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Analysis-suitable T-splines of arbitrary degree: definition and properties

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T-splines are an important tool in IGA since they allow local refinement. In this paper we define analysis-suitable T-splines of arbitrary degree and prove fundamental properties: linear independence of the blending functions and optimal approximation properties of the associated T-spline space. These are corollaries of our main result: a T-mesh is analysis-suitable if and only if it is dual-compatible, a concept already defined and used in [5].

Keywords: Isogeometric analysis; T-splines, analysis-suitable, dual-compatible, linear independence, dual basis, error estimates, Greville sites.

1. Introduction

IsoGeometric Analysis (IGA) is a numerical method for solving partial differential equations (PDEs), introduced by Hughes et al. in [17]. In IGA, B-splines or Non-Uniform Rational B-Splines (NURBS), that typically represent the domain geometry in a Computer Aided Design (CAD) parametrization, become the basis for the solution space of variational formulations of PDEs. IGA methodologies have been studied and applied in many areas such as fluid dynamics, structural mechanics, and electromagnetics (see the book [11] and references therein). During the last years, it has been demonstrated that the use of regular shape functions allow for the writing of new methods enjoying properties which would be hard to obtain with standard finite elements. On the other hand, flexibility in the mesh generation and refinement need to be enhanced by breaking the tensor product structure of

B-splines and NURBS.

Local refinement strategies in IGA are possible thanks to the non-tensor product extensions of B-splines, such as T-splines[23,24]. A T-spline space is spanned by a set of B-spline functions, named T-spline blending functions, that are constructed from a T-mesh. The blending functions are tensor-product of univariate splines, but the T-mesh breaks the global tensor-product structure by allowing so called T-junctions.

T-splines have been recognized as a promising tool for IGA in [3] and have been the object of recent interest in literature (see, e.g., [4, 8, 14, 18, 19, 22, 25]). In particular, in the context of IGA, *analysis-suitable (AS)* T-splines have emerged: introduced in [19] in the bi-cubic case, they are a sub-class of T-splines for which we have fundamental mathematical properties needed in a PDE solver. Linear independence of AS T-splines blending functions has been first shown in [19]. In [5] it is shown that the condition of being AS, which is mainly a condition on the connectivity of the T-mesh, implies that the bi-cubic T-spline basis functions admit a dual basis that can be constructed as in the tensor-product setting. The latter property led to the definition of *dual-compatible (DC) T-splines* on which stable projection operators can be defined. The main consequences of this fact is that DC T-splines (and then also AS T-splines) blending functions are linearly independent and the space enjoys the optimal approximation properties of standard (tensor-product) B-Splines spaces.

It should also be noted that the result in [5] creates an important link between the connectivity property of the T-mesh and the functional property of the T-spline space which was not known before. The present paper generalizes the results of [5] in a fundamental way.

While T-splines are defined for any polinomial degree in [3, 16], everything known in literature about AS T-splines is restricted to the bi-cubic case. The main goal of this paper is to define and study AS T-splines of arbitrary degree, that is, p-degree in one direction and q-degree in the other) for any $p, q \in \mathbb{N}$. This opens the possibility of k-refinement[6,12,15], compatible elements[9,10] and other tools beyond tensor-product based IGA, and allows for more general geometries[21,25].

We give the definition of the class of AS T-splines of p, q-degree, denoted $\mathsf{AS}_{p,q}$, and recall the definition of the class of DC T-splines of p, q-degree, denoted $\mathsf{DC}_{p,q}$ (actually, the concept behind DC T-splines extends with no changes to any degree). Then, we prove that analysis-suitable and dual-compatible are equivalent concepts, that is $\mathsf{AS}_{p,q} = \mathsf{DC}_{p,q}$. This is the second important achievement of the present paper, since DC T-splines enjoy the mathematical structure mentioned before and therefore are "good" spaces for IGA.

The proof that $\mathsf{AS}_{p,q} \subset \mathsf{DC}_{p,q}$ is based on a completely new induction argument with respect to the degrees p and q. It is indeed quicker than the one in [5], and uses the structure of AS T-meshes in a sound way. The proof of $\mathsf{AS}_{p,q} \supset \mathsf{DC}_{p,q}$ is new also for the bi-cubic case and completes the framework by giving a solid theoretical ground for further studies of T-spline spaces.

Moreover, the property characterizing DC T-splines can immediately be extended in multiple space dimension and this paves the way to the definition and the study of T-splines in three space dimensions. Clearly this is the main research challenge in the near future (see also the recent paper [26]).

A T-mesh is typically described in term of the active knots, that is, on the so called parametric domain. Instead, our approach to T-splines is based on the previous one from [5], and on the concept of T-mesh in the index domain (referred as *index* T-mesh in the present paper). Indeed, since we restrict our study to a single-patch bi-variate framework, we can also associate to each knot a pair of integer indices. However, knots and indices are not in a one-to-one correspondence. Indeed, with T-splines, as is usual with B-splines, we allow knot repetition, which is a way to allow variable regularity of the blending functions. Then, it is preferable to represent the T-mesh in the index domain, that is, with respect to the knot indices instead of the knots themselves. In fact, the T-mesh connectivity is given on the indices and both definitions of AS and DC T-mesh are also given at this level. Indeed, we recall that a T-mesh is DC if each couple of associated blending functions shares a common knot vector, at least in one direction (horizontal or vertical). In order to allow knot repetition, the definition and properties are given in terms of local index vectors and blending functions are associated to anchors which are either vertices, edges or elements of the index T-mesh depending on the degree p and q.

The situation above motivates the setting of our paper. We define the index T-mesh, and the classes $\mathsf{AS}_{p,q}$ and $\mathsf{DC}_{p,q}$ of AS and DC T-meshes in Section 2. These concepts depend on the degrees p and q, even though at this stage we have anchors and index vectors but still do not have knots and T-spline functions. Our main result, at this abstract level, is exposed in Sections 3–4, where we investigate the relations between the T-mesh connectivity and the structure of the anchors local indices, coming to $\mathsf{AS}_{p,q} = \mathsf{DC}_{p,q}$. The result is then independent of the global knot vectors that are associated to the indices. In Section 5, we finally introduce knots and T-spline blending functions and show how the abstract theory results in properties of the T-spline space: existence of a dual basis, first of all, and then linear independence of the blending functions and existence of a projector operator with optimal approximation properties.

2. Index T-mesh: definitions and assumptions

An *index T-mesh* (also denoted in literature as T-mesh in the index domain, or simply T-mesh) $\mathcal{M}$ is a rectangular partition of the *index domain* $[\underline{m}, \overline{m}] \times [\underline{n}, \overline{n}]$, with $\underline{m}$, $\overline{m}$, $\underline{n}$, $\overline{n} \in \mathbb{Z}$, such that all element corners, i.e., the *vertices* of $\mathcal{M}$, have integer coordinates. Precisely, $\mathcal{M}$ is the collection of all elements of the partition above, where the elements are taken as open sets. We also denote by $\mathcal{V}$ the collection of all the vertices of $\mathcal{M}$, considered as singletons (subset of $\mathbb{Z}^2$). An *edge* of $\mathcal{M}$ is a segment between vertices of $\mathcal{M}$ that does not intersect any element in $\mathcal{M}$. We further assume that edges do not contain vertices, and in particular that they are open at their endpoints. We denote by $h\mathcal{E}$ (resp., $v\mathcal{E}$) the collection of all horizontal (resp., vertical) edges, and by $\mathcal{E} = h\mathcal{E} \cup v\mathcal{E}$ the collection of all edges. The boundary of an element $Q \in \mathcal{M}$ is denoted by ∂Q, and the union of the two vertices $V^1, V^2 \in \mathcal{V}$ that are endpoints of an edge $e \in \mathcal{E}$ is denoted by ∂e. The *valence* of a vertex $V \in \mathcal{V}$ is the number of edges $e \in \mathcal{E}$ such that $V \subset \partial e$. Since elements are rectangular, only valence three or four is allowed for all vertices $V \subset]\underline{m}, \overline{m}[\times]\underline{n}, \overline{n}[$, that is, so called T-junctions are allowed but L-junctions or I-junctions[19] are not. The horizontal (resp., vertical) *skeleton* of the mesh is denoted by $h\mathcal{S}$ (resp., $v\mathcal{S}$), and is the union of all horizontal (resp., vertical) edges and all vertices. Finally, we denote skeleton the union $\mathcal{S} = h\mathcal{S} \cup v\mathcal{S}$.

We split the index domain $[\underline{m}, \overline{m}] \times [\underline{n}, \overline{n}]$ into an *active region* $AR_{p,q}$ and a *frame region* $FR_{p,q}$, such that

$$AR_{p,q} = [\underline{m} + \lfloor (p+1)/2 \rfloor, \overline{m} - \lfloor (p+1)/2 \rfloor] \times [\underline{n} + \lfloor (q+1)/2 \rfloor, \overline{n} - \lfloor (q+1)/2 \rfloor],$$

and

$$\begin{aligned} FR_{p,q} = &\Big([\underline{m}, \underline{m} + \lfloor (p+1)/2 \rfloor] \;\cup\; [\overline{m} - \lfloor (p+1)/2 \rfloor, \overline{m}] \Big) \times [\underline{n}, \overline{n}] \\ &\cup [\underline{m}, \overline{m}] \times \Big([\underline{n}, \underline{n}, + \lfloor (q+1)/2 \rfloor] \;\cup\; [\overline{n} - \lfloor (q+1)/2 \rfloor, \overline{n}] \Big). \end{aligned}$$

Note that both $AR_{p,q}$ and $FR_{p,q}$ are closed regions. The width of the frame for several choices of p and q is shown in Figure 1.

Definition 2.1. An index T-mesh $\mathcal{M}$ belongs to $\mathsf{AD}_{p,q}$, that is, it is *admissible* for degrees p and q, if $\mathcal{S} \cap FR_{p,q}$ contains the vertical segments

$$\{\ell\} \times [\underline{n}, \overline{n}] \qquad \begin{array}{l} \text{for } \ell = \underline{m}, \dots, \underline{m} + \lfloor (p+1)/2 \rfloor \\ \text{and } \ell = \overline{m} - \lfloor (p+1)/2 \rfloor, \dots, \overline{m}, \end{array}$$

and the horizontal segments

$$[\underline{m}, \overline{m}] \times \{\ell\} \qquad \begin{array}{l} \text{for } \ell = \underline{n}, \dots, \underline{n} + \lfloor (q+1)/2 \rfloor \\ \text{and } \ell = \overline{n} - \lfloor (q+1)/2 \rfloor, \dots, \overline{n}, \end{array}$$

and all vertices $V \subset]\underline{m}, \overline{m}[\times]\underline{n}, \overline{n}[\cap FR_{p,q}$ have valence four.

Definition 2.1 prevents the mesh from having T-junctions in the frame (see Figure 1).

We remark that, in the literature[19], the index T-mesh is sometimes considered as the set $\{Q \in \mathcal{M} : Q \subset AR_{p,q}\} \subset \mathcal{M}$, which is instead denoted as *active* index T-mesh in the present paper. As we will see below, the active index T-mesh carries the anchors, that will be associated in Section 5 to the spline basis functions, while the extra indices of $FR_{p,q}$ will be needed for the definition of the function when the anchor is close to the boundary.

Definition 2.2. An index T-mesh $\mathcal{M} \in \mathsf{AD}_{p,q}$ is said to belong to $\mathsf{AD}^+_{p,q}$ if, for each couple of vertices $V^1 = \{(i_1, j_1)\}$, $V^2 = \{(i_2, j_2)\}$ in $\mathcal{V}$, such that $V^1, V^2 \subset \partial Q$ for some $Q \in \mathcal{M}$, and with $i_1 = i_2$ (resp. $j_1 = j_2$), the open segment $\{i_1\}\times]j_1, j_2[$ (resp., $]i_1, i_2[\times\{j_1\}$) is contained in $\mathcal{S}$.

The condition stated in Definition 2.2 prevents the existence of two facing T-junctions as those in Figure 8.

We are now ready to introduce the *anchors*. For T-splines of arbitrary degree, anchors were associated in [3, 16] to vertices, center of the edges, or center of the elements, depending on p and q. Instead, in the following Definition 2.3, we define anchors as the vertices, edges or elements themselves, for consistency with the next Definition 2.4.

Definition 2.3. Given a T-mesh $\mathcal{M} \in \mathsf{AD}_{p,q}$, we define the set of anchors $\mathcal{A}_{p,q}(\mathcal{M})$ as follows:

- if $p,\ q$ are odd, $\mathcal{A}_{p,q}(\mathcal{M}) = \{A \in \mathcal{V} \,:\, A \subset AR_{p,q}\}$,
- if p is even and q is odd, $\mathcal{A}_{p,q}(\mathcal{M}) = \{A \in h\mathcal{E} \,:\, A \subset AR_{p,q}\}$,
- if p is odd and q is even $\mathcal{A}_{p,q}(\mathcal{M}) = \{A \in v\mathcal{E} \,:\, A \subset AR_{p,q}\}$,
- if $p\,,\ q$ are even $\mathcal{A}_{p,q}(\mathcal{M}) = \{A \in \mathcal{M} \,:\, A \subset AR_{p,q}\}$.

The set of anchors for different values of p and q is represented in Figure 1. We note that the anchors are always contained in the active region.

Anchors $A \in \mathcal{A}_{p,q}$ are of type $a \times b$, where a and b are either singletons subset of $\mathbb{Z}$ or open intervals with integer endpoints.

Let a be either a singleton of $\mathbb{Z}$ or an interval with integer endpoints; we define

$$\begin{aligned} h\mathcal{I}(a) &:= \{i \in \mathbb{Z} \ :\ \{i\} \times a \subset v\mathcal{S}\}, \\ v\mathcal{I}(a) &:= \{j \in \mathbb{Z} \ :\ a \times \{j\} \subset h\mathcal{S}\}, \end{aligned} \tag{2.1}$$

and we assume that these two sets are ordered.

Definition 2.4. Given an anchor $A = a \times b \in \mathcal{A}_{p,q}(\mathcal{M})$, we define its horizontal (vertical) index vector $hv_{p,q}(A)$ ($vv_{p,q}(A)$, resp.) as a subset of $h\mathcal{I}(b)$ ($v\mathcal{I}(a)$, resp.) given by:

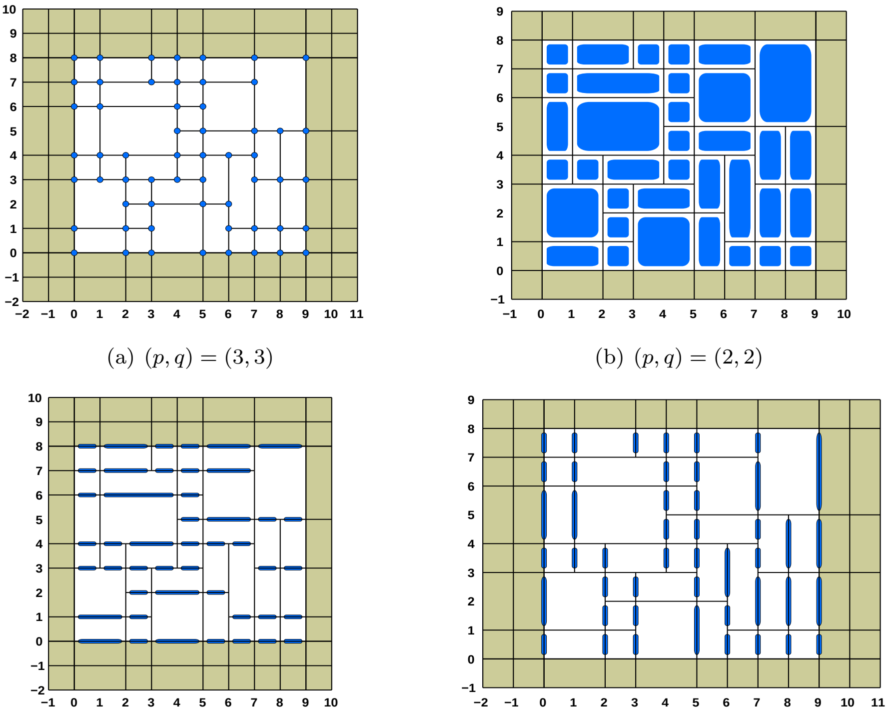
- if p is odd, $hv_{p,q}(A) = (i_1, \ldots, i_{p+2}) \in \mathbb{Z}^{p+2}$ is made of the unique $p+2$ consecutive indices in $h\mathcal{I}(b)$ with $\{i_{(p+1)/2}\} = a$.
- if p is even, $hv_{p,q}(A) = (i_1, \ldots, i_{p+2}) \in \mathbb{Z}^{p+2}$ is made of the unique $p+2$ consecutive indices in $h\mathcal{I}(b)$ such that $]i_{p/2}, i_{p/2+1}[= a$.

The vertical index vector, denoted by $vv_{p,q}(A) = (j_1, \ldots, j_{q+2}) \in \mathbb{Z}^{q+2}$, is constructed in an analogous way.

We also define the *tiled floor* and the *skeleton* of the anchor A as

$$tf_{p,q}(A) = \bigcup_{\substack{\kappa=1,\ldots p+1 \\ \ell=1,\ldots q+1}}]i_\kappa, i_{\kappa+1}[\times]j_\ell, j_{\ell+1}[, \tag{2.2}$$

$$sk_{p,q}(A) = ([i_1, i_{p+2}] \times [j_1, j_{q+2}]) \setminus tf_{p,q}(A). \tag{2.3}$$



(a) $(p, q) = (3, 3)$ (b) $(p, q) = (2, 2)$

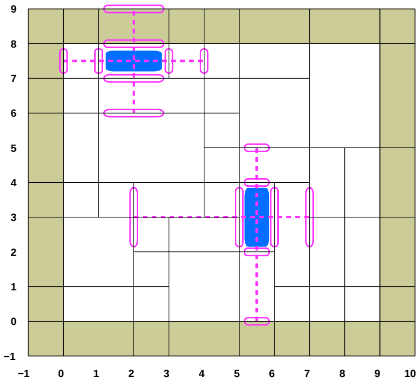
(c) $(p, q) = (2, 3)$ (d) $(p, q) = (3, 2)$

Fig. 1. Admissible meshes and anchors for different values of p and q. The frame region is colored in grey.

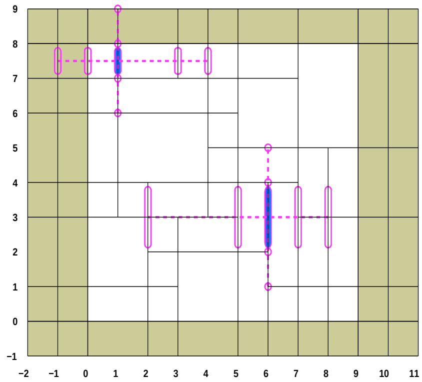
The tiled floor contains the elements of a local tensor product mesh given by the index vectors, which is not necessarily a submesh of $\mathcal{M}$. The skeleton contains all the edges and vertices of this local mesh. In the following, and when no confusion occurs, the subindices p and q will be removed from the previous notations for the sake of clarity.

Some examples of index vectors, for different values of p and q, are given in Figure 2. For example, in Figure 2(a) ($p = q = 2$), the index vectors for the anchor $A_1 =]1, 3[\times]7, 8[$ are $hv(A_1) = (0, 1, 3, 4)$ and $vv(A_1) = (6, 7, 8, 9)$, and for the anchor $A_2 =]5, 6[\times]2, 4[$ they are $hv(A_2) = (2, 5, 6, 7)$ and $vv(A_2) = (0, 2, 4, 5)$. Notice that, since the vertical segments $\{3\}\times]2, 4[$ and $\{4\}\times]2, 4[$ do not belong to the vertical skeleton of the mesh, the values 3 and 4 do not appear in $v\mathcal{I}(]2, 4[)$ (see (2.1)), and are skipped in $hv(A_2)$.

Remark 2.1. In [3] the authors consider the horizontal and the vertical line passing through the center of the anchor, and the index vectors are constructed from the intersections with the orthogonal lines of the mesh. This definition of the index



(a) $(p, q) = (2, 2)$



(b) $(p, q) = (3, 2)$

Fig. 2. Construction of the horizontal and vertical index vector (light magenta), for some values of p and q, and for the anchors marked in light blue.

vectors differs from the one that we give here. For instance, following the construction in [3], the values 3 and 4 would not be skipped by the horizontal index vectors in the examples of Figure 2. However, as we will see in Remark 3.1, both definitions coincide in the classes of T-meshes that we study in this paper.

We are now in the position to define two relevant subclasses of $\mathsf{AD}_{p,q}$.

2.1. *Analysis-suitable T-meshes*

We define $\mathfrak{T}$ as the set of all vertices of valence three in $AR_{p,q}$, denoted *T-junctions*. Following the literature[5,19], we adopt the notation $\perp$, $\top$, $\vdash$, $\dashv$ to indicate the four possible orientations of the T-junctions.

We give now a definition of *analysis-suitable* index T-mesh that extends to any p, q the definition given in [19] for $p = q = 3$. As in [19], we need the notion of T-junction *extension*. T-junctions of type $\vdash$ and $\dashv$ ($\perp$, $\top$, respectively) and their extensions are called horizontal (vertical, resp.). For the sake of simplicity, let us consider a T-junction $T = \{(\bar\imath, \bar\jmath)\} \in \mathfrak{T}$ of type $\dashv$. Clearly, $\bar\imath$ is one of the entries of $h\mathfrak{I}(\{\bar\jmath\})$. We extract from $h\mathfrak{I}(\{\bar\jmath\})$ the $p+1$ consecutive indices $i_1, i_2, \ldots, i_{p+1}$ such that

$$\bar\imath = i_\kappa, \qquad \text{with } \kappa = \lceil (p+1)/2 \rceil.$$

We denote:

$$ext^e_{p,q}(T) = [i_1, \bar\imath] \times \{\bar\jmath\}\,,\ \ ext^f_{p,q}(T) =]\bar\imath, i_{p+1}] \times \{\bar\jmath\}\,,$$
$$ext_{p,q}(T) = ext^f_{p,q}(T) \cup ext^e_{p,q}(T),$$

where $ext^e_{p,q}(T)$ is denoted *edge-extension*, $ext^f_{p,q}(T)$ is denoted *face-extension* and $ext_{p,q}(T)$ is just the *extension* of the T-junction T. This T-junction extension spans

$p+1$ index intervals, referred to as *bays* in the literature. The edge-extension spans $\lceil (p-1)/2 \rceil$ bays, where $[i_{\kappa-1}, i_\kappa] \times \{\bar{\jmath}\}$ is the *first-bay edge-extension*. The face-extension spans $\lfloor (p+1)/2 \rfloor$ bays, with $]i_\kappa, i_{\kappa+1}] \times \{\bar{\jmath}\}$ the *first-bay face-extension*. More precisely, if p is odd, edge and face-extensions span $(p-1)/2$ and $(p+1)/2$ bays, respectively; if p is even, both edge and face-extensions span $p/2$ bays. For the other type of T-junctions, the extensions are defined in a similar way, with the number of bays for vertical extensions depending on the value of q. An example with the length of the extensions is given in Figure 3.

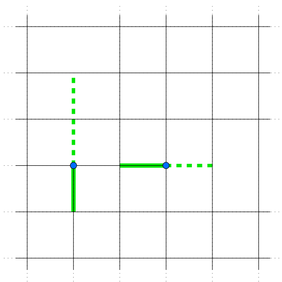


Fig. 3. Extensions for degree $p = 2$ (horizontal) and $q = 3$ (vertical). The dashed lines represent the face extensions.

The union of all extensions is denoted by

$$ext_{p,q}(\mathcal{M}) = \bigcup_{T \in \mathcal{T}} ext_{p,q}(T).$$

Definition 2.5. A T-mesh $\mathcal{M} \in \mathsf{AD}_{p,q}$ is *analysis-suitable* if horizontal T-junction extensions do not intersect vertical T-junction extensions. For given p and q, the class of analysis-suitable T-meshes will be denoted by $\mathsf{AS}_{p,q}$.

Due to the length of the extensions, it is immediate to see that $\mathsf{AS}_{p,q} \subset \mathsf{AS}_{p',q'}$, when $0 \le p' \le p$, $0 \le q' \le q$.

Remark 2.2. As it was already noticed in [19], T-meshes with L-junctions or I-junctions automatically violate the conditions of Definition 2.5, for any p, q. However, there is a type of valence two vertices, represented as , in [5], that could be allowed in AS T-meshes. This situation is not allowed in our present framework. Indeed, while having no practical interest in view of local refinement, these valence two vertices do not play any role in the definition of even degree T-splines and make the theory more involute for odd degree T-splines.

2.2. *Dual-compatible T-meshes*

In the present section we introduce the concept of dual-compatible T-meshes, that is a generalization of the definition found in [5] for the $p = q = 3$ case.

Let $(i_1, i_2, \dots, i_\ell)$ be a vector with ordered entries. By abuse of notation, we write $i \in (i_1, i_2, \dots, i_\ell)$ when $i \in \{i_1, i_2, \dots, i_\ell\}$. We also use the statement $(i_1, i_2, \dots, i_\ell)$ *skips* i, when $i_1 < i < i_\ell$ and $i \notin (i_1, i_2, \dots, i_\ell)$.

Definition 2.6. Let $\mathcal{M} \in \mathsf{AD}_{p,q}$ and let $A^1, A^2 \in \mathcal{A}_{p,q}(\mathcal{M})$ be two anchors, with $hv_{p,q}(A^1) = (i^1_1, i^1_2, \dots, i^1_{p+2})$ and $hv_{p,q}(A^2) = (i^2_1, i^2_2, \dots, i^2_{p+2})$. We say that A^1 and A^2 overlap horizontally, and use the notation $hv_{p,q}(A^1) \asymp hv_{p,q}(A^2)$, if

$$\begin{aligned} &\forall k \in hv_{p,q}(A^1), \quad i^2_1 \le k \le i^2_{p+2} \Rightarrow k \in hv_{p,q}(A^2), \\ &\forall k \in hv_{p,q}(A^2), \quad i^1_1 \le k \le i^1_{p+2} \Rightarrow k \in hv_{p,q}(A^1). \end{aligned} \tag{2.4}$$

Similarly, let $vv_{p,q}(A^1) = (j^1_1, j^1_2, \dots, j^1_{q+2})$ and $vv_{p,q}(A^2) = (j^2_1, j^2_2, \dots, j^2_{q+2})$; we say that A^1 and A^2 overlap vertically, denoted $vv_{p,q}(A^1) \asymp vv_{p,q}(A^2)$, if

$$\begin{aligned} &\forall k \in vv_{p,q}(A^1), \quad j^2_1 \le k \le j^2_{q+2} \Rightarrow k \in vv_{p,q}(A^2), \\ &\forall k \in vv_{p,q}(A^2), \quad j^1_1 \le k \le j^1_{q+2} \Rightarrow k \in vv_{p,q}(A^1). \end{aligned} \tag{2.5}$$

Finally, we say that A^1 and A^2 *partially overlap*, and we denote it by $A^1 \asymp A^2$, if they overlap either vertically or horizontally.

In other words, we say that two anchors overlap in one direction (horizontal or vertical) if their corresponding index vectors can be extracted as consecutive indices from a common global index vector. We remark that if two anchors A^1 and A^2 do not partially overlap, only two situations can occur (up to an exchange of A^1 and A^2):

(i) $hv(A^1)$ skips an index of $hv(A^2)$, and $vv(A^1)$ skips an index of $vv(A^2)$,
(ii) $hv(A^1)$ skips an index of $hv(A^2)$, and $vv(A^2)$ skips an index of $vv(A^1)$.

Definition 2.7. A T-mesh $\mathcal{M} \in \mathsf{AD}_{p,q}$ is *dual-compatible*, and it is denoted by $\mathcal{M} \in \mathsf{DC}_{p,q}$, if all the couples of anchors $A^1, A^2 \in \mathcal{A}_{p,q}(\mathcal{M})$ partially overlap.

3. $\mathsf{AS}_{p,q} \subseteq \mathsf{DC}_{p,q}$

In this section we prove that any analysis-suitable T-mesh is also dual-compatible, thus generalizing the result of [5] to arbitrary degree. The results are proved for T-meshes in $\mathsf{AD}_{p,q}$, but they also hold for T-meshes of class $\mathsf{AD}^+_{p,q}$ (see Definition 2.2), since $\mathsf{AD}^+_{p,q} \subset \mathsf{AD}_{p,q}$. The proofs in this section are based on the following induction argument.

Lemma 3.1. *Let $\wp_{p,q}$ be a proposition such that*

$$\wp_{0,0} \textit{ (holds true)}, \tag{3.1}$$

and, for all $p, q \in \mathbb{N}$,

$$\forall p' \in \mathbb{N} \,|\, 0 \le p' < p,\ \wp_{p',q} \quad \Rightarrow \quad \wp_{p,q}, \tag{3.2}$$

$$\forall q' \in \mathbb{N} \,|\, 0 \le q' < q,\ \wp_{p,q'} \quad \Rightarrow \quad \wp_{p,q}. \tag{3.3}$$

Then for all $p, q \in \mathbb{N}$, $\wp_{p,q}$ holds.

Proof. It follows from the *complete induction* principle. □

Lemma 3.2. *For all $p, q \in \mathbb{N}$ the following statement holds.*
Let $\mathcal{M} \in \mathsf{AS}_{p,q}$ and $A \in \mathcal{A}_{p,q}(\mathcal{M})$. Then

(a) $\forall V \in \mathcal{V}$, $V \cap tf_{p,q}(A) = \emptyset$, that is, there are no vertices of $\mathcal{M}$ in $tf_{p,q}(A)$;
(b) $\mathcal{S} \cap tf_{p,q}(A) \subset ext_{p,q}(\mathcal{M})$, that is, any portion of the edges of $\mathcal{M}$ that intersect $tf_{p,q}(A)$ is contained in some T-junctions extension of $\mathcal{M}$.
(c) $sk_{p,q}(A) \subset (\mathcal{S} \cup ext_{p,q}(\mathcal{M}))$.

Proof. Let $\wp(p, q)$ the statement of the lemma, that we want prove for all $p, q \in \mathbb{N}$ by induction (Lemma 3.1). Observe that (3.1) is trivial, because $tf_{0,0}(A)$ is one element of the mesh $\mathcal{M}$.

Then, we prove (3.2) for p odd and $q \in \mathbb{N}$. Let $\mathcal{M} \in \mathsf{AS}_{p,q}$ and $A \in \mathcal{A}_{p,q}(\mathcal{M})$ be given; from Definition 2.3, A is either a vertex or a vertical edge of $\mathcal{M}$, when q is odd or even, respectively. Let γ be the vertical line of $sk_{p,q}(A)$ that contains A (see Figure 4). Consider the T-junctions of kind $\vdash$ or $\dashv$ on γ, and let $\bar{\mathcal{M}}$ be the T-mesh which is obtained refining $\mathcal{M}$ by *adding* the first-bay face-extension of the T-junctions above, as depicted in Figure 5. Clearly $\mathcal{S} \subseteq \bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ denotes the skeleton of $\bar{\mathcal{M}}$.

From Definition 2.1, and observing that $FR_{p-1,q} \subsetneq FR_{p,q}$ for an odd p, it is immediate to see that the added extensions will not affect the frame, thus $\bar{\mathcal{M}} \in \mathsf{AD}_{p-1,q}$. Moreover, from Definition 2.5, $\bar{\mathcal{M}} \in \mathsf{AS}_{p-1,q}$: indeed when adding a horizontal edge as above, either the new edge connects two T-junctions of $\mathcal{M}$, that become vertices of valence four in $\bar{\mathcal{M}}$, or the corresponding T-junction T of $\mathcal{M}$ is replaced by a new T-junction $\bar{T}$ of $\bar{\mathcal{M}}$ (see Figure 5). In the latter case, it is immediate to see that $ext_{p-1,q}(\bar{T}) \subset ext_{p,q}(T)$, and new intersections of extensions cannot appear.

Let $\bar{A}^1, \bar{A}^2$ be the two (left, right) anchors in $\mathcal{A}_{p-1,q}(\bar{\mathcal{M}})$ such that $A \subseteq \partial\bar{A}^1 \cap \partial\bar{A}^2$, as depicted in Figure 5. We denote by $hv_{p-1,q}(\bar{A}^i)$ and $vv_{p-1,q}(\bar{A}^i)$ the horizontal and vertical index vectors of $\bar{A}^i$ with respect to the T-mesh $\bar{\mathcal{M}}$. By construction, there are no T-junctions of kind $\vdash$ or $\dashv$ on $\gamma \cap \bar{\mathcal{M}}$. Moreover, by the induction hypothesis $\wp(p-1, q)$ point (a), each horizontal line of $\bar{\mathcal{M}}$ crossing γ cannot terminate in $tf_{p-1,q}(\bar{A}^i)$, whence $vv_{p-1,q}(\bar{A}^i) = vv_{p,q}(A)$. Furthermore, $hv_{p-1,q}(\bar{A}^1)$ (resp., $hv_{p-1,q}(\bar{A}^2)$) gets the leftmost (resp., rightmost) $p + 1$ entries of $hv_{p,q}(A)$. As a consequence, we have

$$tf_{p,q}(A) = tf_{p-1,q}(\bar{A}^1) \cup tf_{p-1,q}(\bar{A}^2), \tag{3.4}$$

$$sk_{p,q}(A) = sk_{p-1,q}(\bar{A}^1) \cup sk_{p-1,q}(\bar{A}^2). \tag{3.5}$$

Point (a) of $\wp(p, q)$ follows using (3.4) and point (a) of $\wp(p-1, q)$ for $\bar{\mathcal{M}}$, because the vertices of $\mathcal{M}$ are also vertices of $\bar{\mathcal{M}}$.

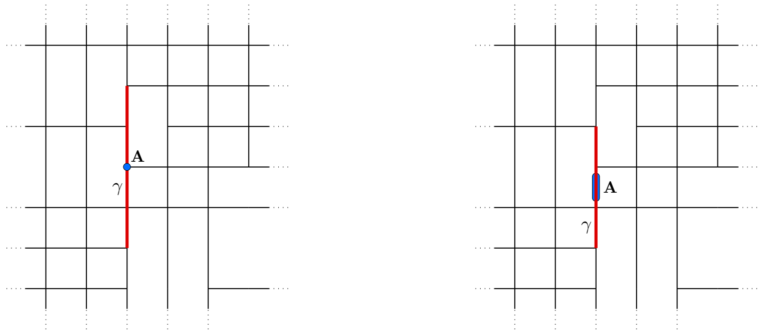


Fig. 4. Examples of $A \in \mathcal{A}_{p,q}(\mathcal{M})$, with p odd and $q = 3$ (left) and $q = 2$ (right). The anchor A is represented in blue, while the vertical line γ selected in the proof of Lemma 3.2 is highlighted in red.

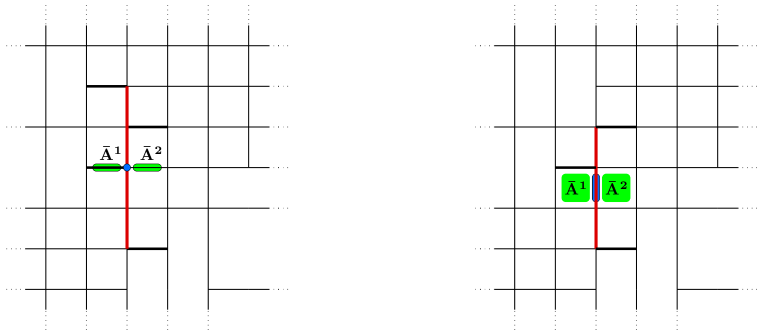


Fig. 5. From Figure 4, the first-bay horizontal face-extensions are added in order to obtain $\bar{\mathcal{M}}$. The added lines either replace T-junctions with new ones, or they connect two T-junctions. The anchors $\bar{A}^1, \bar{A}^2 \in \mathcal{A}_{p-1,q}(\bar{\mathcal{M}})$ in the proof of Lemma 3.2 are represented in light green.

Point (b) of $\wp(p,q)$ follows, first using $\mathcal{S} \subseteq \bar{\mathcal{S}}$ and (3.4), then point (b) of $\wp(p-1,q)$ for $\bar{\mathcal{M}}$, and then $ext_{p-1,q}(\bar{\mathcal{M}}) \subset ext_{p,q}(\mathcal{M})$:

$$\mathcal{S} \cap tf_{p,q}(A) \subseteq \bar{\mathcal{S}} \cap (tf_{p-1,q}(\bar{A}^1) \cup tf_{p-1,q}(\bar{A}^2)) \subset ext_{p-1,q}(\bar{\mathcal{M}}) \subset ext_{p,q}(\mathcal{M}).$$

Point (c) of $\wp(p,q)$ follows from (3.5), then point (c) of $\wp(p-1,q)$ for $\bar{\mathcal{M}}$, and then $(\bar{\mathcal{S}} \cup ext_{p-1,q}(\bar{\mathcal{M}})) \subset (\mathcal{S} \cup ext_{p,q}(\mathcal{M}))$:

$$sk_{p,q}(A) = sk_{p-1,q}(\bar{A}^1) \cup sk_{p-1,q}(\bar{A}^2) \subset (\bar{\mathcal{S}} \cup ext_{p-1,q}(\bar{\mathcal{M}})) \subset (\mathcal{S} \cup ext_{p,q}(\mathcal{M})).$$

The proof of (3.2) for p even, $p \geq 2$, and $q \in \mathbb{N}$, is similar and we only give an outline pointing out the differences. Take $\mathcal{M} \in \mathsf{AS}_{p,q}$ and $A \in \mathcal{A}_{p,q}(\mathcal{M})$ (see Figure 6). Since $\mathcal{M} \in \mathsf{AS}_{0,q}$ and $A \in \mathcal{A}_{0,q}(\mathcal{M})$ as well, the induction hypothesis $\wp(0,q)$ point (a) guarantees that there are no T-junctions in $tf_{0,q}(A)$:

$$\{T \in \mathcal{T} \,:\, T \subset tf_{0,q}(A)\} = \emptyset. \tag{3.6}$$

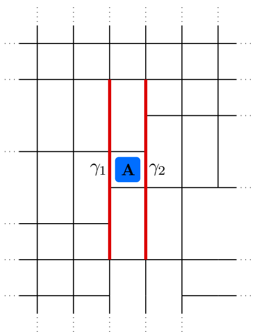
Let γ_1 (resp., γ_2) the left (resp., right) vertical line of $sk_{0,q}(A)$ (see Figure 6). We now construct a *coarser* T-mesh $\bar{\mathcal{M}}$ by the following procedure: consider any T-junctions $T = \{(\bar{\imath}, \bar{\jmath})\}$ of kind $\dashv$ contained in γ_1 (resp., of kind $\vdash$ contained in γ_2). Observe that, since $\mathcal{M} \in \mathsf{AD}_{p,q}$, $\bar{\imath} \geq \underline{m} + \frac{p}{2} + 1$ and $\bar{\imath} \leq \overline{m} - \frac{p}{2} - 1$. Then,

- if $\bar{\imath} > \underline{m} + \frac{p}{2} + 1$ (resp., $\bar{\imath} < \overline{m} - \frac{p}{2} - 1$), *remove* the first-bay edge-extension of T,
- if $\bar{\imath} = \underline{m} + \frac{p}{2} + 1$ (resp., $\bar{\imath} = \overline{m} - \frac{p}{2} - 1$), *remove* the line $]\underline{m}, \underline{m} + \frac{p}{2} + 1[\times \{\bar{\jmath}\}$ (resp., $]\overline{m} - \frac{p}{2} - 1, \overline{m}[\times \{\bar{\jmath}\}$).

That is, in general we remove the first-bay edge-extension (see Figure 7), and if this first-bay edge-extension arrives up to the frame, we also remove the horizontal line in the frame to maintain the valence condition in Definition 2.1.

The construction above leads to and admissible T-mesh $\bar{\mathcal{M}} \in \mathsf{AD}_{p-1,q}$: indeed, a horizontal edge-extension of $\mathcal{M}$ does not intersect any vertical T-junction ($\perp$ or $\top$) by the condition $\mathcal{M} \in \mathsf{AS}_{p,q}$, then by removing edges as above we only produce valence three or four vertices in $\bar{\mathcal{M}}$. Moreover, we have $\bar{\mathcal{M}} \in \mathsf{AS}_{p-1,q}$, and each vertex of $\bar{\mathcal{M}}$ on γ_1 is now connected by a horizontal edge to a vertex on γ_2, and viceversa (use $(\bar{\mathcal{S}} \cap tf_{0,q}(A)) = (\mathcal{S} \cap tf_{0,q}(A))$ and (3.6)).

Let us consider $\bar{A}^1, \bar{A}^2 \in \mathcal{A}_{p-1,q}(\bar{\mathcal{M}})$ such that $\bar{A}^i \subseteq \partial A$ (see Figure 7). By construction, $tf_{p,q}(A) = tf_{p-1,q}(\bar{A}^1) \cup tf_{p-1,q}(\bar{A}^2)$, $sk_{p,q}(A) = sk_{p-1,q}(\bar{A}^1) \cup sk_{p-1,q}(\bar{A}^2)$ and $ext_{p-1,q}(\bar{\mathcal{M}}) \subset ext_{p,q}(\mathcal{M})$; moreover $\mathcal{S} \cap tf_{p,q}(A)$ is the union of $\bar{\mathcal{S}} \cap tf_{p,q}(A)$ and the edge-extensions of $\mathcal{M}$ that have been removed. Reasoning as before, we get the three points of $\wp(p, q)$.



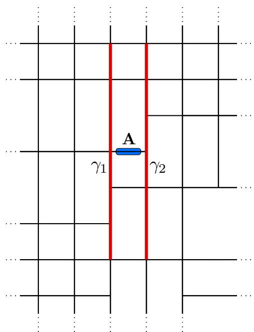


Fig. 6. Examples of $A \in \mathcal{A}_{p,q}(\mathcal{M})$, with p even, $q = 2$ (left) and $q = 3$ (right). The anchor A is represented in blue, while the regions γ_1 and γ_2 selected in the proof of Lemma 3.2 are highlighted in red.

Eventually, (3.3) is proved analogously to (3.2), and not detailed. □

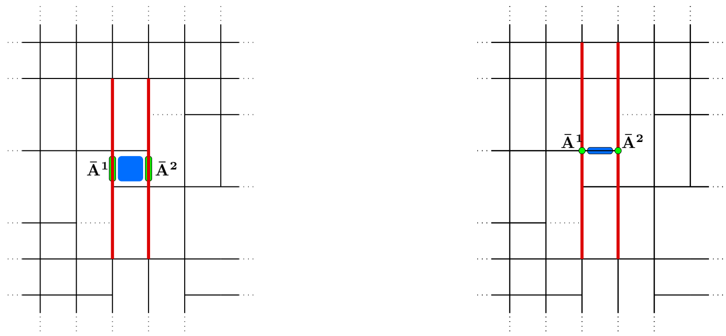


Fig. 7. From Figure 6, the first-bay edge-extensions (dotted lines) are removed in order to obtain $\bar{\mathcal{M}}$. The anchors $\bar{A}^1, \bar{A}^2 \in \mathcal{A}_{p-1,q}(\bar{\mathcal{M}})$ are represented in light green.

Remark 3.1. As observed in Remark 2.1, our index vector definition (Definition 2.4) differs from the one given in [3] for a generic T-mesh. However, it is easy to check that the two definitions coincide when property (a) of Lemma 3.2 holds. Indeed, the property implies that, given an anchor $A \in \mathcal{A}_{p,q}(\mathcal{M})$, there are no lines of the skeleton $\mathcal{S}$ that terminate in a T-junction which lies in a rectangle of the tiled floor $tf_{p,q}(A)$.

The major consequence of Lemma 3.2 is the following theorem.

Theorem 3.1. *If* $\mathcal{M} \in \mathsf{AS}_{p,q}$ *then* $\mathcal{M} \in \mathsf{DC}_{p,q}$.

Proof. The proof is given by contradiction, assuming $\mathcal{M} \in \mathsf{AS}_{p,q}$ and $\mathcal{M} \notin \mathsf{DC}_{p,q}$. Since $\mathcal{M} \notin \mathsf{DC}_{p,q}$, from Definition 2.7 there exist $A^1, A^2 \in \mathcal{A}_{p,q}(\mathcal{M})$ such that $A^1 \not\asymp A^2$, and, as we have already mentioned, one of these two cases may occur:

(i) $hv(A^1)$ skips a value of $hv(A^2)$, and $vv(A^1)$ skips a value of $vv(A^2)$,
(ii) $hv(A^1)$ skips a value of $hv(A^2)$, and $vv(A^2)$ skips a value of $vv(A^1)$.

In the first case there exists a vertical line and a horizontal line of $sk_{p,q}(A^2)$, that intersect in $tf_{p,q}(A^1)$. In the second case, there exists a vertical line of $sk_{p,q}(A^2)$ that passes through $tf_{p,q}(A^1)$, and a horizontal line of $sk_{p,q}(A^1)$ that passes through $tf_{p,q}(A^2)$, and the two lines intersect. Both cases are in contradiction with $\mathcal{M} \in \mathsf{AS}_{p,q}$. Indeed, from points (b) and (c) in Lemma 3.2, the skeleton of an anchor contained in the tiled floor of another anchor is part of an extension of some T-junction. Therefore, in the two cases there are vertical and horizontal extensions that intersect. □

4. $\mathsf{DC}_{p,q} \subseteq \mathsf{AS}_{p,q}$

In this section, we want to prove that any dual-compatible mesh is also analysis-suitable, thus showing that the two concepts are equivalent. Here we restrict ourselves to the class of T-meshes $\mathsf{AD}^+_{p,q}$ as provided in Definition 2.2. This is not a technical issue, but rather a needed assumption as it is clear from the mesh $\mathcal{M}$ in Figure 8: $\mathcal{M}$ is a tensor product mesh to which two edges have been removed. Let $p, q \geq 2$. If both p and q are odd, the anchors and the index vectors are exactly the ones of the corresponding tensor product mesh. As a consequence, $\mathcal{M}$ is dual-compatible, i.e., $\mathcal{M} \in \mathsf{DC}_{p,q}$ but clearly we have $\mathcal{M} \notin \mathsf{AS}_{p,q}$ since extensions intersect. It should also be mentioned that if p and q are even $\mathcal{M} \notin \mathsf{DC}_{p,q}$, because the two anchors associated with the rectangular elements do not partially overlap.

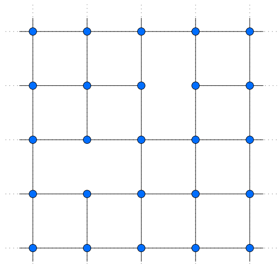


Fig. 8. A mesh $\mathcal{M}$ which does not belong to the class $\mathsf{AD}^+_{p,q}$

We first state and prove the main result of this section for odd degrees p and q.

Theorem 4.1. *Let $\mathcal{M} \in \mathsf{AD}^+_{p,q}$ with p and q odd. If $\mathcal{M} \in \mathsf{DC}_{p,q}$ then $\mathcal{M} \in \mathsf{AS}_{p,q}$.*

Proof. We are given with a T-mesh $\mathcal{M} \in \mathsf{AD}^+_{p,q}$ such that $\mathcal{M} \notin \mathsf{AS}_{p,q}$ and we prove that $\mathcal{M} \notin \mathsf{DC}_{p,q}$, that is, we find two anchors that do not overlap. If $\mathcal{M} \notin \mathsf{AS}_{p,q}$, there is a horizontal T-junction $T^1 = \{(i^1, j^1)\}$ and a vertical T-junction $T^2 = \{(i^2, j^2)\}$ such that $ext_{p,q}(T^1) \cap ext_{p,q}(T^2) \neq \emptyset$. We assume, without loss of generality, that T_1 is of type $\dashv$ while T^2 is of type $\top$. Let $\kappa = (p+1)/2$ and $\ell = (q+1)/2$. Following the definition of the face and edge extension, we extract from $h\mathcal{I}(j^1)$ the $p+1$ consecutive indices $i^1_1, \ldots, i^1_{p+1}$ such that $i^1_\kappa = i^1$ and analogously, we extract from $v\mathcal{I}(i^2)$ the $q+1$ consecutive indices $j^2_1, \ldots, j^2_{q+1}$ such that $j^2_\ell = j^2$. We recall that

$$\begin{aligned} ext^e_{p,q}(T^1) &= [i^1_1, i^1_\kappa] \times \{j^1\}, \qquad & ext^f_{p,q}(T^1) &=]i^1_\kappa, i^1_{p+1}] \times \{j^1\}, \\ ext^e_{p,q}(T^2) &= \{i^2\} \times [j^2_1, j^2_\ell], \qquad & ext^f_{p,q}(T^2) &= \{i^2\} \times]j^2_\ell, j^2_{q+1}]. \end{aligned} \tag{4.1}$$

Since p and q are odd we define $A^1 = T^1$ and $A^2 = T^2$ and have $A^1, A^2 \in \mathcal{A}_{p,q}(\mathcal{M})$. The extensions above are related with the index vectors $hv(A^1)$ and $vv(A^2)$. Indeed, by definition, there exist two indices $i^1_0 < i^1_1$ and $j^2_0 < j^2_1$ such that:

$$hv(A^1) = (i^1_0, i^1_1, \ldots, i^1_{p+1}), \qquad vv(A^2) = (j^2_0, j^2_1, \ldots, j^2_{q+1}). \tag{4.2}$$

We remark first that $\{(i^1_{\kappa+1}, j^1)\} \notin h\mathcal{S}$ but it belongs to a vertical edge e^1, and $\{(i^2, j^2_{\ell+1})\} \notin v\mathcal{S}$ but it belongs to a horizontal edge e^2. This follows from the definition of $h\mathcal{I}(j^1)$ and $v\mathcal{I}(i^2)$, considering that the two points cannot be associated to vertices of the mesh thanks to the assumption $\mathcal{M} \in \mathsf{AD}^+_{p,q}$. The boundary of both e^1 and e^2 is made of two vertices, i.e., $B^1_- \cup B^1_+ = \partial e^1$ and $B^2_- \cup B^2_+ = \partial e^2$, see Figure 9(a).

By construction, $vv(A^1) \not\asymp vv(B^1_-)$, and $vv(A^1) \not\asymp vv(B^1_+)$, since $vv(B^1_-)$ and $vv(B^1_+)$ skip j^1; analogously, $hv(A^2) \not\asymp hv(B^2_-)$, and $hv(A^2) \not\asymp hv(B^2_+)$, since $hv(B^2_-)$ and $hv(B^2_+)$ skip i^2. If $hv(A^1) \not\asymp hv(B^1_-)$ then A^1 and B^1_- do not partially overlap; if $hv(A^1) \not\asymp hv(B^1_+)$ then A^1 and B^1_+ do not partially overlap; if $vv(A^2) \not\asymp vv(B^2_-)$ then B^2_- and A^2 do not partially overlap; and if $vv(A^2) \not\asymp vv(B^2_+)$ then B^2_+ and A^2 do not partially overlap. In these four cases the proof is finished. From now on, let $B^1 = B^1_+$, $B^2 = B^2_+$, we suppose that

$$hv(A^1) \asymp hv(B^1)\,,\ vv(A^2) \asymp vv(B^2). \tag{4.3}$$

We consider now all possible cases.

Case 1) We consider the case when $ext_{p,q}(T^1) \cap ext_{p,q}(T^2) \not\subset \mathcal{S}$. Here it is immediate to check that $A^1 = T^1$ and $A^2 = T^2$ do not overlap. In fact, $hv(A^1)$ skips i^2 while $vv(A^2)$ skips j^1.

Case 2) We consider the case when $ext_{p,q}(T^1) \cap ext_{p,q}(T^2) = V \in \mathcal{V}$. We can assume without loss of generality that edge extensions intersect, i.e. (see Figure 9(a))

$$ext^e_{p,q}(T^1) \cap ext^e_{p,q}(T^2) = V \in \mathcal{V}. \tag{4.4}$$

If this is not the case, it is easy to see that we can select another couple of T-junctions $\tilde{T}^1 \subset ext_{p,q}(T^1)$ and $\tilde{T}^2 \subset ext_{p,q}(T^2)$, with $\tilde{T}^1$ of kind $\vdash$ or $\dashv$ and $\tilde{T}^2$ of kind $\perp$ or $\top$, such that $ext^e_{p,q}(\tilde{T}^1) \cap ext^e_{p,q}(\tilde{T}^2) = V$; then select $A^1 = \tilde{T}^1$ and $A^2 = \tilde{T}^2$, and repeat a construction similar to case 1) above. Since (4.4),

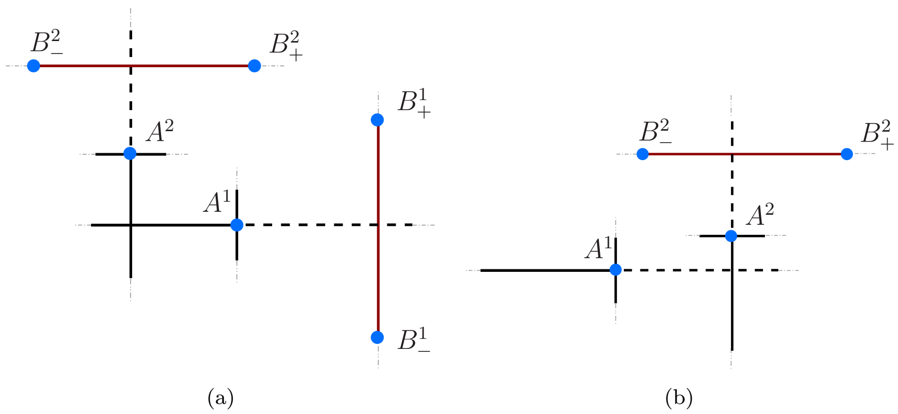


Fig. 9. T-junctions T^1 and T^2 for p and q odd. (a) $ext^e_{p,q}(T^1) \cap ext^e_{p,q}(T^2)$ is a vertex of the T-mesh. (b) $ext^f_{p,q}(T^1) \cap ext^e_{p,q}(T^2)$ belongs to a vertical edge of the mesh.

$$\exists n \quad 1 \leq n \leq \kappa \ : \ i^2 = i^1_n \ , \qquad \exists m \quad 1 \leq m \leq \ell \ : \ j^1 = j^2_m. \tag{4.5}$$

Recalling (4.3) ($hv(A^1) \asymp hv(B^1)$), (4.2) and (4.5), we have that $hv(B^1)$ contains i^2. By construction, we have that $hv(B^2)$ skips i^2. Thus,

$$hv(B^1) \not\asymp hv(B^2).$$

Using (4.3) ($vv(A^2) \asymp vv(B^2)$), (4.2) and (4.5), we have that $vv(B^2)$ contains j^1. By construction $vv(B^1)$ skips j^1. Thus

$$vv(B^1) \not\asymp vv(B^2).$$

Hence B^1 and B^2 do not partially overlap.

Case 3) We consider the case when $ext_{p,q}(T^1) \cap ext_{p,q}(T^2)$ is contained in a vertical edge $e \in v\mathcal{E}$. Without loss of generality (arguing as in Case 2) we can further assume that

$$ext_{p,q}(T^1) \cap ext_{p,q}(T^2) = ext^f_{p,q}(T^1) \cap ext^e_{p,q}(T^2) \subset e, \quad e \in v\mathcal{E}; \tag{4.6}$$

see Figure 9(b).

From (4.6), $hv(A^1)$ contains i^2, and, by construction, $hv(B^2)$ skips i^2. Thus

$$hv(A^1) \not\asymp hv(B^2).$$

From (4.1)–(4.3) we have that the first $(p+1)/2$ entries of $vv(B^2)$ correspond to the indices of $ext^e_{p,q}(T^2)$. Therefore, due to (4.6), $vv(B^2)$ skips j^1. Whence

$$vv(A^1) \not\asymp vv(B^2).$$

We conclude that A^1 and B^2 do not overlap.

Case 4) We consider the case when $ext_{p,q}(T^1) \cap ext_{p,q}(T^2)$ is contained in a horizontal edge. Arguing as before we can assume $ext_{p,q}(T^1) \cap ext_{p,q}(T^2) = ext^e_{p,q}(T^1) \cap ext^f_{p,q}(T^2) \subset e$, $e \in h\mathcal{E}$, and the proof is analogous to that of Case 3. □

Now, we want to consider the case of even or mixed degrees p and q. We start proving the next Lemma which shows that the Property (a) of Lemma 3.2 holds also for dual-compatible meshes.

Lemma 4.1. *Let $\mathcal{M} \in \mathsf{DC}_{p,q}$ and $A \in \mathcal{A}_{p,q}(\mathcal{M})$. Then $tf_{p,q}(A)$ does not contain vertices, i.e., for all $V \in \mathcal{V}$, it holds: $V \cap tf_{p,q}(A) = \emptyset$.*

Proof. Let us suppose that there exists an anchor $A \in \mathcal{A}_{p,q}(\mathcal{M})$ and a vertex $V \in \mathcal{V}$, $V = \{(i,j)\}$, such that $V \subset tf_{p,q}(A)$. By definition, V is either a T-junction or a vertex of valence 4. We will choose an anchor $A' \in \mathcal{A}_{p,q}(\mathcal{M})$ such that $A' \not\asymp A$. This choice depends on p and q:

- if p and q are odd, it is trivial and it is enough to choose $A' = V$;
- if p is odd and q is even (p is even and q is odd) we can choose A' as a vertical (horizontal) edge e such that $V \subset \partial e$.

- if q and p are even, we choose A' as any of the elements $Q \in \mathcal{M}$ such that V is one of the four corners of Q.

In all cases $hv(A')$ contains i while $hv(A)$ skips it, and $vv(A')$ contains j while $vv(A)$ skips it. Thus, $A \not\asymp A'$. □

We are now ready to extend Theorem 4.1 also to the cases when p and q are not odd. We first prove the result when p and q are both even, and then for the mixed odd/even and even/odd cases.

Theorem 4.2. *Let $\mathcal{M} \in \mathsf{AD}^+_{p,q}$ with p and q even. If $\mathcal{M} \in \mathsf{DC}_{p,q}$ then $\mathcal{M} \in \mathsf{AS}_{p,q}$.*

Proof. The proof follows the same steps as for Theorem 4.1 and will be therefore presented more briefly.

Let T^1 and T^2 be two T-junctions as in the proof of Theorem 4.1, with the notation (4.1). Let $Q^\ell \subset \mathcal{M}$, $\ell = 1, 2$ be the only element of the mesh whose closure contains the first-bay face extension of the T-junction T^ℓ.

We associate anchors A^ℓ and B^ℓ, $\ell = 1, 2$ to the T-junctions as follows (see Figure 10(a)):

i) A^ℓ is one of the two anchors in $\mathcal{A}_{p,q}(\mathcal{M})$ such that the first-bay edge extension of T^ℓ belongs to ∂A^ℓ.

ii) $B^\ell = Q^\ell$, $B^\ell \in \mathcal{A}_{p,q}(\mathcal{M})$.

As in Theorem 4.1, it is easy to see that $vv(B^1) \not\asymp vv(A^1)$ and $hv(B^2) \not\asymp hv(A^2)$. We are then left with the case:

$$hv(B^1) \asymp hv(A^1) \qquad \text{and} \qquad vv(B^2) \asymp vv(A^2). \tag{4.7}$$

As in Theorem 4.1, the proof splits in the next four cases, and all the other possibilities can be recast in these cases up to possible alternate choice of T-junctions.

Case 1) Let $ext_{p,q}(T^1) \cap ext_{p,q}(T^2) \not\subset \mathcal{S}$. It is immediate to check that $A^1 \not\asymp A^2$.

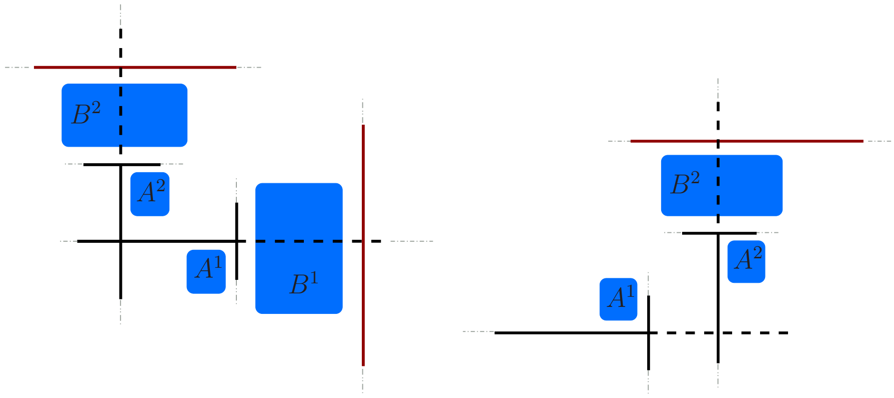
Case 2) Let $ext^e_{p,q}(T^1) \cap ext^e_{p,q}(T^2) = V \in \mathcal{V}$. In view of (4.7), with the same argument as in Theorem 4.1, the index vector $hv(B^2)$ skips i^2 while $hv(B^1)$ contains i^2, and $vv(B^1)$ skips j^1 while $vv(B^2)$ contains j^1. Whence $B^1 \not\asymp B^2$, see Figure 10(a).

Case 3) Let $ext^f_{p,q}(T^1) \cap ext^e_{p,q}(T^2) \subset e$, $e \in v\mathcal{E}$. In view of (4.7), with the same arguments as in the proof of Theorem 4.1, $hv(B^2)$ skips i^2 and $vv(B^2)$ skips j^1 while $hv(A^1)$ contains i^2 and $vv(A^1)$ contains j^1. Whence $A^1 \not\asymp B^2$, see Figure 10(b).

Case 4) The case $ext^e_{p,q}(T^1) \cap ext^f_{p,q}(T^2) \subset e$, $e \in h\mathcal{E}$ is analogous to Case 3. □

Finally, we propose the same Theorem in the case of mixed degrees, i.e., p being even and q odd or viceversa.

Theorem 4.3. *Let $\mathcal{M} \in \mathsf{AD}^+_{p,q}$ with p odd and q even or viceversa. If $\mathcal{M} \in \mathsf{DC}_{p,q}$ then $\mathcal{M} \in \mathsf{AS}_{p,q}$.*



(a) (b)

Fig. 10. T-junctions T^1 and T^2 for p and q even. (a) $ext^e_{p,q}(T^1) \cap ext^e_{p,q}(T^2)$ is a vertex of the T-mesh. (b) $ext^f_{p,q}(T^1) \cap ext^e_{p,q}(T^2)$ belongs to a vertical edge of the mesh.

Proof. The proof of this result uses the same argument as in Theorems 4.1 and 4.2 and we just sketch it briefly. Let T^1 and T^2 be two T-junctions as in Theorem 4.1, and let $Q^\ell, \ell = 1, 2$, be the elements defined as in Theorem 4.2.

We concentrate on the case p odd and q even, the other one being analogous. We associate the anchors A^ℓ and B^ℓ to T^ℓ, $\ell = 1, 2$ as follows (see Figure 11):

i) $A^\ell \in \mathcal{A}_{p,q}(\mathcal{M})$ is one anchor (possibly unique) such that $T^\ell \subset \partial A^\ell$.
ii) B^1 is the only anchor that intersects the first-bay face-extension of T^1. It is unique due to $\mathcal{M} \in \mathsf{AD}^+_{p,q}$.
iii) B^2 is one anchor such that $B^2 \subset \partial Q^2$.

As before, the proof splits in four cases:

Case 1) Let $ext_{p,q}(T^1) \cap ext_{p,q}(T^2) \not\subset \mathcal{S}$. It holds $A^1 \not\asymp A^2$.
Case 2) Let $ext^e_{p,q}(T^1) \cap ext^e_{p,q}(T^2) = V \in \mathcal{V}$. It holds $B^1 \not\asymp B^2$, see Figure 11(a).
Case 3) Let $ext^f_{p,q}(T^1) \cap ext^e_{p,q}(T^2) \subset e$, $e \in v\mathcal{E}$. It holds $A^1 \not\asymp B^2$, see Figure 11(b).
Case 4) Let $ext^e_{p,q}(T^1) \cap ext^f_{p,q}(T^2) \subset e$, $e \in h\mathcal{E}$. It holds $A^2 \not\asymp B^1$, see Figure 11(c).

The case p even and q odd is proved in a similar way, reminding that the anchors are the horizontal edges, and exchanging the way to choose B^1 and B^2. $\square$

5. Properties of DC T-splines

In this section we finally define T-splines blending functions over T-meshes in the classes $\mathsf{AS}_{p,q} \equiv \mathsf{DC}_{p,q}$, and we analyze how the structure of dual-compatible T-meshes has an impact on the mathematical properties of the T-spline blending function and the space they span. We start by introducing T-spline spaces of general polynomial degree (p, q), that is a space of real valued functions living on the

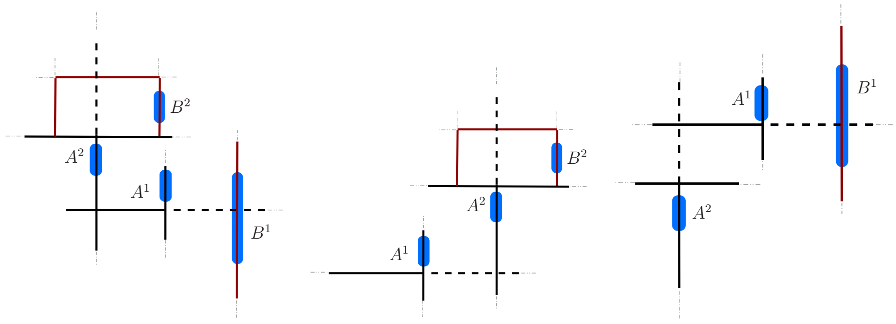


Fig. 11. T-junctions T^1 and T^2 for p odd and q even. (a) $ext^e_{p,q}(T^1) \cap ext^e_{p,q}(T^2) = V \in \mathcal{V}$, (b) $ext^f_{p,q}(T^1) \cap ext^e_{p,q}(T^2) \subset e$, $e \in v\mathcal{E}$, (c) $ext^e_{p,q}(T^1) \cap ext^f_{p,q}(T^2) \subset e$, $e \in h\mathcal{E}$.

parametric domain $[0,1] \times [0,1]$. Let $\mathcal{M} \in \mathsf{DC}_{p,q}$ be a given T-mesh as defined in Section 2. Let moreover $\Xi_s = (s_{\underline{m}}, \ldots, s_{\overline{m}})$ be an open $(p-)$knot vector on the interval $[0,1]$, that is,

$$0 = s_{\underline{m}} = s_{\underline{m}+1} = \ldots = s_{\underline{m}+p} < s_{\underline{m}+p+1} \leq \ldots \leq s_{\overline{m}-p-1} < s_{\overline{m}-p} = \ldots = s_{\overline{m}} = 1.$$

with all internal knots having multiplicity less than or equal to $p+1$. Analogously, let $\Xi_t = (t_{\underline{n}}, \ldots, t_{\overline{n}})$ be an open $(q-)$knot vector on $[0,1]$, that is

$$0 = t_{\underline{n}} = t_{\underline{n}+1} = \ldots = t_{\underline{n}+q} < t_{\underline{n}+q+1} \leq \ldots \leq t_{\overline{n}-q-1} < t_{\overline{n}-q} = \ldots = t_{\overline{n}} = 1.$$

with all internal knots having multiplicity less than or equal to $q+1$.

We define $\mathcal{M}^{\text{ext}}$ as the T-mesh obtained by adding to $\mathcal{M}$ all the T-junction extensions, that is, $\mathcal{S}^{\text{ext}} = \mathcal{S} \cup ext_{p,q}(\mathcal{M})$, where $\mathcal{S}^{\text{ext}}$ is the skeleton of $\mathcal{M}^{\text{ext}}$. We call this mesh the *extended T-mesh* in the index domain. Notice that in general $\mathcal{M}^{\text{ext}} \notin \mathsf{AD}_{p,q}$. Finally, we denote by $\mathtt{M}^{\text{ext}}$ the (extended) mesh in the parametric space, which is defined as the collection of *non empty* elements of the form

$$\mathtt{Q} =]s_{i_1}, s_{i_2}[\times]t_{j_1}, t_{j_2}[\neq \emptyset, \text{ with } Q =]i_1, i_2[\times]j_1, j_2[\in \mathcal{M}^{\text{ext}}.$$

Given a polynomial degree p' and a local knot vector $(c_1, c_2, \ldots, c_{p'+2})$, let in the following $N_{p'}(c_1, c_2, \ldots, c_{p'+2}; \cdot)$ denote the associated standard univariate B-spline of degree p', see for instance [20, 13]. We associate to each anchor $A = a \times b \in \mathcal{A}_{p,q}(\mathcal{M})$ a bivariate B-spline function of degree p in the first coordinate s and of degree q in the second coordinate t. Such function is referred to as T-*spline blending function*, and is defined as follows:

$$B^A_{p,q}(s,t) := N_p(s_{i_1}, s_{i_2}, \ldots, s_{i_{p+2}}; s)\, N_q(t_{j_1}, t_{j_2}, \ldots, t_{j_{q+2}}; t) \quad \forall (s,t) \in [0,1]^2, \tag{5.1}$$

where $hv(A) = (i_1, \ldots, i_{p+2})$ and $vv(A) = (j_1, \ldots, j_{q+2})$ denote the horizontal and vertical index vectors associated to the anchor A, see Definition 2.4.

The T-spline space $\mathtt{B}_{p,q} = \mathtt{B}_{p,q}(\mathcal{M}, \Xi_s, \Xi_t)$ is finally given as the span

$$\mathtt{B}_{p,q}(\mathcal{M}, \Xi_s, \Xi_t) := \mathrm{span}\{B^A_{p,q} \ : \ A \in \mathcal{A}_{p,q}(\mathcal{M})\}. \tag{5.2}$$

The functions in $\mathtt{B}_{p,q}(\mathcal{M}, \Xi_s, \Xi_t)$ are real valued, live in $[0,1]^2$, and, thanks to Lemma 3.2-point (c), their restrictions to any $\mathtt{Q} \in \mathtt{M}^{\mathrm{ext}}$ are bivariate polynomials of degree (p,q). The regularity across the edges of the mesh $\mathtt{M}^{\mathrm{ext}}$ depends on the multiplicity of the knots in Ξ_s or Ξ_t associated to the edge, as in standard B-splines.

To each $A = a \times b \in \mathcal{A}_{p,q}(\mathcal{M})$ we also associate the functional $\lambda^A_{p,q}$ defined as:

$$\lambda^A_{p,q} = \lambda_p(s_{i_1}, s_{i_2}, \ldots, s_{i_{p+2}}) \otimes \lambda_q(t_{j_1}, t_{j_2}, \ldots, t_{j_{q+2}}), \tag{5.3}$$

where the two one-dimensional functionals $\lambda_p(s_{1_1}, s_{i_2}, \ldots, s_{i_{p+2}})$ and $\lambda_q(t_{j_1}, t_{j_2}, \ldots, t_{j_{q+2}})$ are those defined in Theorem 4.41 of [20]. Finally, for every anchor $A \in \mathcal{A}_{p,q}(\mathcal{M})$ we represent with the symbol $\mathtt{Q}^A$ the open support of $B^A_{p,q}$ or, equivalently, of $\lambda^A_{p,q}$.

The following fundamental result holds.

Proposition 5.1. *Let $\mathcal{M} \in \mathsf{DC}_{p,q}$. Then the set of functionals*

$$\{\lambda^A_{p,q} \ : \ A \in \mathcal{A}_{p,q}(\mathcal{M})\} \tag{5.4}$$

is a set of dual functionals for the set

$$\{B^A_{p,q} \ : \ A \in \mathcal{A}_{p,q}(\mathcal{M})\}. \tag{5.5}$$

Proof. Let any two anchors $A^1 = a^1 \times b^1$ and $A^2 = a^2 \times b^2$ in $\mathcal{A}_{p,q}(\mathcal{M})$. We then need to show

$$\lambda^{A^1}_{p,q}(B^{A^2}_{p,q}) = \begin{cases} 1 & \text{if } a^1 = a^2,\ b^1 = b^2 \ , \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Since the mesh $\mathcal{M} \in \mathsf{DC}_{p,q}$, the two anchors A^1 and A^2 overlap either in the horizontal or vertical direction. Without loss of generality, we assume the anchors overlap in the horizontal direction. As a consequence, since (as observed in Section 2.2) their corresponding horizontal index vectors can be extracted as consecutive indices from a common global index vector, following Definition 2.4 and using the obvious notation for $hv(A^1), hv(A^2)$, it is immediate to check that

$$\lambda(s_{i^1_1}, \ldots, s_{i^1_{p+2}}) \left(N(s_{i^2_1}, \ldots, s_{i^2_{p+2}}; \cdot)\right) = \delta_{(a^1,a^2)}$$

where the (generalized) Kronecker symbol $\delta_{(a^1,a^2)} = 1$ if $a^1 = a^2$, and vanishes otherwise.

Therefore by definition (5.3) we have

$$\lambda^{A^1}_{p,q}(B^{A^2}_{p,q}) = \delta_{(a^1,a^2)}\, \lambda(t_{j^1_1}, \ldots, t_{i^1_{q+2}}) \left(N(t_{j^2_1}, \ldots, t_{j^2_{q+2}}; \cdot)\right) .$$

The above identity immediately proves (5.6) in the case $a^1 \neq a^2$. If, on the contrary, $a^1 = a^2$, then by definition (2.1) it holds $v\mathfrak{I}(a^1) = v\mathfrak{I}(a^2)$, and then A^1 and A^2 also overlap in the vertical direction. Therefore the result follows since in such case, again by the same argument above,

$$\lambda(t_{j_1^1},\ldots,t_{i^1_{q+2}})\left(N(t_{j_1^2},\ldots,t_{j^2_{q+2}};\cdot)\right)=\delta_{(b^1,b^2)}. \qquad \square$$

Since in the previous sections we proved that $\mathsf{AS}_{p,q}\subset\mathsf{DC}_{p,q}$, the result of Proposition 5.1 applies also to analysis-suitable T-meshes. Therefore, the existence of dual functionals immediately implies a set of important properties for AS T-spline spaces, i.e. spaces of T-splines (5.2) generated by a T-mesh $\mathcal{M}\in\mathsf{AS}_{p,q}$. We list such properties in the following propositions and remarks. The derivations below follow with the same identical arguments used in [5] for the bi-cubic case. Therefore we do not include the (simple) proofs of the following propositions and refer the reader to [5].

The first result is the linear independence of set of functions appearing in (5.2), therefore forming a *basis*.

Proposition 5.2. *Given any T-mesh $\mathcal{M}\in\mathsf{AS}_{p,q}$ and knots Ξ_s,Ξ_t, the blending functions $\{B^A\}_{A\in\mathcal{A}_{p,q}(\mathcal{M})}$ are linearly independent. Therefore* (5.5) *constitutes a basis for $\mathtt{B}_{p,q}(\mathcal{M},\Xi_s,\Xi_t)$ and* (5.4) *is a corresponding dual basis.*

An important consequence of Proposition 5.1 is that we can build a projection operator $\Pi: L^2(]0,1[^2)\to\mathtt{B}_{p,q}(\mathcal{M},\Xi_s,\Xi_t)$, defined by

$$\Pi(f)(s,t)=\sum_{A\in\mathcal{A}_{p,q}(\mathcal{M})}\lambda^A_{p,q}(f)\,B^A_{p,q}(s,t)\qquad\forall f\in L^2(]0,1[^2),\ \forall(s,t)\in[0,1]^2. \quad (5.7)$$

Due to Proposition 5.1 it is immediate to check that Π is a projection operator. Such operator allows, for instance, to prove straightforwardly the partition of unity property under a very mild additional assumption on the T-spline space.

Lemma 5.1. *Let $\mathcal{M}\in\mathsf{AS}_{p,q}$ and let Ξ_s,Ξ_t be two given knot vectors. Assume that the constant function is in the space $\mathtt{B}_{p,q}(\mathcal{M},\Xi_s,\Xi_t)$. Then the basis functions $\{B^A\}_{A\in\mathcal{A}_{p,q}(\mathcal{M})}$ are a partition of unity.*

In addition, the existence of a dual basis grants a very powerful tool to prove approximation properties for AS T-spline spaces. Approximation properties are a fundamental condition for any discrete space to be used in the analysis of PDE problems; while the approximation properties of tensor product splines and NURBS is at a more advanced stage[20,2,6,7], there is almost no result for T-splines[4,5]. We here below limit ourselves to showing why the tool provided by Proposition 5.1 is so promising. We postpone to future publications a deeper analysis of the approximation properties of T-splines of arbitrary degree.

The following result will make use of the notion of extended patch $\widetilde{\mathtt{Q}}$ associated to a generic element $\mathtt{Q}$, reminding that $\mathtt{Q}^A$ denotes the open support of $B^A_{p,q}$:

$$\widetilde{\mathtt{Q}}=\bigcup_{A\in\mathcal{T}_{p,q}(\mathtt{Q})}\mathtt{Q}^A,\qquad \mathcal{T}_{p,q}(\mathtt{Q})=\{A\in\mathcal{A}_{p,q}(\mathcal{M})\text{ such that }\mathtt{Q}^A\cap\mathtt{Q}\neq\emptyset\}.$$

Furthermore, we will denote by $\mathtt{R}_{\widetilde{\mathtt{Q}}}$ the smallest rectangle in $[0,1]^2$ containing $\widetilde{\mathtt{Q}}$.

The following result holds.

Proposition 5.3. *Let $\mathcal{M} \in \mathsf{AS}_{p,q}$ and let Ξ_s, Ξ_t be two given knot vectors. Assume that the constant function belongs to the space $\mathtt{B}_{p,q}(\mathcal{M}, \Xi_s, \Xi_t)$. Then the projection operator Π is (locally) $h-$uniformly continuous in the L^2 norm. More precisely, it exists a constant C independent of $\mathcal{M}, \Xi_s, \Xi_t$ such that*

$$||\Pi(f)||_{L^2(\mathtt{Q})} \leq C||f||_{L^2(\widetilde{\mathtt{Q}})} \quad \forall \mathtt{Q} \in \mathtt{M}^{\mathrm{ext}},\ \forall f \in L^2(]0,1[^2).$$

Note that the constant C may depend on p and q.

The above continuity property implies for instance the following approximation result in the L^2 norm.

Proposition 5.4. *Let $\mathcal{M} \in \mathsf{AS}_{p,q}$ and let Ξ_s, Ξ_t be two given knot vectors. Assume that the space of global polynomials of degree $\min\{p,q\}$ living on $[0,1]^2$ are included in the T-spline space $\mathtt{B}_{p,q}(\mathcal{M}, \Xi_s, \Xi_t)$. Then it exists a constant C' independent of $\mathcal{M}, \Xi_s, \Xi_t$ such that for $r \in [0, \min\{p,q\}+1]$*

$$||f - \Pi(f)||_{L^2(\mathtt{Q})} \leq C'(h_{\mathtt{R}_{\widetilde{\mathtt{Q}}}})^r |f|_{H^r(\mathtt{R}_{\widetilde{\mathtt{Q}}})} \quad \forall \mathtt{Q} \in \mathtt{M}^{\mathrm{ext}},\ \forall f \in H^r(]0,1[^2).$$

where $H^r(]0,1[^2)$ indicates the Sobolev space of order r and $h_{\mathtt{R}_{\widetilde{\mathtt{Q}}}}$ represents the diameter of $\mathtt{R}_{\widetilde{\mathtt{Q}}}$. The constant C' may depend on p and q.

Note that the inclusion property for $\mathtt{B}_{p,q}(\mathcal{M}, \Xi_s, \Xi_t)$ in Proposition 5.4 is the minimal one required in order to obtain an $O(h^{\min\{p,q\}+1})$ convergence rate in the mesh size.

We conclude the present section with a final observation. For given degrees p and q, to each T-spline basis function $B^A_{p,q}$, with knot vectors $(s_{i_1}, s_{i_2}, \ldots, s_{i_{p+2}})$ and $(t_{j_1}, t_{j_2}, \ldots, t_{j_{q+2}})$, one can associate the Greville site $\mathbf{g}^A$ in $[0,1]^2$ of coordinates (s^A, t^A)

$$s^A = \frac{\sum_{k=2}^{p+1} s_{i_k}}{p}\ , \quad t^A = \frac{\sum_{k=2}^{q+1} t_{j_k}}{q}.$$

The interest in Greville sites is mainly related to interpolation with splines and to collocation methods. We defer the interested reader to e.g., [13, 1] and references therein. Clearly, a key condition for the well posedness of interpolation or collocation at Greville sites is that all points $\{\mathbf{g}^A\}_{A \in \mathcal{A}_{p,q}(\mathcal{M})}$ are distinct. Such result, that is obvious in the tensor product case, is not guaranteed for general T-spline spaces. Nevertheless, again due to the dual-compatibility property, we can derive the following proposition.

Proposition 5.5. *Let $\mathcal{M} \in \mathsf{AS}_{p,q}$ and let Ξ_s, Ξ_t be two given knot vectors with knot multiplicity equal or less than p and q, respectively. Then, all Greville sites $\{\mathbf{g}^A\}_{A \in \mathcal{A}_{p,q}(\mathcal{M})}$ are distinct.*

Proof. First, we remind that in one space dimension, Greville sites for a given degree p' and constructed from a common knot vector, are distinct as soon as knots have multiplicity at most p' times. Given two anchors A^1 and A^2 in $\mathcal{A}_{p,q}(\mathcal{M})$ such that $\mathsf{Q}^{A_1} \cap \mathsf{Q}^{A_2} \neq \emptyset$, then $A_1 \asymp A_2$ since $\mathcal{M} \in \mathsf{DC}_{p,q}$. If A^1 and A^2 overlap vertically then the ordinates of the corresponding Greville sites are different. If A^1 and A^2 overlap horizontally, then the abscissae of the corresponding Greville sites are different. $\square$

References

1. F. Auricchio, L. Beirão da Veiga, T.J.R. Hughes, A. Reali, and G. Sangalli. Isogeometric collocation methods. *Math. Models Methods Appl. Sci.*, 20(11):2075–2107, 2010.
2. Y. Bazilevs, L. Beirão da Veiga, J. A. Cottrell, T. J. R. Hughes, and G. Sangalli. Isogeometric analysis: approximation, stability and error estimates for h-refined meshes. *Math. Models Methods Appl. Sci.*, 16(7):1031–1090, 2006.
3. Y. Bazilevs, V.M. Calo, J.A. Cottrell, J.A. Evans, T.J.R. Hughes, S. Lipton, M.A. Scott, and T.W. Sederberg. Isogeometric analysis using T-splines. *Comput. Methods Appl. Mech. Engrg.*, 199(5-8):229 – 263, 2010.
4. L. Beirão da Veiga, A. Buffa, D. Cho, and G. Sangalli. IsoGeometric analysis using T-splines on two-patch geometries. *Comput. Methods Appl. Mech. Engrg.*, 200(21-22):1787–1803, 2011.
5. L. Beirão da Veiga, A. Buffa, D. Cho, and G. Sangalli. Analysis-Suitable T-splines are Dual-Compatible. *To appear in Comput. Methods Appl. Mech. Engrg.*, 2012.
6. L. Beirão da Veiga, A. Buffa, J. Rivas, and G. Sangalli. Some estimates for h-p-k-refinement in isogeometric analysis. *Numer. Math.*, 118(2):271–305, 2011.
7. L. Beirão da Veiga, D. Cho, and G. Sangalli. Anisotropic NURBS approximation in Isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 209-2011:1–11, 2012.
8. A. Buffa, D. Cho, and G. Sangalli. Linear independence of the T-spline blending functions associated with some particular T-meshes. *Comput. Methods Appl. Mech. Engrg.*, 199(23–24):1437–1445, 2010.
9. A. Buffa, C. de Falco, and G. Sangalli. Isogeometric Analysis: stable elements for the 2D Stokes equation. *Internat. J. Numer. Methods Fluids*, 65(11-12):1407–1422, 2011.
10. A. Buffa, J. Rivas, G. Sangalli, and R. Vázquez. Isogeometric discrete differential forms in three dimensions. *SIAM J. Numer. Anal.*, 49(2):818–844, 2011.
11. J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs. *Isogeometric Analysis: toward integration of CAD and FEA*. John Wiley & Sons, 2009.
12. J.A. Cottrell, T.J.R. Hughes, and A. Reali. Studies of refinement and continuity in isogeometric structural analysis. *Comput. Methods Appl. Mech. Engrg.*, 196:4160–4183, 2007.
13. C. de Boor. *A practical guide to splines*, volume 27 of *Applied Mathematical Sciences*. Springer-Verlag, New York, revised edition, 2001.
14. M.R. Dörfel, , B. Jüttler, and B. Simeon. Adaptive isogeometric analysis by local h-refinement with T-splines. *Comput. Methods Appl. Mech. Engrg.*, 199(5-8):264 – 275, 2010.
15. J.A. Evans, Y. Bazilevs, I. Babuška, and T.J.R. Hughes. n-width, sup-infs, and optimality ratios for the k-version of the isogeometic finite element method. *Comput. Methods Appl. Mech. Engrg.*, 198:1726–1741, 2009.
16. G.T. Finnigan. Arbitrary degree T-splines. Master's thesis, Brigham Young University, 2008.
17. T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, 194(39-41):4135–4195, 2005.

18. X. Li and M.A. Scott. On the nesting behavior of T-splines. Technical Report 11-13, ICES, 2011.
19. X. Li, J. Zheng, T.W. Sederberg, T.J.R. Hughes, and M.A. Scott. On linear independence of T-spline blending functions. *Comput. Aided Geom. Design*, 29(1):63 – 76, 2012.
20. Larry L. Schumaker. *Spline functions: basic theory*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2007.
21. M.A. Scott, M.J. Borden, C.V. Verhoosel, T.W. Sederberg, and T.J.R. Hughes. Isogeometric finite element data structures based on Bézier extraction of T-splines. *Internat. J. Numer. Methods Engrg.*, 88(2):126–156, 2011.
22. M.A. Scott, X. Li, T.W. Sederberg, and T.J.R. Hughes. Local refinement of analysis-suitable T-splines. *Comput. Methods Appl. Mech. Engrg.*, 213 - 216:206 – 222, 2012.
23. T.W. Sederberg, D.L. Cardon, G.T. Finnigan, N.S. North, J. Zheng, and T. Lyche. T-spline simplication and local refinement. *ACM Trans. Graph.*, 23(3):276–283, 2004.
24. T.W. Sederberg, J. Zheng, A. Bakenov, and A. Nasri. T-splines and T-NURCCSs. *ACM Trans. Graph.*, 22(3):477–484, 2003.
25. W. Wang, Y. Zhang, M.A. Scott, and T.J.R. Hughes. Converting an unstructured quadrilateral mesh to a standard T-spline surface. *Comput. Mech.*, 48(4):477–498, 2011.
26. Y. Zhang, W. Wang, and T.J.R. Hughes. Solid T-spline construction from boundary representations for genus-zero geometry. *Comput. Methods Appl. Mech. Engrg.*, (0):–, 2012.