# Lecture 3
# Box-partitions and dimension of spline spaces over Box-partition

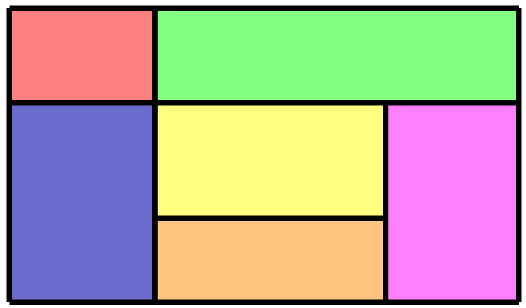
Definition of LR B-splines
some geometric properties

Tor Dokken

# Box-partitions

- Box-partitions - Rectangular subdivision of regular domain $d$-box $\mathbb{R}^d$

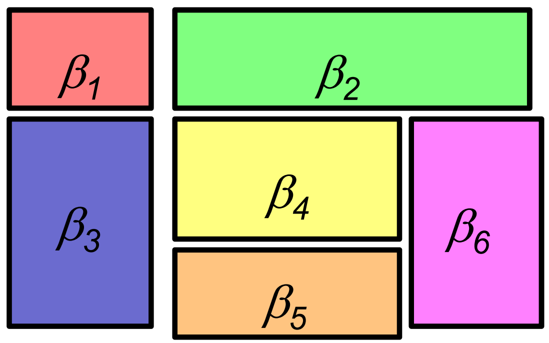
$$\Omega \subseteq \mathbb{R}^d$$
$$\Omega = [a_1, b_1] \times \cdots \times [a_d, b_d]$$
$$a_i < b_i,\ 1 \leq i \leq d$$



$\Omega \subseteq \mathbb{R}^2$

- Subdivision of $\Omega$ into smaller $d$ -boxes

$$\mathcal{E} = \{\beta_1, \ldots, \beta_1\}$$
$$\beta_1 \cup \beta_2 \cup \cdots \cup \beta_n = \Omega$$
$$\beta_i^o \cap \beta_j^o = \emptyset,\ i \neq j$$




$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$

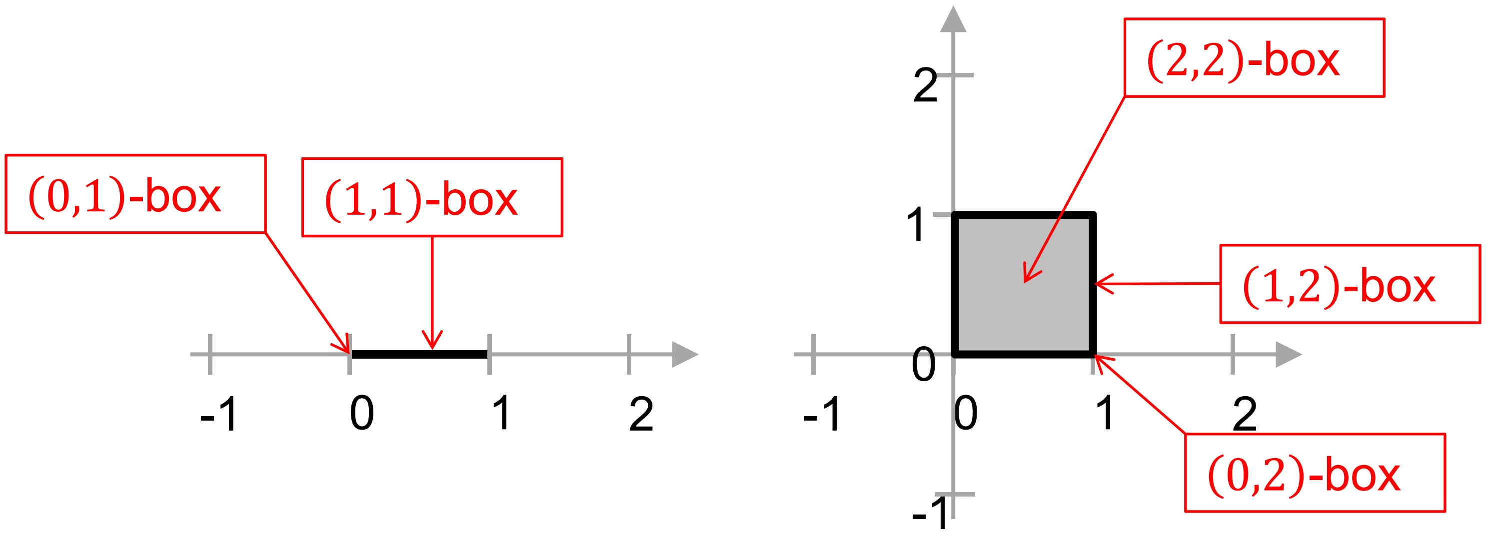
# A box in $\mathbb{R}^d$

Given an integer $d \geq 0$. A box in $\mathbb{R}^d$ is a Cartesian product

$$\beta = J_1 \times \cdots \times J_d \subseteq \mathbb{R}^d,$$

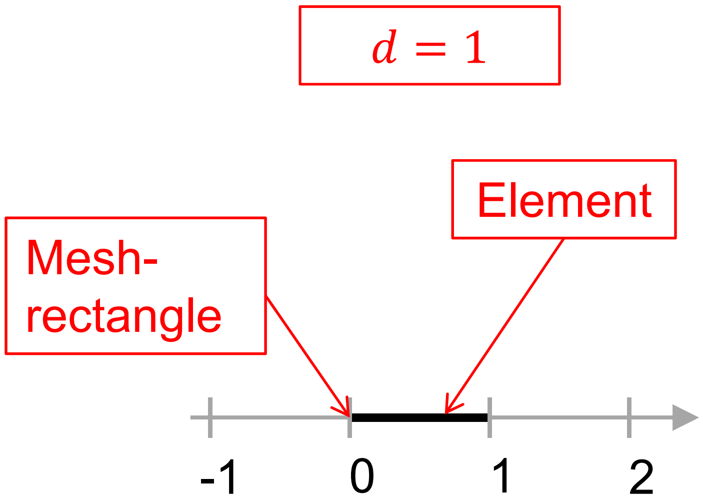
where each $J_k = [a_k, b_k]$ with $a_k \leq b_k$ is a closed finite interval in $\mathbb{R}^d$. We also write $\beta = [\boldsymbol{a}, \boldsymbol{b}]$, where $\boldsymbol{a} = [a_1, \dots, a_d]$, and $\boldsymbol{b} = [b_1, \dots, b_d]$.

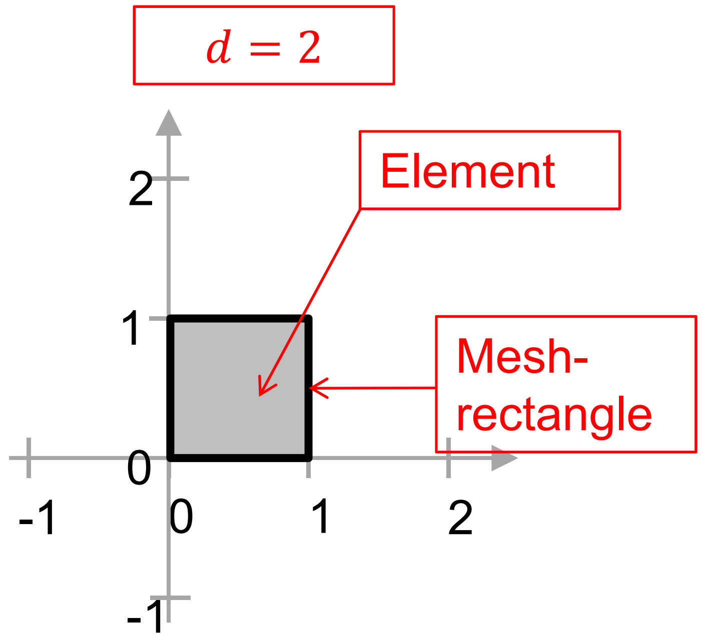
- The interval $J_k$ is said to be **trivial** if $a_k = b_k$ and **non-trivial** otherwise.
- The **dimension** of $\beta$, denoted $\dim \beta$, is the number of non-trivial intervals $J_k$ in $\beta$.
  - We call $\beta$ an $l$-box or and $(l, d)$-box if $\dim \beta = l$.



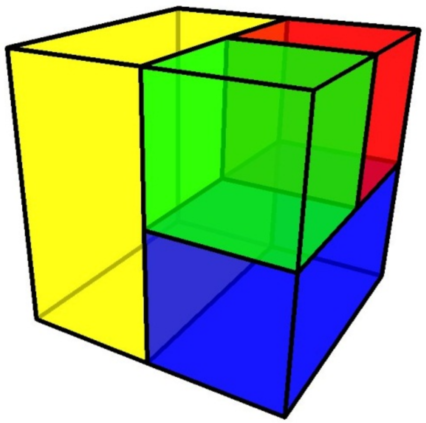
# Important boxes

- If $\dim\beta = d$ then $\beta$ is called an **element**.
- If $\dim\beta = d - 1$ there exists exactly one $k$ such that $J_k = [a]$ is trivial. Then $\beta$ is called a **mesh-rectangle**, a $k$-**mesh-rectangle** or a $(k, a)$-**mesh-rectangle**

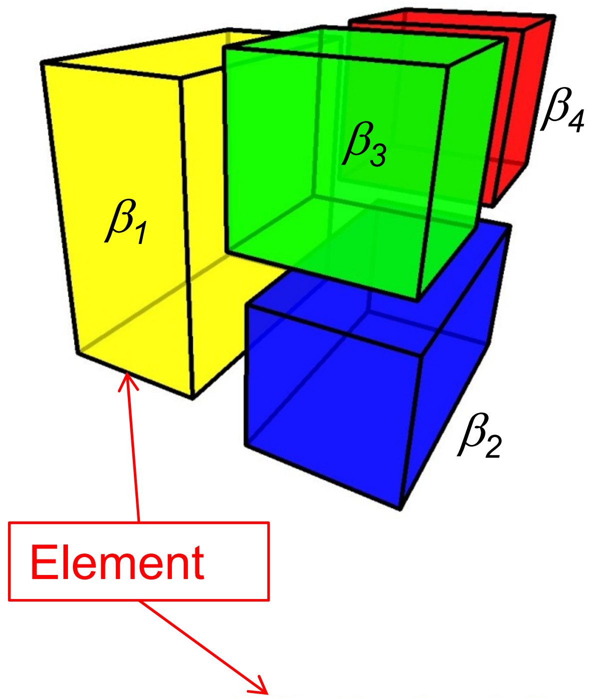


# Example: 3D-mesh



$\Omega \subseteq \mathbb{R}^3$




$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$

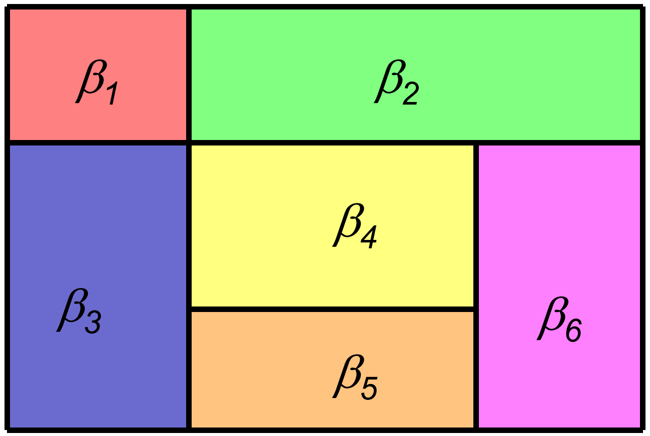
Illustration by: Kjell Fredrik Pettersen, SINTEF

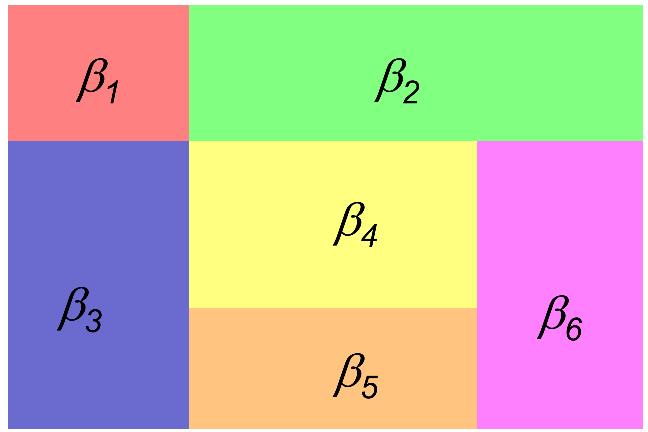
# Lower-dimensional boxes in the mesh

- $\mathcal{F}_l(\mathcal{E})$ is the set of $l$ -boxes describing the mesh topology, $0 \leq l \leq d$
- $\mathcal{F}_d(\mathcal{E})$ is the same as $\mathcal{E}$
- For $l < d$: $\mathcal{F}_l(\mathcal{E})$ is the set of $l$-boxes where higher-dimensional boxes in $\mathcal{E}$ intersect, or at boundary of $\Omega$

# Example: 2D-mesh

dim = 2 (Elements)



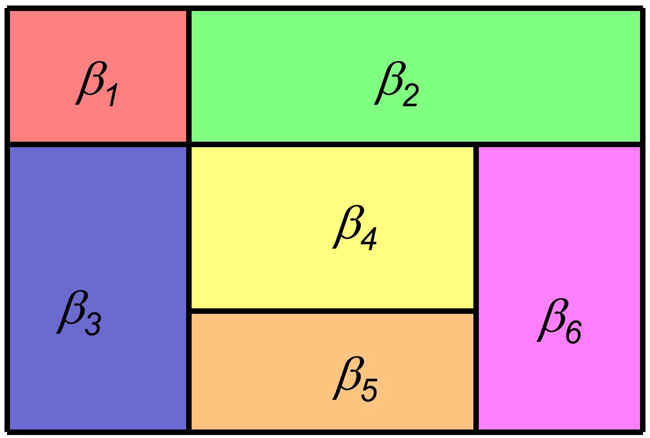




$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$$

$$\mathcal{F}_2(\mathcal{E}) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\} = \mathcal{E}$$

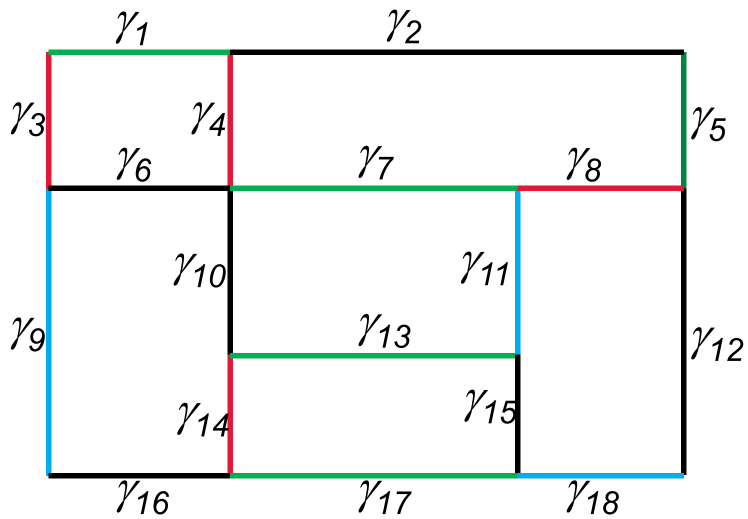
Illustration by: Kjell Fredrik Pettersen, SINTEF

# Example: 2D-mesh




$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$$

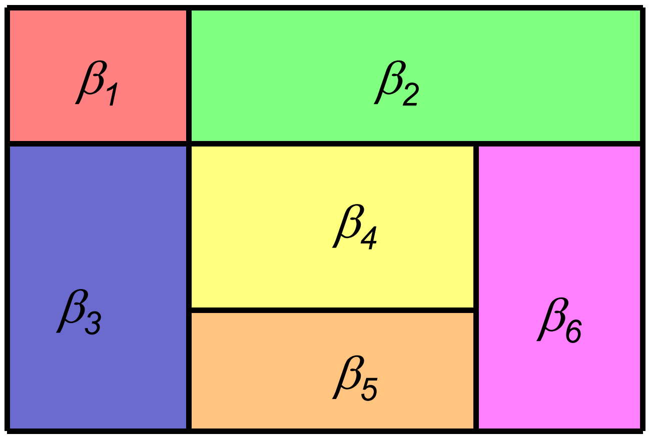
dim = 1 (Mesh-rectangles)




$$\mathcal{F}_1(\mathcal{E}) = \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{18}\}$$

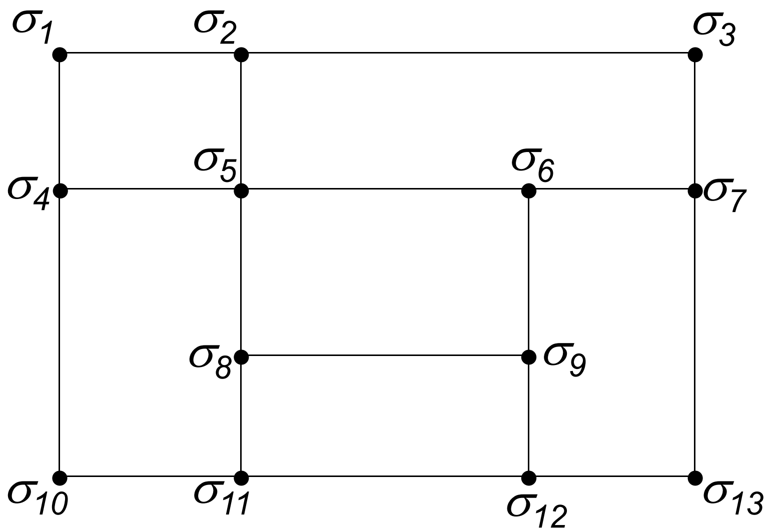
Illustration by: Kjell Fredrik Pettersen, SINTEF

# Example: 2D-mesh




$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$$

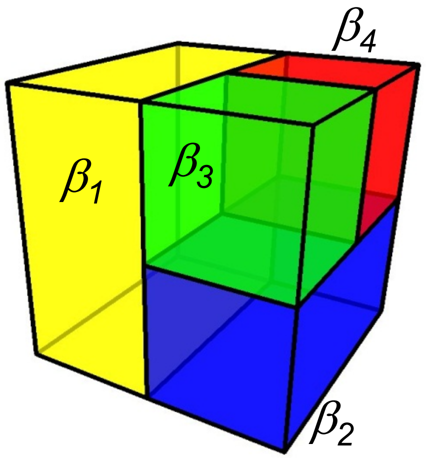
dim = 0




$$\mathcal{F}_0(\mathcal{E}) = \{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{13}\}$$

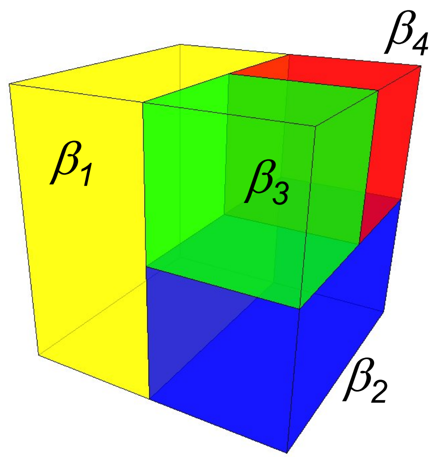
Illustration by: Kjell Fredrik Pettersen, SINTEF

# Example: 3D-mesh




$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

dim = 3 (Elements)

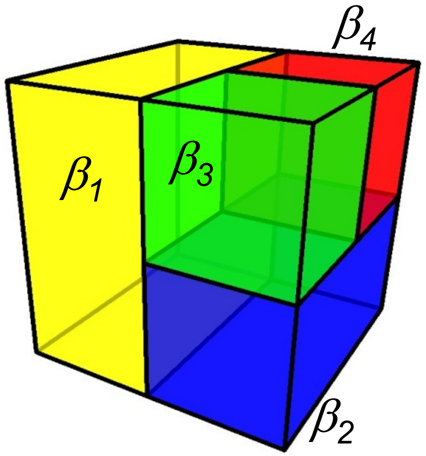



$$\mathcal{F}_3(\mathcal{E}) = \{\beta_1, \beta_2, \beta_3, \beta_4\} = \mathcal{E}$$

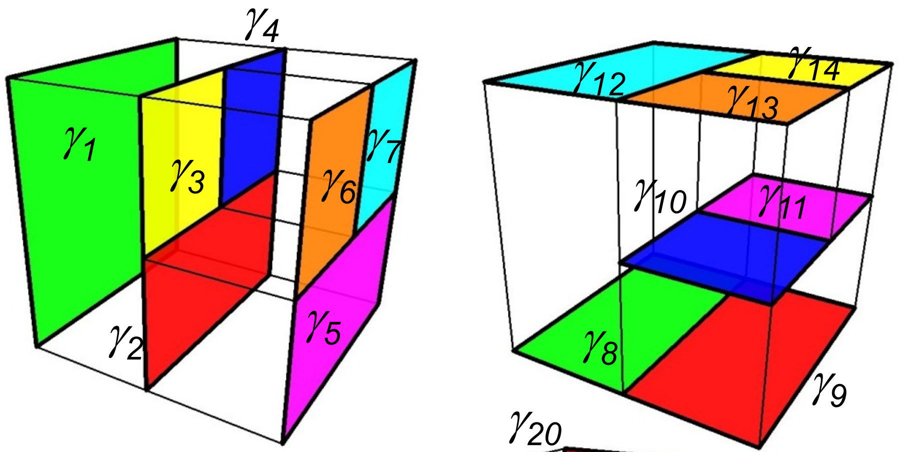
Illustration by: Kjell Fredrik Pettersen, SINTEF

# Example: 3D-mesh

dim = 2 (Mesh-rectangles)




$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$




$$\mathcal{F}_2(\mathcal{E}) = \{\gamma_1, \ldots, \gamma_{21}\}$$

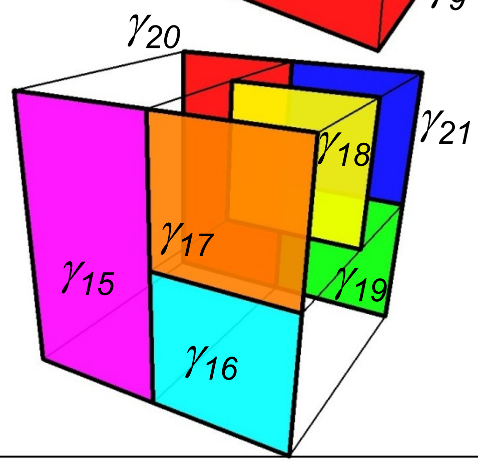
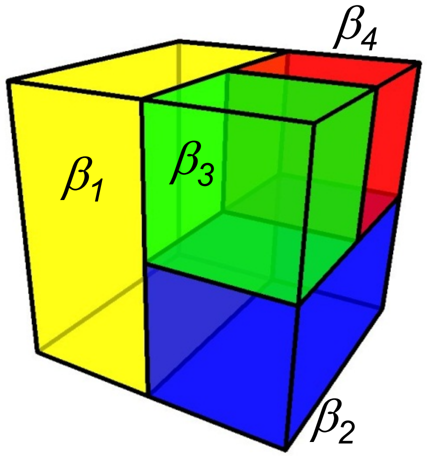


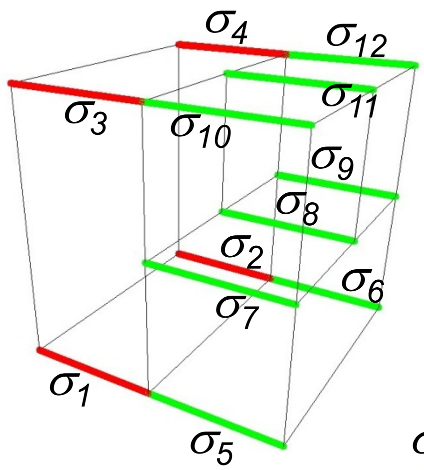

Illustration by: Kjell Fredrik Pettersen, SINTEF

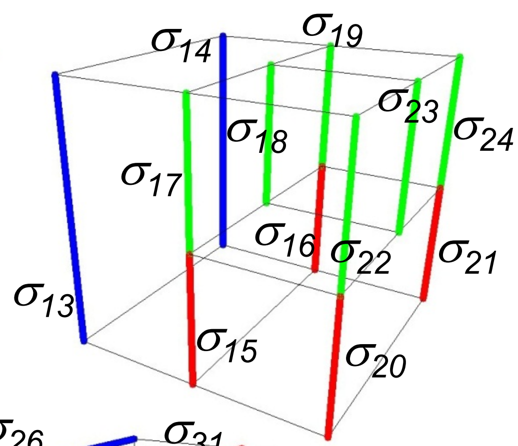
# Example: 3D-mesh




$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

dim = 1




$$\mathcal{F}_1(\mathcal{E}) = \{\sigma_1, \ldots, \sigma_{36}\}$$

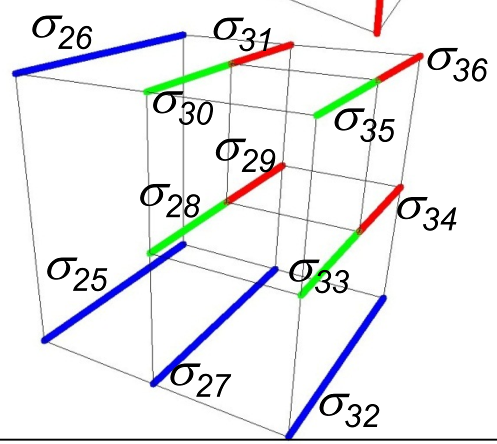
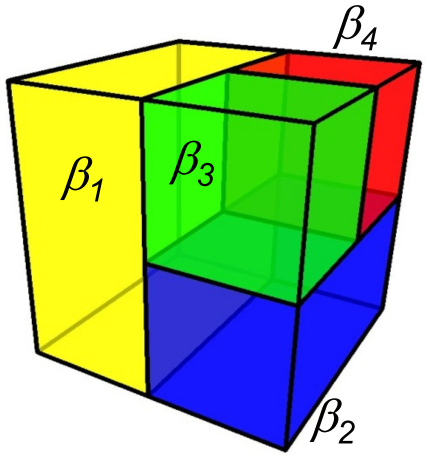


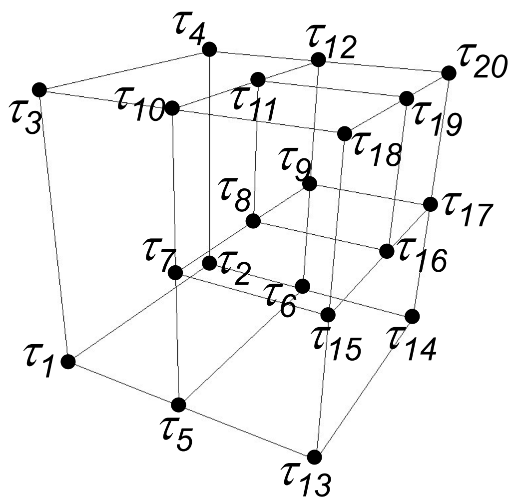

Illustration by: Kjell Fredrik Pettersen, SINTEF

# Example: 3D-mesh




$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$

dim = 0

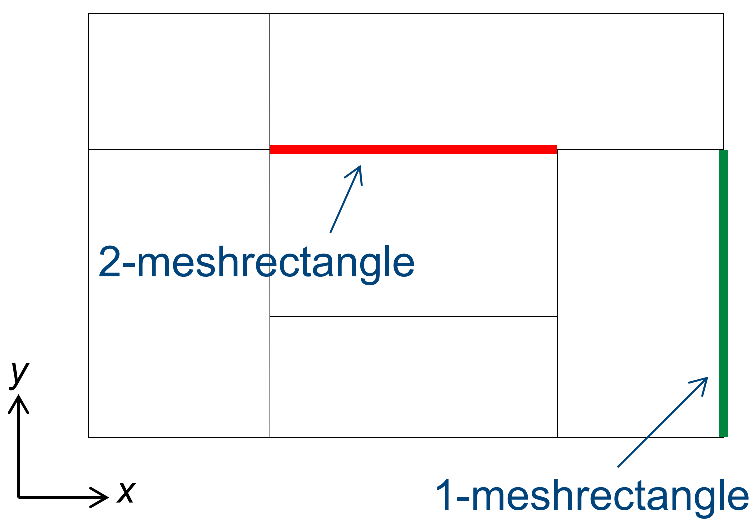



$\mathcal{F}_0(\mathcal{E}) = \{\tau_1, \ldots, \tau_{20}\}$

Illustration by: Kjell Fredrik Pettersen, SINTEF

# Mesh-rectangles

Example, 2D




Example, 3D

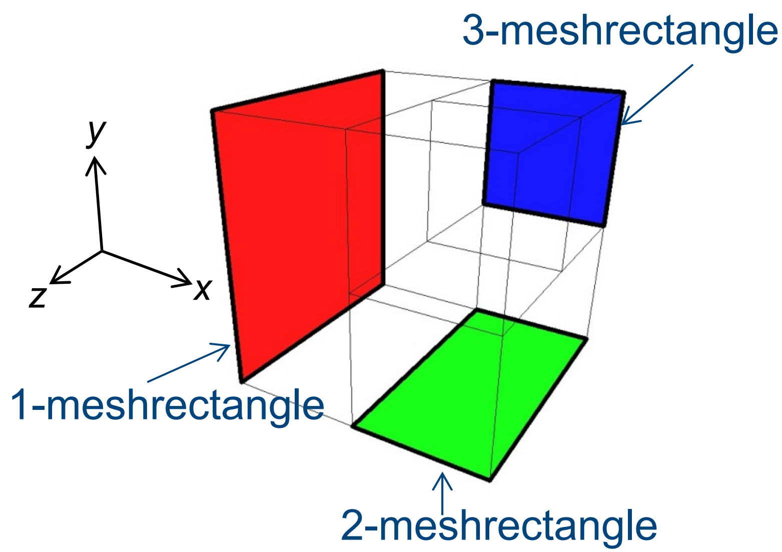
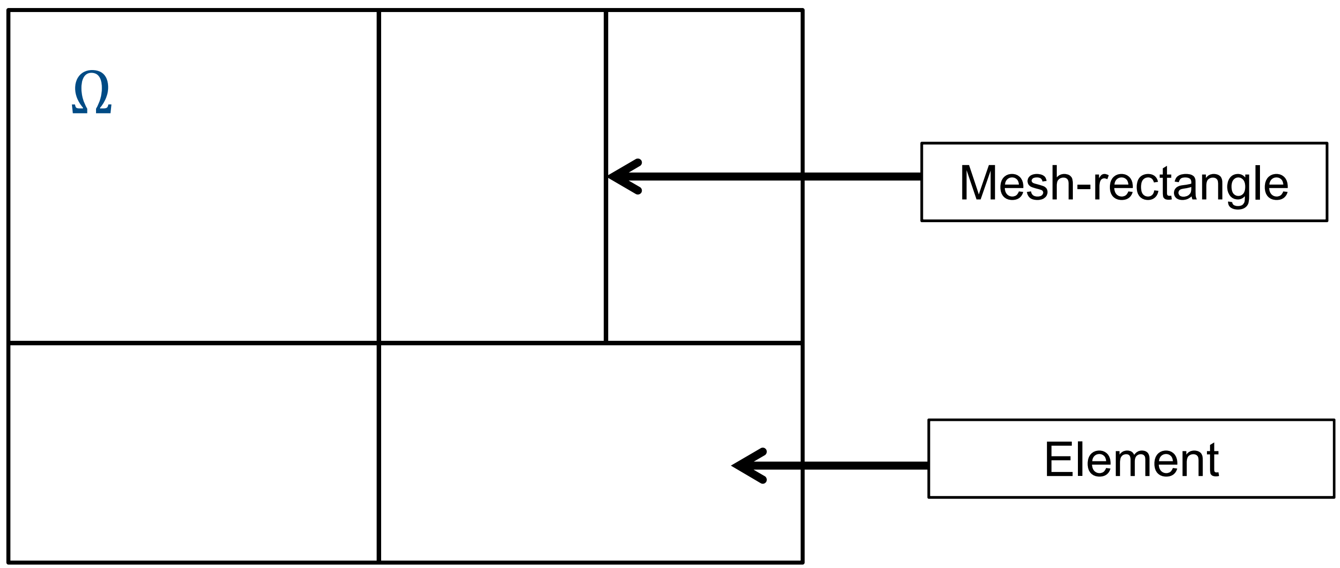


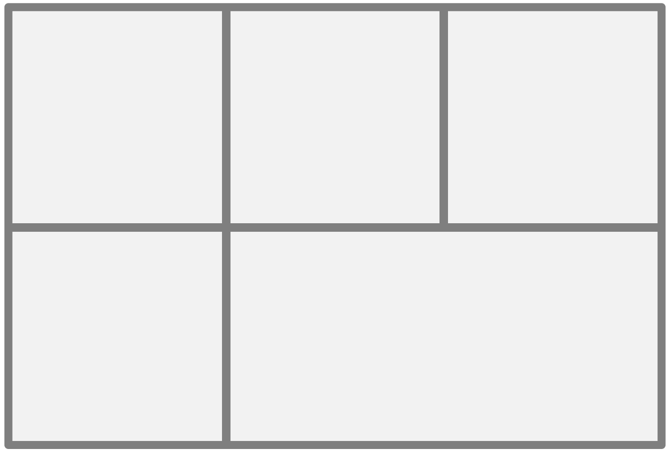

Illustration by: Kjell Fredrik Pettersen, SINTEF

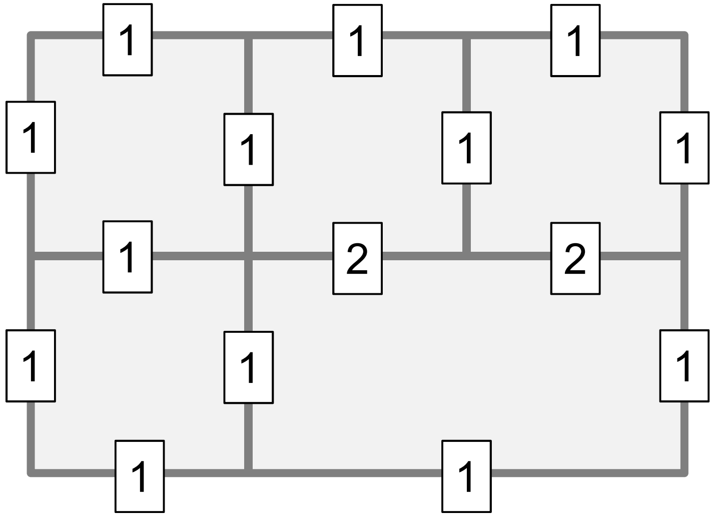
# Box-partition

- $\Omega \subseteq \mathbb{R}^d$ a $d$-box in $\mathbb{R}^d$.
- A finite collection $\mathcal{E}$ of $d$-boxes in $\mathbb{R}^d$ is said to be a **box partition** of $\Omega$ if
  1. $\beta_1^o \cap \beta_2^o = \emptyset$ for any $\beta_1^o, \beta_2^o \in \mathcal{E}$, where $\beta_1^o \neq \beta_2^o$.
  2. $\bigcup_{\beta \in \mathcal{E}} \beta = \Omega$.



# $\mu$-extended box-mesh (adding multiplicities)






- A multiplicity $\mu$ is assigned to each mesh-rectangle
- Supports variable knot multiplicity for Locally Refined B-splines, and local lower order continuity across mesh-rectangles.

# Comment

- When our work started out we used the index grid for knots.
  - However, this posed challenges with respect to mesh-rectangles of multiplicity higher than one and the uniqueness of refinement.
  - The generalized dimension formula uses multiplicity, thus to ensure
  - Not straight forward to understand what happens with spline space dimensionality when two knotline segments converge and optain the same knot value.
- To make a consistent theory we discarded the index grid.

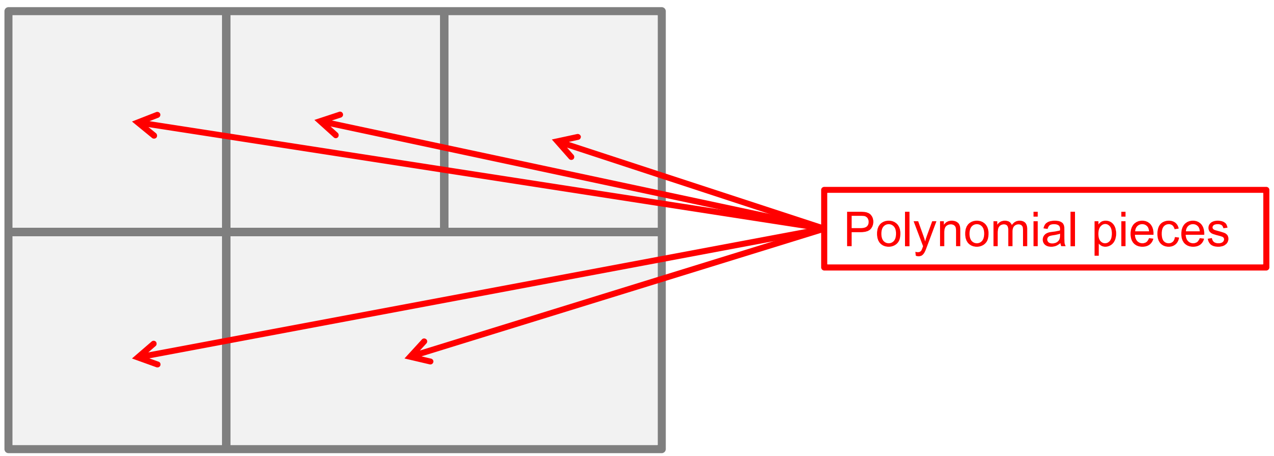
# Polynomials of component degree

On each element the spline is a polynomial.

We define polynomials of component degree at most $p_k, k = 1, \dots, d$ by:

$$\Pi_{\boldsymbol{p}}^d = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f(\boldsymbol{x}) = \sum_{\boldsymbol{0} \le \boldsymbol{i} \le \boldsymbol{p}} c_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}}, c_{\boldsymbol{i}} \text{ in } \mathbb{R} \text{ for all } \boldsymbol{i} \right\}.$$

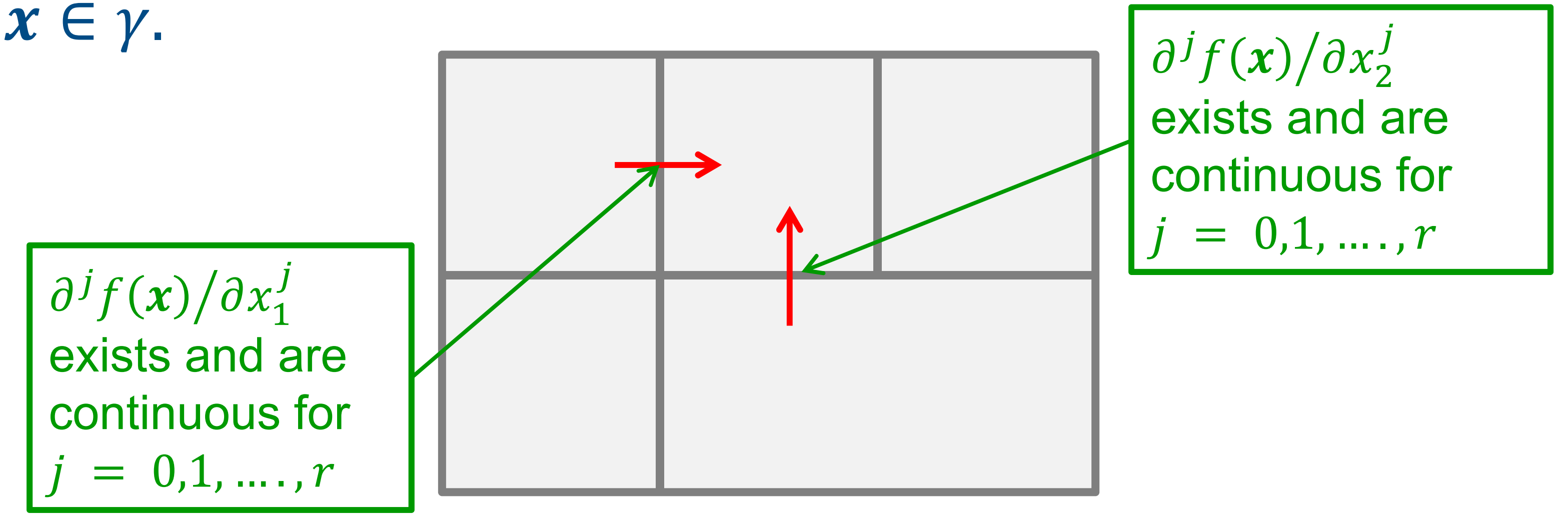
$\boldsymbol{p} = (p_1, \dots, p_d)$
$\boldsymbol{i} = (i_1, \dots, i_d)$



# Continuity across mesh-rectangles

Given a function $f: [\boldsymbol{a}, \boldsymbol{b}] \rightarrow \mathbb{R}$, and let $\gamma \in \mathcal{F}_{d-1,k}(\mathcal{E})$ be any $k$-mesh-rectangle in $[\boldsymbol{a}, \boldsymbol{b}]$ for some $1 \leq k \leq d$.

We say that $f \in C^r(\gamma)$ if the partial derivatives $\partial^j f(\boldsymbol{x})/\partial x_k^j$ exists and are continuous for $j = 0,1, \ldots, r$ and all $\boldsymbol{x} \in \gamma$.

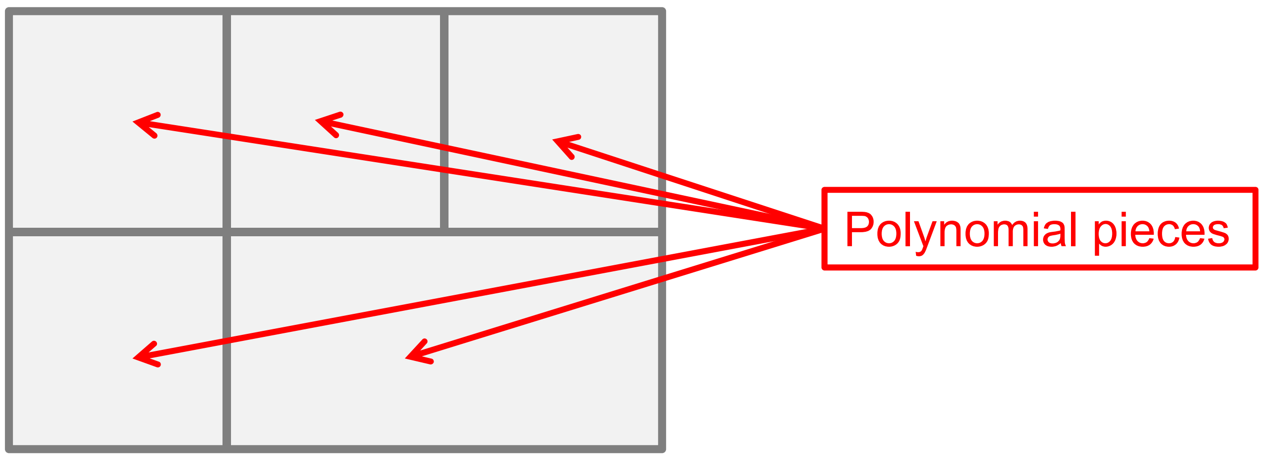


# Piecewise polynomial space

We define the piecewise polynomial space

$$\mathbb{P}_{\boldsymbol{p}}(\boldsymbol{\mathcal{E}}) = \{f: [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}: f|_{\beta} \in \Pi^d_{\boldsymbol{p}}, \beta \in \tilde{\mathcal{E}}\},$$

where $\mathcal{E}$ is obtained from $\boldsymbol{\mathcal{E}}$ using half-open intervals as for univate B-splines.



# Spline space

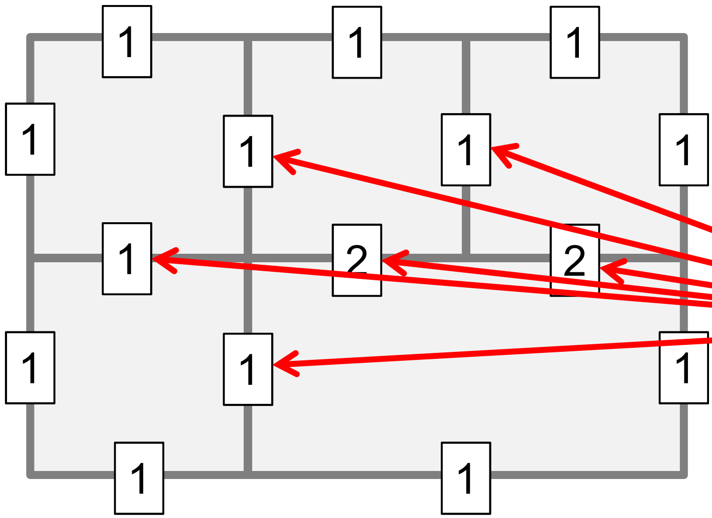
Polynomial degree in direction $k$

Continuity across $k$-mesh-rectangle $\gamma$

We define the spline space

$$\mathbb{S}_{\boldsymbol{p}}(\mathcal{M},\mu) = \left\{ f \in \Pi_{\boldsymbol{p}}^{d}\left(\mathcal{E}(\mathcal{M})\right) : f \in C^{p_k-\mu(\gamma)}(\gamma), \; \forall \gamma \in \mathcal{F}_{d-1,k}\left(\mathcal{E}(\mathcal{M})\right), k = 1, \dots, d \right\}$$

All $k$-mesh-rectangles




Specify multiplicity, e.g., continuity across mesh-rectangle

# How to measure dimensional of spline space of degree $p$ over a $\mu$-extended box partition $(\mathcal{M}, \mu)$.

- Dimension formula developed (Mourrain, Pettersen)

$$\dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}, \boldsymbol{\mu}) = \sum_{\ell=0}^{d} (-1)^{d-\ell} \left( \sum_{\beta \in \mathcal{F}_\ell(\mathcal{M})} \prod_{k=1}^{d} (p_k - \mu_k(\beta) + 1) \right) - \sum_{q=0}^{d-1} (-1)^{d-q} \dim H_q(\widetilde{\mathfrak{S}}(\mathcal{N}))$$

Summing over all $l$-boxes of all dimensions

Dimension influenced by mesh-rectangle multiplicity

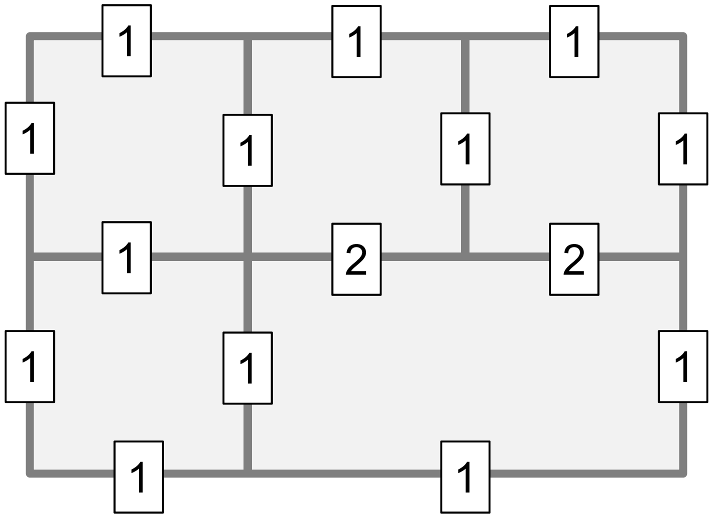
Combinatorial values calculated from topological structure

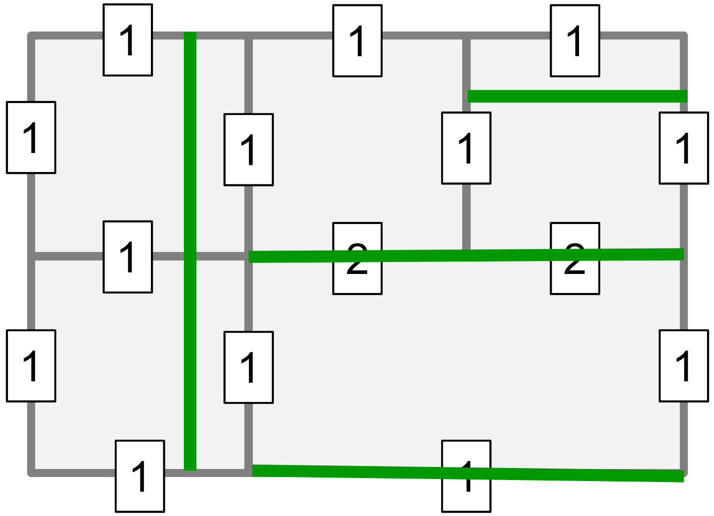
Homology terms

- In the case of 2-variate LR-splines always zero

# LR B-splines over $\mu$-extended box-mesh

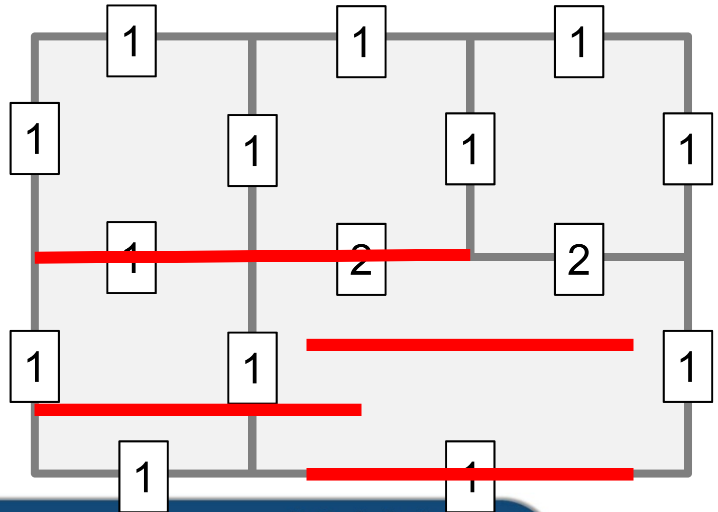
# Refinement by inserting mesh-rectangles giving a constant split




Constant split

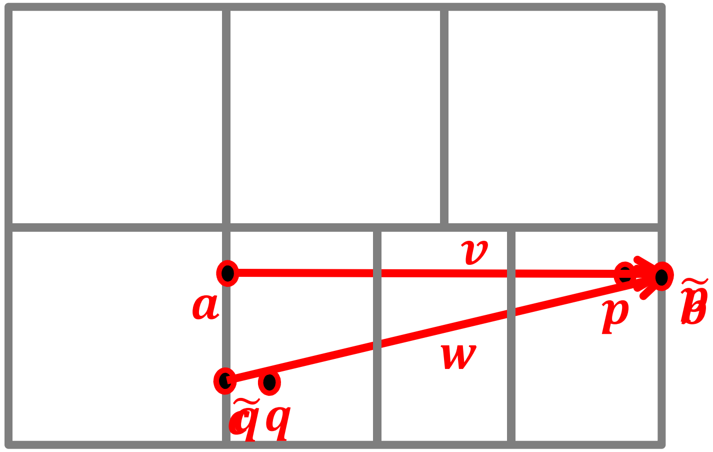
Not constant split



# Interactively specifying mesh-rectangles – 2-variate case

Idea developed together with Peter Nørtoft Nielsen and Odd Andersen, SINTEF

- A mesh-rectangle is defined by the two the extreme corners
  - $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}, \boldsymbol{a} + \boldsymbol{v})$
  - Where $\boldsymbol{v} = (v_1, v_2)$, with just one of $v_1, v_2$ zero
- Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ as input. Snap $\boldsymbol{p}$ to the nearest mesh-rectangle to create $\widetilde{\boldsymbol{p}}$, remember the class of $k$-mesh-rectangles snapped to, $k \in \{1,2\}$. Snap $\boldsymbol{q}$ to the nearest parallel $k$-meshrectangle to create $\widetilde{\boldsymbol{q}}$.
  - Make $\boldsymbol{c}$ where $c_j = \min(\widetilde{p}_j, \widetilde{q}_j)$
  - Make $\boldsymbol{w}$ where $w_j = |\widetilde{p}_j - \widetilde{q}_j|$
  - Define $\boldsymbol{a} = (a_1, a_2)$, $a_j = \begin{cases} p_k, j \neq k \\ c_j, j = k \end{cases}$
  - Define $\boldsymbol{v} = (v_1, v_2)$, $v_j = \begin{cases} 0, j \neq k \\ w_j, j = k \end{cases}$

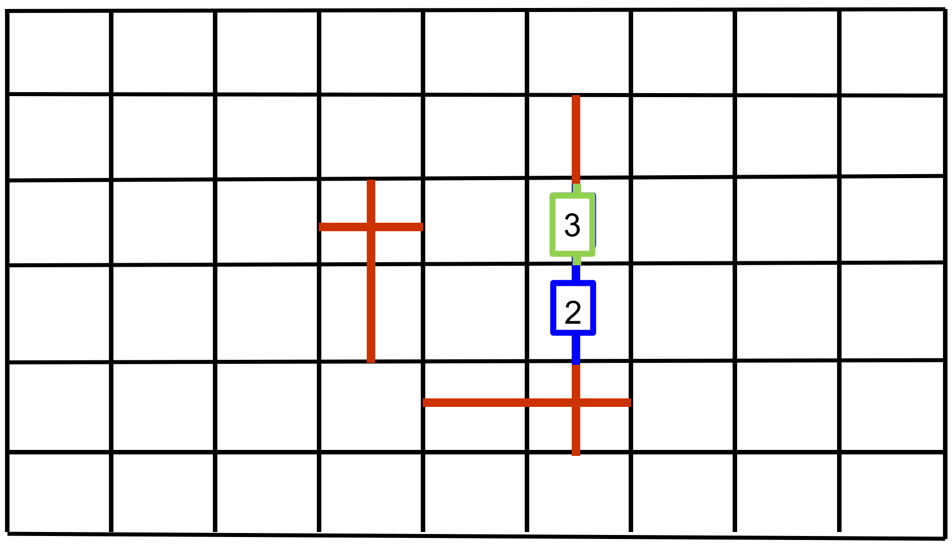



- Provided $v_j > 0$, for $j = k$, we have a well defined mesh-rectangle define by $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}, \boldsymbol{a} + \boldsymbol{v})$
- When creating mesh-rectangles automatic checks can be run with respect to the increase of dimensionality, and if the resulting B-splines form a basis.

# $\mu$-extended LR-mesh

A $\mu$ -extended LR-mesh is a $\mu$-extended box-mesh $(\mathcal{M},\mu)$ where either

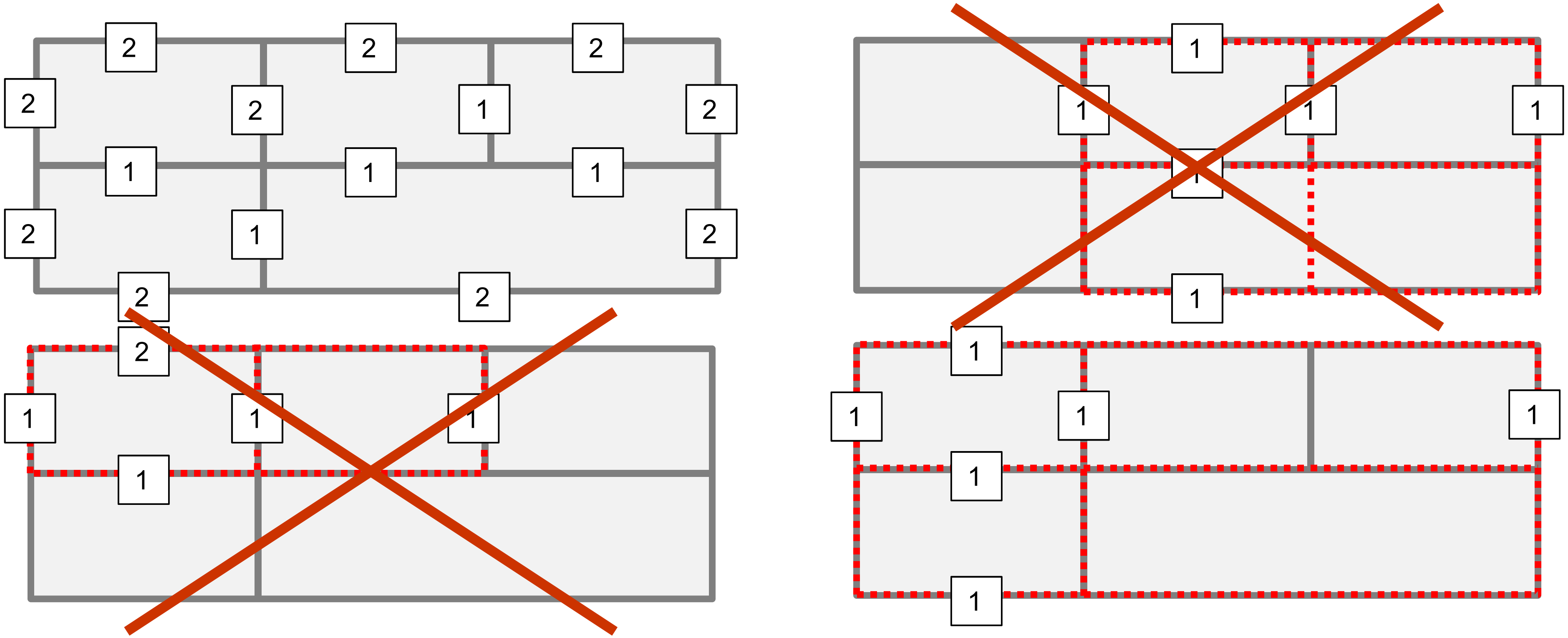
1. $(\mathcal{M},\mu)$ is a tensor-mesh with knot multiplicities or
2. $(\mathcal{M},\mu) = \left(\widetilde{\mathcal{M}} + \gamma, \tilde{\mu}_\gamma\right)$ where $\left(\widetilde{\mathcal{M}}, \tilde{\mu}\right)$ is a $\mu$ -extended LR-mesh and $\gamma$ is a constant split of $\left(\widetilde{\mathcal{M}}, \tilde{\mu}\right)$.




*All multiplicities not shown are 1.*

# LR B-spline

Let $(M, \mu)$ be an $\mu$-extended LR-mesh in $\mathbb{R}^d$. A function $B: \mathbb{R}^d \to \mathbb{R}$ is called an LR B-spline of degree $\boldsymbol{p}$ on $(\mathcal{M}, \mu)$ if $B$ is a tensor-product B-spline with minimal support in $(\mathcal{M}, \mu)$.



# Splines on a $\mu$-extended LR-mesh

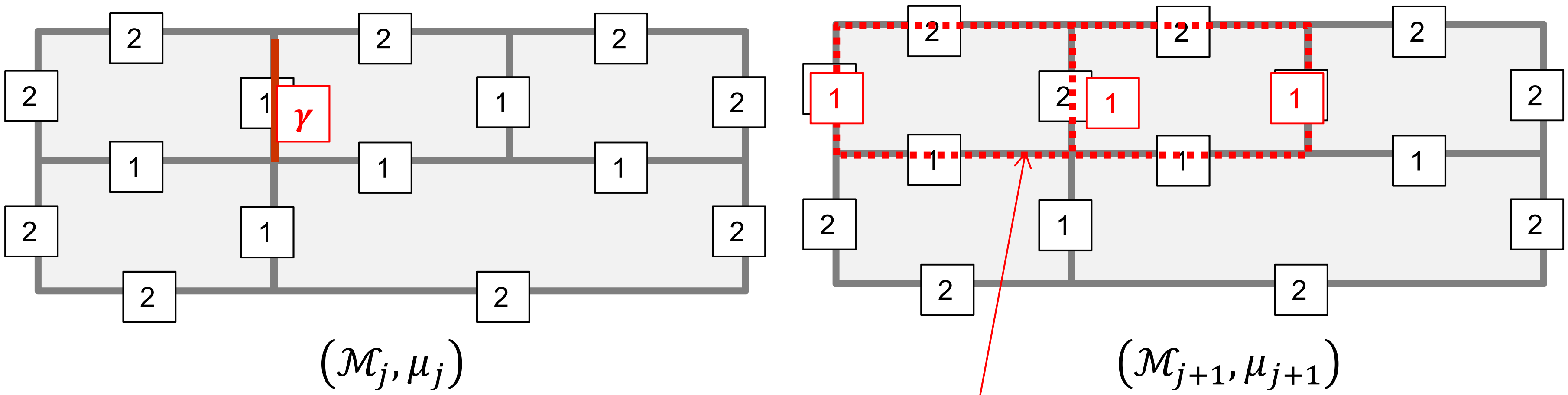
We define as sequence of $\boldsymbol{\mu}$-extended LR-meshes $(\mathcal{M}_1, \mu_1), \ldots, (\mathcal{M}_q, \mu_q)$ with corresponding collections of minimal support B-splines $\mathcal{B}_1, \ldots, \mathcal{B}_q$.

For $j = 1, \ldots, q-1$ creating $(\mathcal{M}_{j+1}, \mu_{j+1}) = (\mathcal{M}_j + \gamma_j, \mu_{j,\gamma_j})$ from $(\mathcal{M}_j, \mu_j)$ involves inserting a mesh-rectangles $\gamma_j$ that increases the number of B-splines. More specifically:

- $\gamma_j$ splits $(\mathcal{M}_j, \mu_j)$ in a constant split.
- at least on B-spline in $\mathcal{B}_j$ does not have minimal support in $(\mathcal{M}_{j+1}, \mu_{j+1})$.

After inserting $\gamma_j$ we start a process to generate a collection of minimal support B-splines $\mathcal{B}_{j+1}$ over $(\mathcal{M}_{j+1}, \mu_{j+1})$ from $\mathcal{B}_j$.

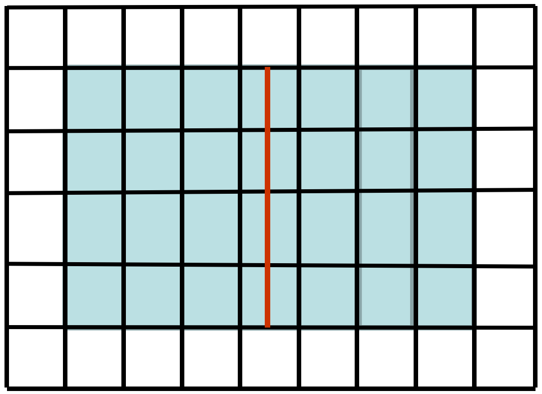
# Going from $(\mathcal{M}_j, \mu_j)$ to $(\mathcal{M}_{j+1}, \mu_{j+1})$

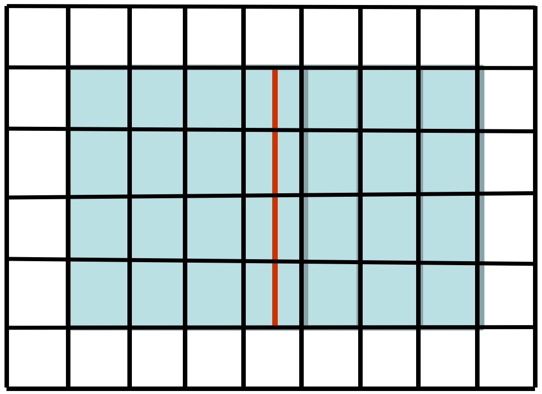



B-spline from $\mathcal{B}_j$ that has to be split to generate $\mathcal{B}_{j+1}$

# LR B-Spline Refinement step
# Cubic example: One line

- Insert knot line segments that at least span the width of one basis function





Four B-splines functions that do not have minimal Support in the refined mesh

- 4 B-splines to be removed
- 5 B-splines to be added
- Dimension increase 1

# Refinement of LR B-splines is focused on the tensor product B-splines

Let $d$ be a positive integer, suppose $\boldsymbol{p} = (p_1, \dots, p_d)$ has nonnegative components (the degrees), and let $\boldsymbol{y}_k; = (y_{k,1}, \dots, y_{k,p_k+2})$ be a nondecreasing (knot) sequence $k = 1, \dots, d$. We define a tensor product B-spline $B[\boldsymbol{y}_1, \dots, \boldsymbol{y}_d]: \mathbb{R}^d \to \mathbb{R}$ by

$$B[\boldsymbol{y}_1, \dots, \boldsymbol{y}_d](x_1, \dots, x_d) := \prod_{k=1}^{d} B[\boldsymbol{y}_k]\,(x_k).$$

Tensor product B-spline.

Product of Univariate B-splines.

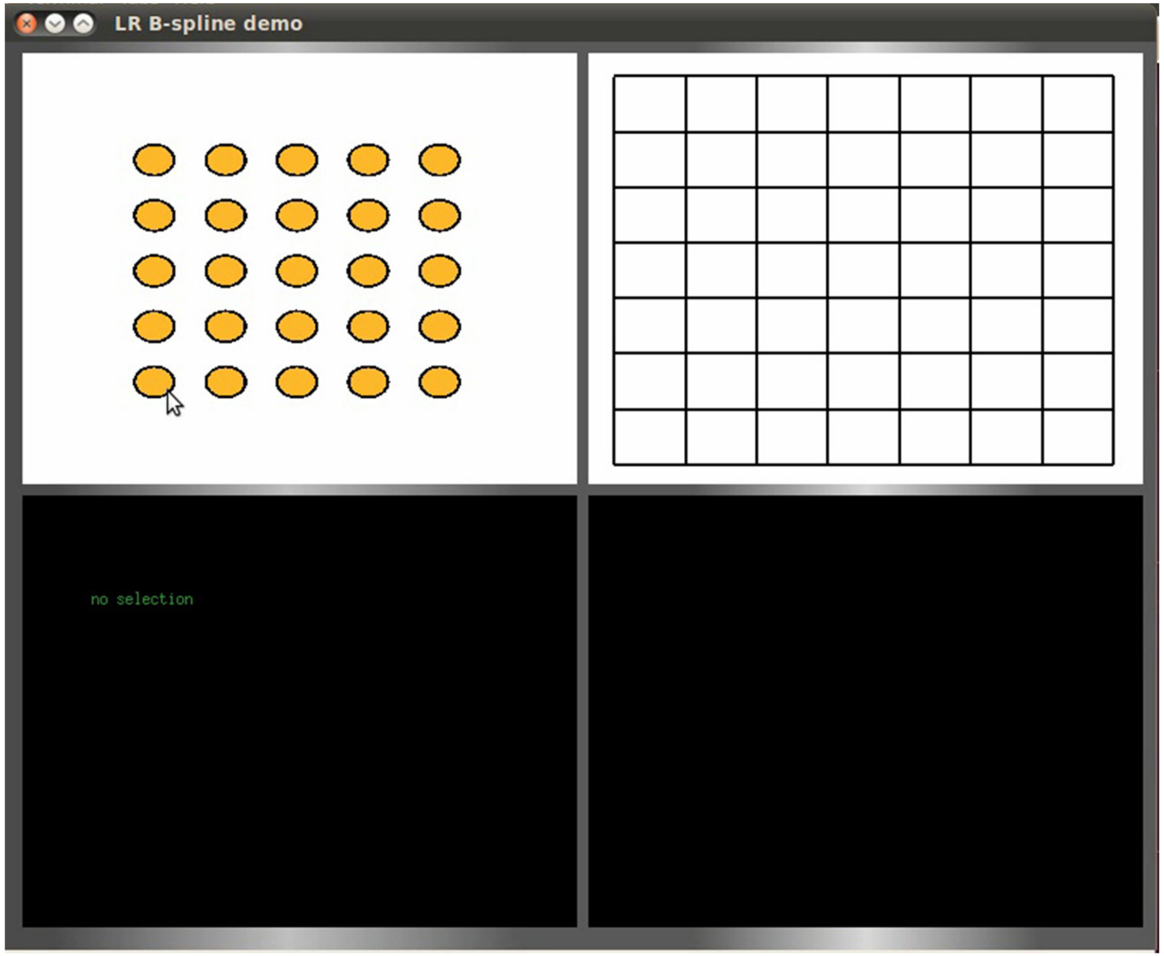
# Refinement of a tensor product B-spline

- The support of $B$ is given by the cartesian product
$$\mathrm{supp}(B) := \left[y_{1,1}, \dots, y_{1,p_k+2}\right] \times \cdots \times \left[y_{d,1}, \dots, y_{d,p_k+2}\right].$$
- Suppose we insert a knot $z$ in $\left(y_{k,1}, \dots, y_{k,p_k+2}\right)$ for some $1 \le k \le d$. Then
$$B[\boldsymbol{Y}] = \alpha_1 B[\boldsymbol{Y}_1] + \alpha_2 B[\boldsymbol{Y}_2]$$
- Where $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are the knot vectors of the resulting tensor product B-splines, and
$$\alpha_1 := \begin{cases} 1 & y_{k,p_k+1} \le z < y_{k,p_k+2} \\ \dfrac{z - y_{k,1}}{y_{k,p_k+1} - y_{k,1}} & y_{k,1} < z < y_{k,p_k+1} \end{cases}$$
$$\alpha_2 := \begin{cases} 1 & y_{k,1} \le z \le y_{k\ 2} \\ \dfrac{y_{k,p_k+2} - z}{y_{k,p_k+2} - y_{k,2}} & y_{k,2} < z < y_{k,p_k+2} \end{cases}$$
- $\alpha_1$ and $\alpha_2$ calculated by Oslo Algorithm/Boehms algorithm

# LR B-splines and partition of unity

- The LR B-spline refinement starts from a partition of unity tensor product B-spline basis.
- By accumulating the weights $\alpha_1$ and $\alpha_2$ as scaling factors for the LR B-splines, partition of unity is maintained throughout the refinement for the scaled collection of tensor product B-splines
- The partition of unity properties gives the coefficients of LR B-splines the same geometric interpretation as B-splines and T-splines.
  - However, the spatial interrelation of the coefficients is more intricate than for T-splines as the refinement strategies are more generic than for T-splines.
  - This is, however, no problem as in general algorithms calculate the coefficients both in FEA and CAD.

# Example LR B-spline refinement




Video by PhD fellow Kjetil A. Johannessen, NTNU, Trondheim, Norway.