy and momentum balance equations is consistent with the first and the second principles of thermodynamics; mathematical analysis of the model is work in progress.
Conclusion

- Rainflow method for fatigue evaluation in elastoplastic materials (uniaxial cyclic loading) allow to consider dissipated energy as a measure for fatigue

- The solution cannot be expected to exist globally: singularities (thermal shocks) occur in finite time

- Phase transition in the model accounts also for decreasing fatigue rate

- The time of failure can be shifted and considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

- The resulting full system of energy and momentum balance equations is consistent with the first and the second principles of thermodynamics; mathematical analysis of the model is work in progress