# Second lecture
# Splines spaces over box partitions

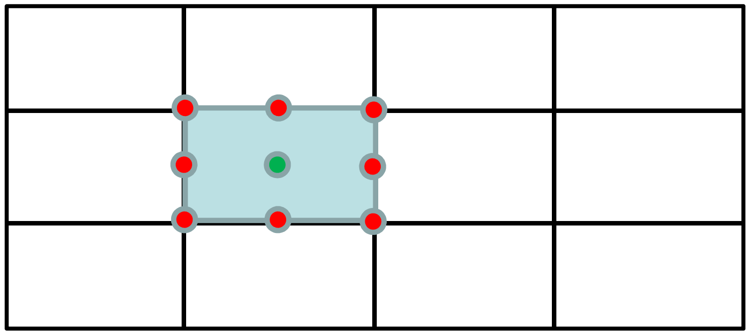
Block structured grids, local refinement and spline bases

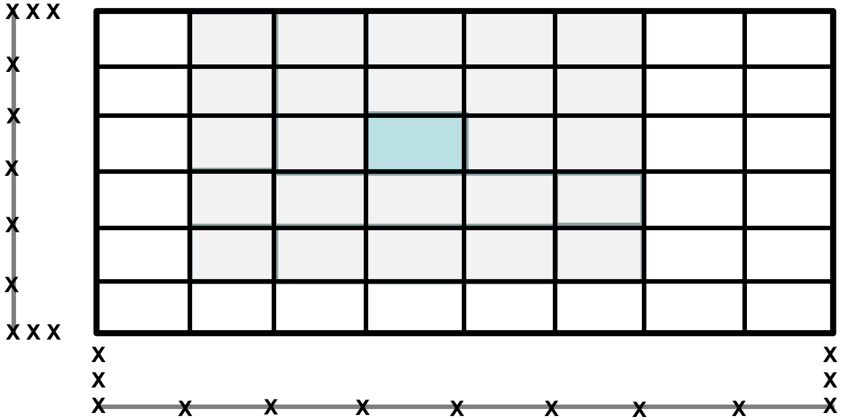
Tor Dokken

# Isogeometric Analysis challenge traditional FEA approaches to block structured grids.

- Block structured FEA
  - Composed of coupled blocks of meshes with a regular structure
- Spline function represented by tensor product B-splines
  - The B-splines space defined by univariate knot vectors $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ one in each parameter direction and the polynomial degrees $p_1, p_2$. A regular structure of not lines
  - Continuity defined by multiplicity of values
  - A tensor product B-spline not local to an element
- Higher order continuity a challenge when blocks meet in an extraordinary point
  - Simple when four surfaces meet
  - Challenge independent of T-splines, LR B-splines….

# Bi-quadratic example





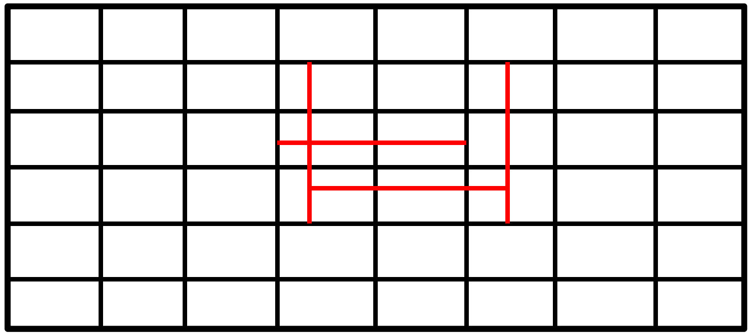

## Bi-quadratic finite element

- Shape functions local to each element
- $C^0$- continuity by matching edge nodes
- $C^1$- continuity imposes many relations between nodes

## Bi-quadratic B-spline

- Shape functions (B-splines) not local to a single element
- Continuity embedded in the B-spline definition through the knot vectors
- $C^1$- continuity straight forward

# Local refinement of a block

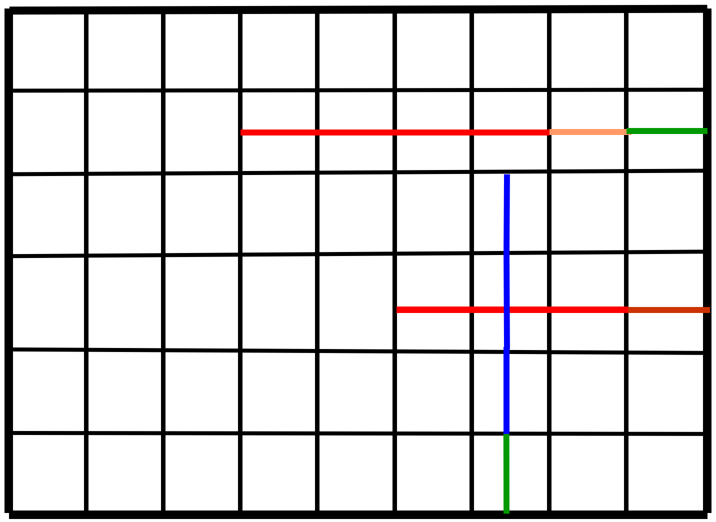


- We want to insert new local line segment in the grid that result in new elements (and degrees of freedom) located where they are needed.
- Questions:
  - What is the dimension of the resulting spline space?
  - How to find a suitable basis?
  - Which restrictions has to be imposed on the refinement?
- The questions are independent of the approach is T-splines Locally Refined B-splines, PHT-splines or Hierarchical B-splines.

# Dimension increase for local refinement in the 2-variate case well understood.

Example dimension increase of spline space when

- Bi-cubic $p = 3$.
- $C^2$-continuity across interior mesh lines



Mesh line segment length 1 starting at the boundary, T-joint at other end. Dimension increase 1.

Mesh line segment length 1 extending existing mesh line segment of length 4 or more, T-joint at other end. Dimension increase 1.

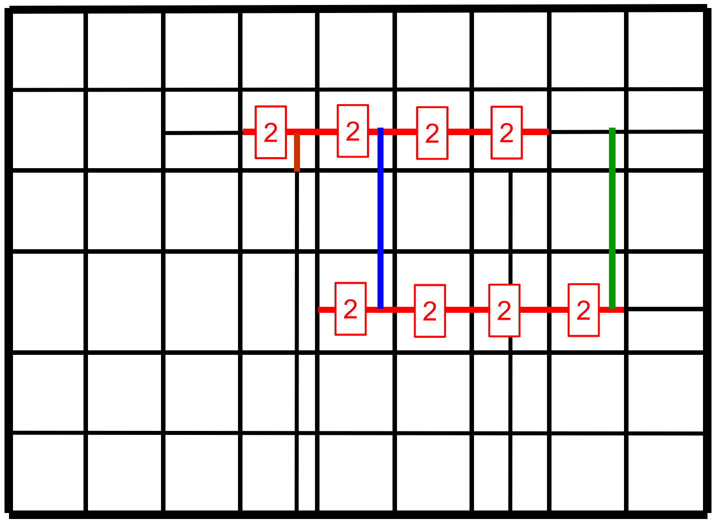
Interior mesh line segment length 4, T-joints at both ends. Dimension increase 1.

Mesh line segment length 1 gap filling. Dimension increase 4.

Mesh line segment length 1 extension of existing mesh line segment to the boundary. Dimension increase 4

# Increasing interior multiplicity in the bi-cubic case

- When multiplicity is 2 we have $C^1$-continuity across interior mesh lines in the bi-cubic case.
- The edges on the boundary have multiplicity 4.



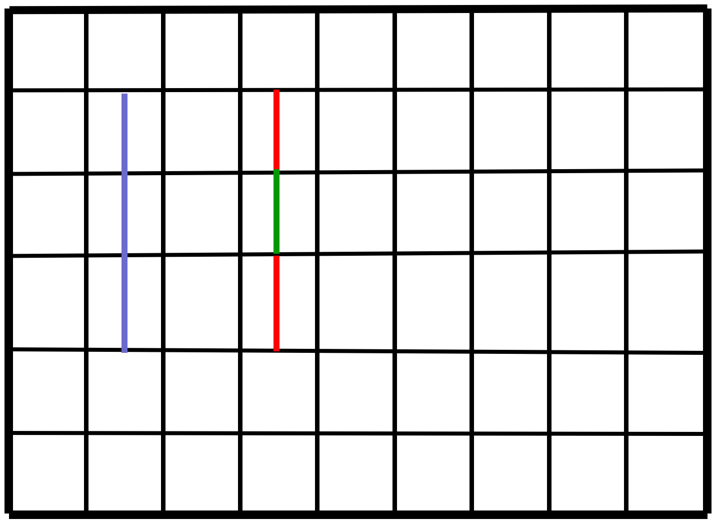

Interior mesh line segment length 4, increase multiplicity to 2, lower multiplicity at both ends, dimension increase 1.

Interior mesh line segment length 3, ending in T-joints with orthogonal mesh line segment , one with multiplicity 1,and one with multiplicity 2, dimension increase 1.

Interior mesh line segment length 3, ending in T-joints with orthogonal mesh line segment of multiplicity 2, dimension increase 2.

Extend existing mesh line segment by length 1, ending in T-joint with orthogonal mesh line segment with multiplicity 2, dimension increase 2,

# Short lines do not give dimension increase bi-cubic example



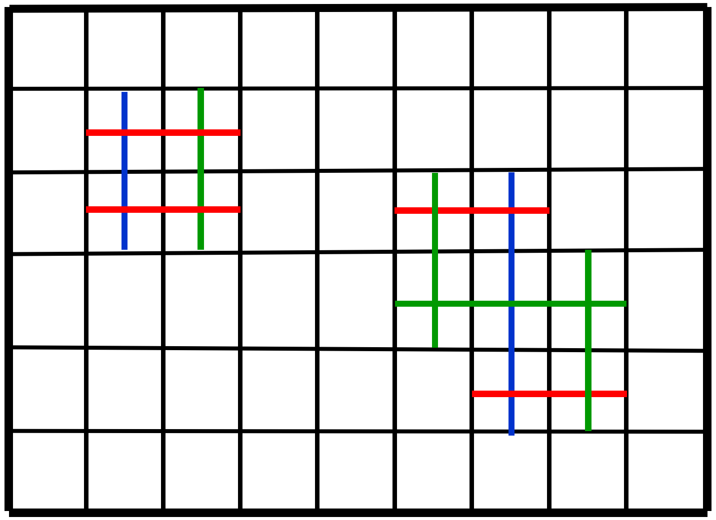
Interior mesh line segment length 3 (or shorter), T-joints at both ends. No dimension increase.

Interior mesh line segment length 3 (or shorter), T-joints at both ends. No dimension increase.

Gap filling mesh line segment length 1, resulting mesh line segment less than 4 (or shorter), T-joints at both ends. No dimension increase.

- Mesh line segment ending in T-joints has to have length $(p + 1)$ to give dimension increase by 1.

# Using short lines to localize refinement bi-cubic example



Interior mesh line segment length 2, T-joints at both ends. No dimension increase.

Interior mesh line segment length 3, T-joints at both ends. No dimension increase.

Interior mesh line segment length 4, T-joints at both ends. Dimension increase 1.

- Note: we are only addressing refinement of the spline space. Currently no relation to T-spline, LR B-splines, PHT-splines, Hierarchical B-splines etc.

# Bernard Mourrain: *On the dimension of spline spaces on planer T-subdivisions*

- A T-subdivision is a 2-variate specialization of what we will denote an $\mu$-extended box-mesh.
- Dimension of space of bivariate functions, piecewise polynomial of bidegree $(p_1, p_2)$ and class $C^{r_1,r_2}$ over a planar T-subdivision:

$$\dim \mathcal{S}_{p_1,p_2}^{r_1,r_2} = (p_1+1)(p_2+1)f_2 - (p_1+1)(r_2+1)f_1^v \\ -(p_2+1)(r_1+1)f_1^h + (r_1+1)(r_2+1)f_0 + h_{p_1,p_2}^{r_1,r_2}$$

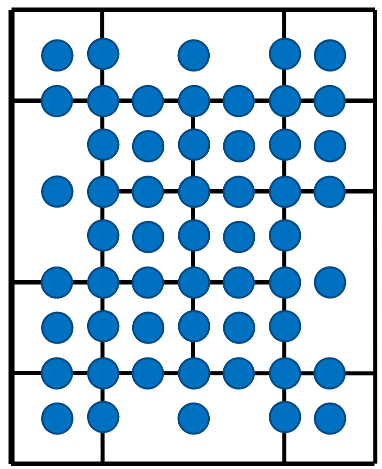
- $f_2$, $f_1^v$, $f_1^h$, $f_0$ are respectively the number of 2-faces(elements), horisontal interior edges, vertical interior edges and interior vertices
- $h_{p_1,p_2}^{r_1,r_2}$ is a homology term that is zero provided all sequences of interior edges that are not attached to the boundary resepectively consists of at least $p_1$ or $p_2$ consequtive edges.

# Dimension formula, example

- Dimension of T-mesh space, $(p_1 = p_2 = 3,\ r_1 = r_2 = 2)$:

$$\dim \mathcal{S}_{p_1,p_2}^{r_1,r_2} = (p_1+1)(p_2+1)f_2 - (p_1+1)(r_2+1)f_1^v - (p_2+1)(r_1+1)f_1^h + (r_1+1)(r_2+1)f_0$$

- $f_2$ = number of 2-faces
- $f_1^h$ = number of interior horisontal edges
- $f_1^v$ = number of interior vertical edges
- $f_0$ = number of interior vertices



$f_0$=12

$$\dim \mathcal{S}_{3,3}^{2,2} = 4\times 4\times 16 - 4\times 3\times 13 - 4\times 3\times 14 + 3\times 3\times 12 = 40$$

# Find a piecewise polynomial basis for the locally refined block mesh (1)

- Using Mourrain's dimension formula when it applicable or the more general formula to be presented soon enables in the cases where the homology terms are zero to calculate the spline space over the extended box-mesh.
- We now want to find a basis that span this box splines.
- Some candidate approaches for bases:
  - Hierarchical B-splines
  - PHT-splines
  - T-splines
  - LR B-splines

# Find a piecewise polynomial basis for the locally refined block mesh (2)

- **Hierarchical B-splines** have nested levels of refinement, with a nested selection of regions to use at each level.
  - The spline spaced by the extended box-mesh produced by merging the trimmed knotlines of all levels (the element structure) is not spanned by the Hierarchical B-spline basis.
- Hierarchical B-splines do not offer a basis than spans the spline space of the extended box-mesh.
- **PHT-splines** are special basis produced for specific degrees and continuities for very general box-meshes.
  - There exists, e.g., a $C^{1,1}$ bi-cubic PHT-spline basis.
  - PHT-splines do not, as far as I know, offer a general solution.

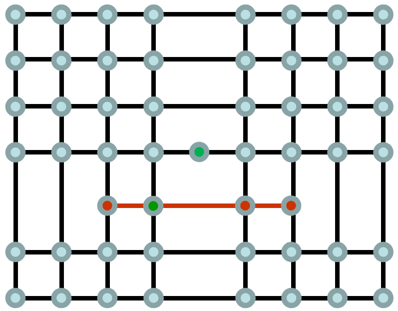
# Find a piecewise polynomial basis for the locally refined block mesh (3)

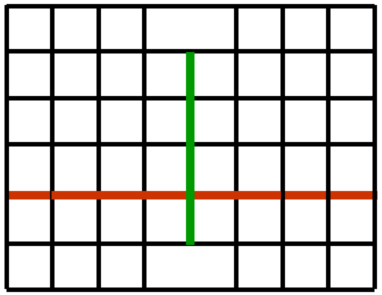
- **T-splines**
  - Refinement defined in the T-spline vertex T-grid following the T-spline rules, recently new extension.
  - Box-mesh (Bezier surfaces) inferred from T-grid
  - Rational scaling of B-splines to form partition of unity
- **LR B-splines**
  - Refinement directly on B-splines in box-mesh, mesh-lines projected on to the surface provides a T-grid structure
  - More choices of refinements than T-splines
  - Scaled B-splines to form partition of unity
- Same challenges of both methods to ensure:
  - Spanning of spline space defined by the Box-mesh
  - Linear independence of the B-splines

# Direct modeling by vertices

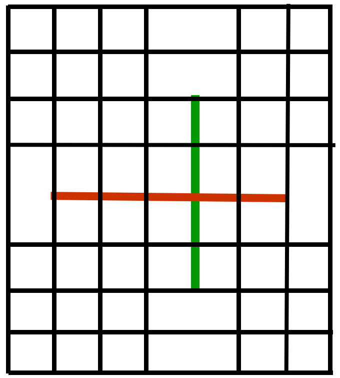
- Both T-splines an LR B-splines have bases that are a partition of unity.
  - Consequently the coefficient/vertices have a geometric interpretation, and the surfaces can be directly manipulated by these.
- The refinement allowed for T-splines is restricted by the requirement that the vertex grid of T-splines, T- joint, recently extensions by L-joints and I-joints.
  - No such structure is imposed on the coefficients of the general LR B-splines. The only restriction, at least one B-spline has to be refined.
  - Direct modeling by vertices is difficult, so in CAD the location of vertices is most often calculated by algorithms.

# The refinement approaches of T-spline and LR-spline, bi-cubic example

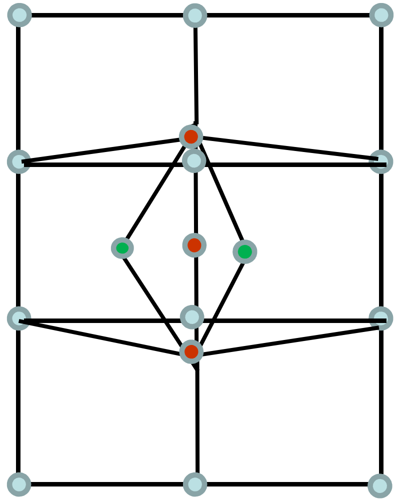




Parameter domain,
Polynomial segments



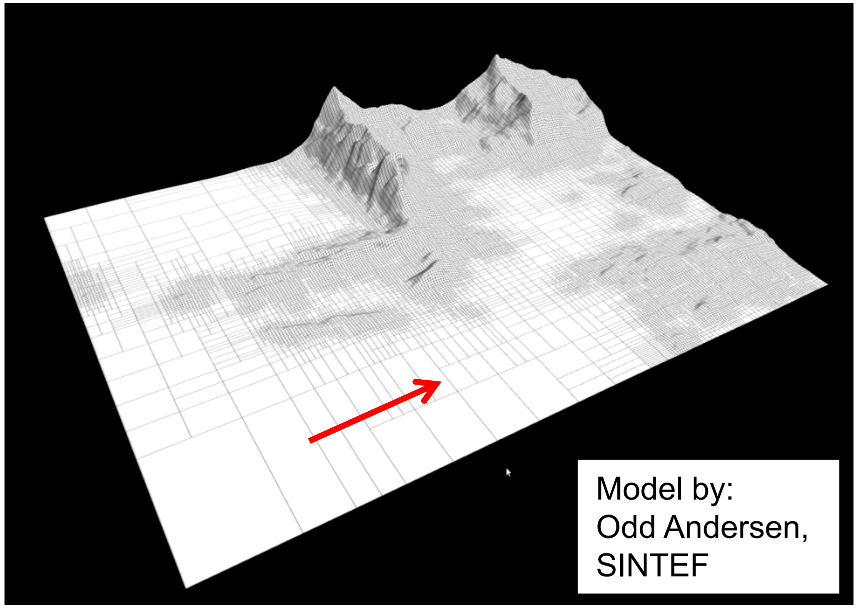
Parameter domain

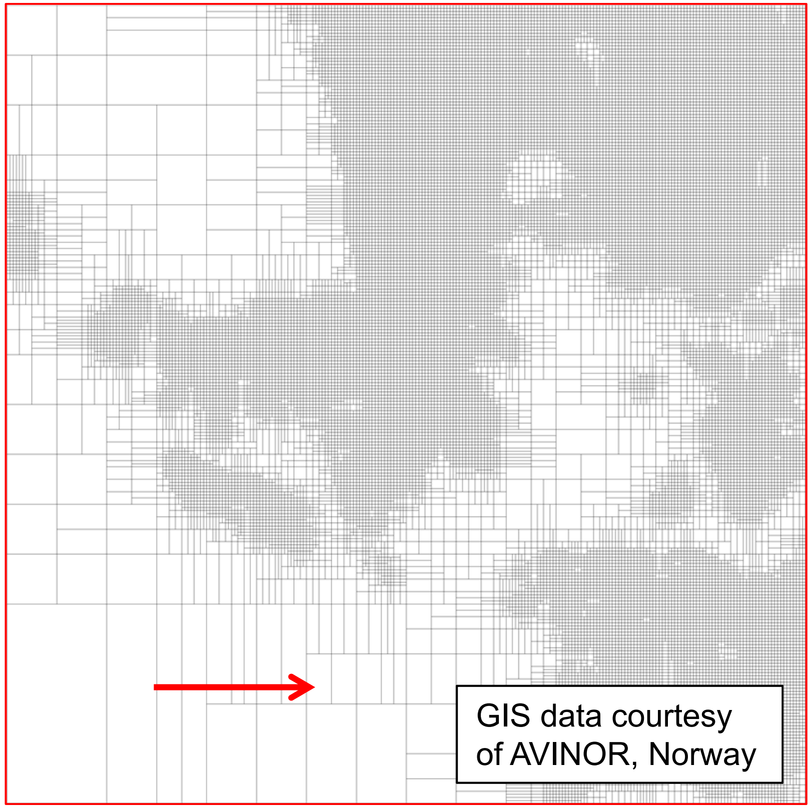


- Adding two close vertices by T-spline refinement creates 11 new polynomial segments and 5 vertices (original T-spline rules)

- Adding a minimal “+” structure by LR B-splines creates 2 vertices and 8 new polynomial segments
- Position of vertices in parameter domain average of internal knots

# Local refinement for LR B-splines

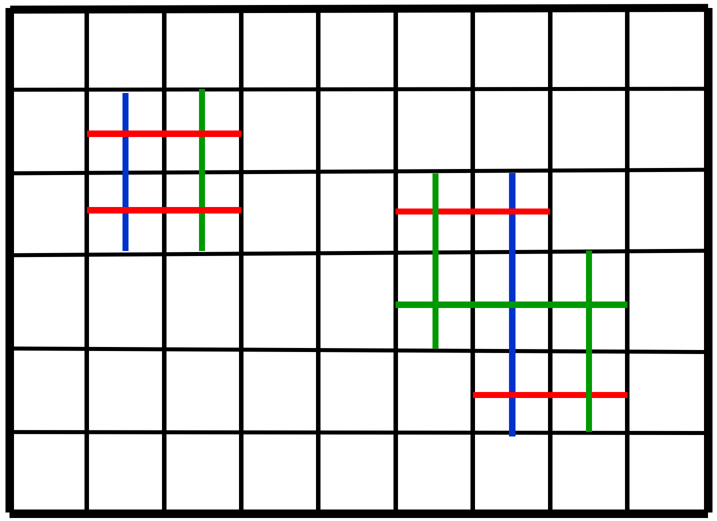



- Refine in mesh project on 3D surface
- Refine in 2D mesh in parameter domain
- Vector specifiying refinement knotline

- Automatic checking:
  - Spline space filled?
  - LR B-spline basis exists?
  - Automatic corrections possible

# Support of short lines to localize refinement



- In the $C^{1,1}$ bi-cubic case possibly supported by PHT-splines.
- Currently not supported by T-splines or LR B-splines
  - Challenge: How to extend LR B-spline theory to allow minimal support B-splines in the mesh that are not a result of B-spline refinement.

# Summary

| Method: | Use B-splines | Fill the spline space of the mesh | Guaranteed basis | Geometric approach to refinement |
|---|---|---|---|---|
| PHT-splines | No | For specific continuities | For specific continuities | In parameter domain (mapped on surface) |
| Hierarchal B-splines | Yes | No | Yes | Nested regions |
| T-splines | Yes | Often | If analysis suitable | In vertex T-grid |
| LR B-splines | Yes | Often | Tools exist ensue that a basis is made | In parameter domain (mapped on surface) |

The theory of LR B-splines applicable to T-splines