Isogeometric compatible discretizations

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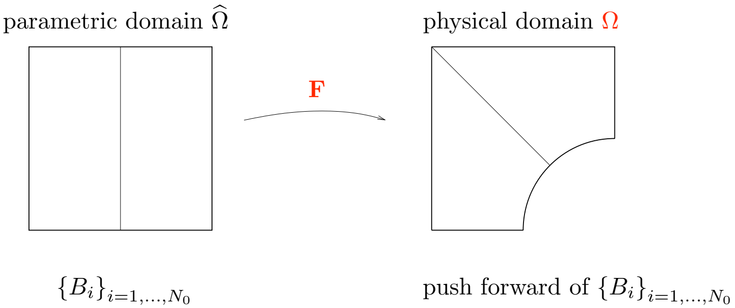
1. **Approximation spaces**
   - Definition and refinement
   - Basic estimates

2. **Vector fields compatible approximations**
   - De Rham Diagram : Maxwell and Darcy
   - New applications
   - The T-Spline complex

3. **The GeoPDEs Library**

Construction of approximation spaces



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  general geometry: unstructured collection of "patches".

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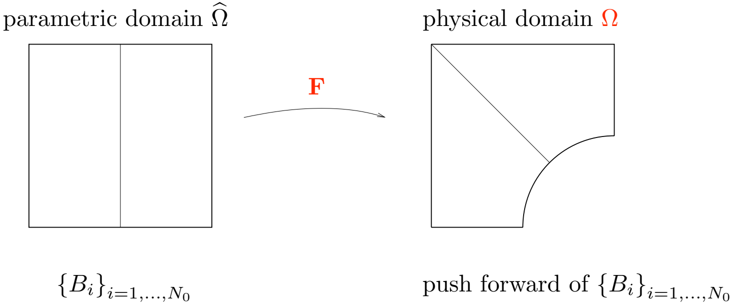

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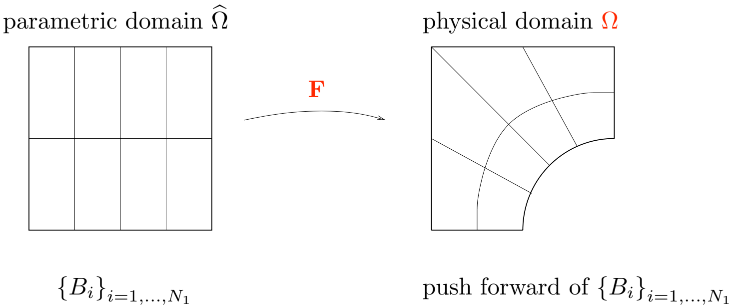

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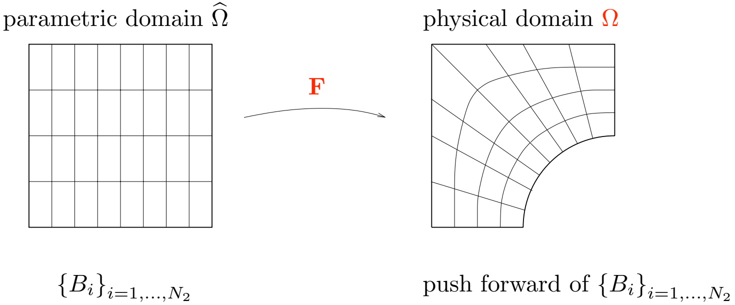

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- The discrete space on Ω is the *push-forward* of Spline/NURBS
- Refinement by **knot insertion** and **degree elevation**, geometry unchanged.
- Tensor product structure

Approximation results in 1D

Given $\Xi = \{\xi_0, \xi_1, ..., \xi_{n+p}, \xi_{n+p+1}\}$ be an open knot vector:

$$\Xi = \{\underbrace{\zeta_1, \ldots, \zeta_1}_{p+1 \text{ times}}, \underbrace{\zeta_2, \ldots, \zeta_2}_{r_2 \text{ times}}, \ldots, \underbrace{\zeta_m, \ldots, \zeta_m}_{p+1 \text{ times}}\},$$

with $\sum_{i=1}^{m} r_i = n + p + 1$.

$$\begin{aligned}\widehat{S}(\Xi) &= \{s \in L^2(0,1) \quad s_{|(\zeta_i,\zeta_{i+1})} \in \mathbf{P}^p \\ &\qquad D_-^k s(\zeta_i) = D_+^k s(\zeta_i) \quad \forall k = 0, \ldots, p - r_i \quad \forall\, 1 \le i \le m\}. \\ &= \mathrm{span}\{B_\ell\}_{\ell=1,..,n}.\end{aligned}$$

Given a mapping $\mathbf{F} \in \widehat{S}(\Xi)$, $\mathbf{F} : (0,1) \to \Omega$,

$$V_h = \{s \ : \ s \circ \mathbf{F}^{-1} \in \widehat{S}(\Xi)\}\}$$

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Given $u : \Omega \to \mathbb{R}$ we want to find an approximation in the space V_h; then, we need a spline $s \in S(\Xi)$ such that

$$\begin{array}{ll} u \approx s \circ \mathbf{F}^{-1} & \text{in } \Omega \\ \Downarrow & \\ u \circ \mathbf{F} \approx s & \text{in } (0,1) \end{array}$$

BUT $u \in H^s(\Omega) \quad \not\Rightarrow \quad u \circ \mathbf{F} \in H^s(0,1)$!!

Approximation results in 1D

$$u \in H^s(\Omega) \quad \not\Rightarrow \quad u \circ \mathbf{F} \in H^s(0,1)$$

Standard estimates does not work. In fact it would look as follows:

$$\inf_{s \in \widehat{S}(\Xi)} \|u \circ \mathbf{F} - s\|_{L^2(0,1)} \leq C h^{p+1} |u \circ \mathbf{F}|_{H^{p+1}(0,1)}$$

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Instead, $u \circ \mathbf{F} \in \mathcal{H}^s(0,1)$ with

$$\begin{aligned} \mathcal{H}^s(0,1) \quad &= \{ v \in L^2(0,1) \quad u_{|(\zeta_i,\zeta_{i+1})} \in H^s(\zeta_i,\zeta_{i+1}) \\ &\qquad D^k_- v(\zeta_i) = D^k_+ v(\zeta_i) \quad \forall k = 0,\ldots,\min\{s-1, p-r_i\} \quad \forall\, i\}. \end{aligned}$$

The semi-norm for this space is: $|v|^2_{\mathcal{H}^s} := \sum_{i=1}^{m-1} |v|^2_{H^s(\zeta_i,\zeta_{i+1})}$

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and $\forall v \in \mathcal{H}^s(0,1)\,,\ \exists\ \Gamma(v) \in \widehat{S}(\Xi)\ :\ v - \Gamma(v) \in H^s(0,1)$!!

Approximation results in 1D

Schumaker book 2007, Beirao da Veiga et al 2008 - 2011

- $\Pi_S : L^2([0,1)) \to \widehat{S}(\Xi)$ local projection operator:

$$\Pi_S f = \sum_{\ell=1}^{n} \lambda_\ell(f)\, B_\ell \quad \text{with } \lambda_\ell(B_k) = \delta_{\ell,k}.$$

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- Let $\widehat{u} = u \circ \mathbf{F}$. We can easily obtain: $Q = \mathbf{F}(\widehat{Q})$, $\widehat{Q} = (\zeta_i, \zeta_{i+1})$

$$\begin{aligned} |u - \Pi_S(u \circ \mathbf{F}) \circ \mathbf{F}^{-1}|_{H^m(Q)} &\leq C|\widehat{u} - \Pi_S(\widehat{u})|_{H^m(\widehat{Q})} \\ &\leq C|\widehat{u} - \Gamma(\widehat{u}) - (\Pi_S(\widehat{u}) - \Gamma(\widehat{u}))|_{H^m(\widehat{Q})} \end{aligned}$$

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&\leq C' \, (h_Q)^{p+1-m} \sum_{\widehat{Q}' \subset \widetilde{\widehat{Q}}} |\widehat{u} - \Gamma(\widehat{u})|_{H^{p+1}(\widehat{Q}')}
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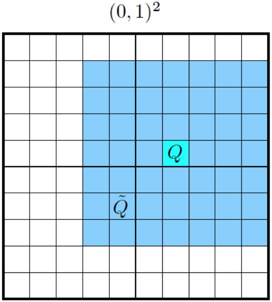
&= C''\,(h_Q)^{p+1-m} \sum_{Q' \subset \widetilde{Q}} |u|_{H^{p+1}(Q')}.
\end{aligned}$$

Approximation results in 2D and 3D

Schumaker book 2007, Beirao da Veiga et al 2006 - 2010 - 2011

- $\Pi_S = \Pi_S^1 \otimes \Pi_S^2 \otimes \Pi_S^3$
- Exploiting the tensor product structure: Given $u \in H^{p+1}(\Omega)$

$$|u - \Pi(u \circ \mathbf{F}) \circ \mathbf{F}^{-1}|_Q \leq C(\mathbf{F}, p) h^{p+1-m} \sum_{Q' \in \widetilde{Q}} |u|_{H^{p+1}(Q')}$$



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- Dependance of the constant in p still open. Only partial results for reduced regularity splines: e.g., C^1 cubics, C^2 quintic ..
- For NURBS, the projection becomes ...

Approximation results in 2D and 3D

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$$u \approx \left(\frac{s}{w}\right) \circ \mathbf{F}^{-1} \quad \text{in } \Omega$$

$$\Downarrow$$

$$u \circ \mathbf{F} \approx \frac{s}{w} \quad \text{in } (0,1)^2 = \widehat{\Omega}$$

$$\Downarrow$$

$$w(u \circ \mathbf{F}) \approx s \quad \text{in } \widehat{\Omega}$$

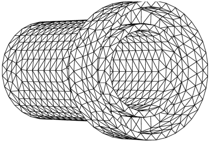
and the estimate works exactly the same way.

Compute Vector fields - compatible discretizations

Finite elements

$$H(d,\Omega) = \{v \in L^2(\Omega) \mid d\,v \in L^2(\Omega)\}$$

$$\begin{cases} \mathbf{curl}\,\mathbf{u} = \mathbf{f} \\ \mathrm{div}\,\mathbf{a}\mathbf{u} = \mathbf{g} \end{cases}$$



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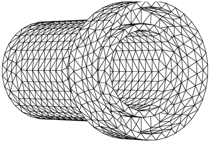

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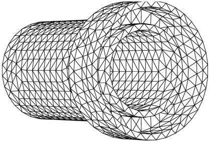

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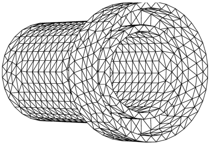

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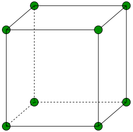
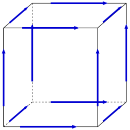
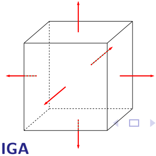
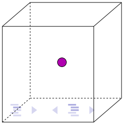

Compute Vector fields - compatible discretizations

Finite elements, Nédélec '80

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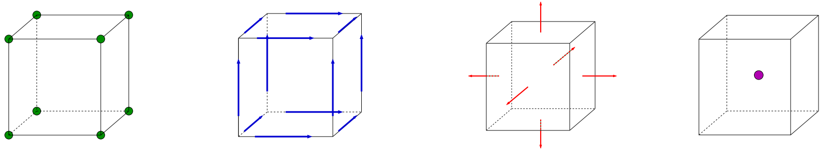

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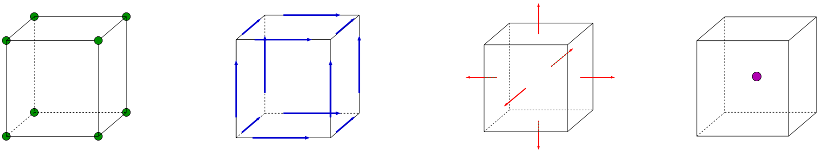

$$\Lambda_h^0 = \mathbb{Q}^{1,1,1}$$

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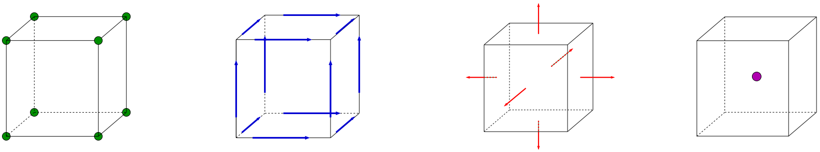

$$\Lambda_h^1 = (\mathbb{Q}^{0,1,1},\ \mathbb{Q}^{1,0,1},\ \mathbb{Q}^{1,1,0})$$

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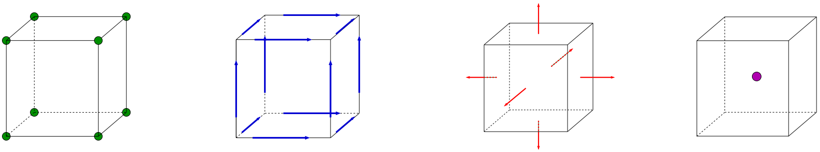

$$\Lambda_h^2 = (\mathbb{Q}^{1,0,0},\ \mathbb{Q}^{0,,1,0},\ \mathbb{Q}^{0,0,1})$$

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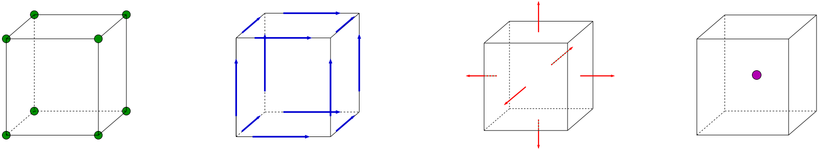

$$\Lambda_h^3 = \mathbb{Q}^{0,0,0}$$

Compatible discretizations

- Needed for problems with a geometric structure which has to be preserved at the discrete level to obtain spurious free discretizations

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- Literature: *Finite Element Exterior Calculus* ...
  started around 2000 and now is a rather mature theory
  Arnold, Boffi, Bossavit, B., Costabel, Christiansen, Demkovicz, Dauge, Falk, Hiptmair, Winther ....
- Conjugate results from differential geometry, functional analysis and numerical analysis.

The Spline Complex

B., Sangalli, Vazquez 2009-2011

Let $\mathbf{F} : \widehat{\Omega} \to \Omega$ be the parametrization of Ω
Let $\widehat{S}^p$ be the space of splines of degree p in 1D and set in $\widehat{\Omega}$:

$$\widehat{S}^{p,p,p} = \widehat{S}^p \otimes \widehat{S}^p \otimes \widehat{S}^p$$

The Spline Complex

B., Sangalli, Vazquez 2009-2011

Let $\mathbf{F} : \widehat{\Omega} \to \Omega$ be the parametrization of Ω
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$$\widehat{\Lambda}_h^0 = \widehat{S}^{p,p,p}$$

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$$\widehat{\Lambda}_h^1 = (\widehat{S}^{p-1,p,p}, \widehat{S}^{p,p-1,p}, \widehat{S}^{p,p,p-1})$$

$$\partial_x : \widehat{S}^{p,p,p} \to \widehat{S}^{p-1,p,p}$$

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$$\widehat{\Lambda}_h^2 = (\widehat{S}^{p,p-1,p-1}, \widehat{S}^{p-1,p,p-1}, \widehat{S}^{p-1,p-1,p})$$

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$$\Lambda_h^0 = \iota_{\mathbf{F}}^0 \, [\, \widehat{\Lambda}_h^0 \,] \qquad \iota_{\mathbf{F}}^0(\widehat{\phi}) \circ \mathbf{F} = \widehat{\phi}$$

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$$\Lambda_h^1 = \iota_{\mathbf{F}}^1 \; [\; \widehat{\Lambda}_h^1 \;] \qquad \iota_{\mathbf{F}}^1(\widehat{\mathbf{u}}) \circ \mathbf{F} = D\mathbf{F}^{-T}\widehat{\mathbf{u}}$$

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\end{array}$$

$$\Lambda_h^2 = \iota_{\mathbf{F}}^2 \left[\widehat{\Lambda}_h^2 \right] \qquad \iota_{\mathbf{F}}^2(\widehat{\mathbf{u}}) \circ \mathbf{F} = (\det D\mathbf{F})^{-1} D\mathbf{F}\, \widehat{\mathbf{u}}$$

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- how one constructs projectors?

Simple 1D complex

$$\begin{array}{ccc} H^1(0,1) & \xrightarrow{d/dx} & L^2(0,1) \\ \Pi \big\downarrow & & \Pi_D \big\downarrow \\ S^p & \xrightarrow{d/dx} & S^{p-1} \end{array}$$

Π and Π_D chosen so that the diagram commutes $\Pi_D \frac{d}{dx}\phi = \frac{d}{dx}\Pi\phi$.

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- $\Pi_D = \Pi_S \Rightarrow$ generates a non local and nasty Π_0 :-(

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- $\Pi_D = \Pi_S \Rightarrow$ generates a non local and nasty Π_0 :-(
- $\Pi = \Pi_S \Rightarrow$ generates a local and nice Π_1 :-)

$$\Pi_D v = \frac{d}{dx}\Pi_S \int_0^t v(s)\, ds$$

Π_D commutes, and it is a stable projector:

$$\widehat{\Pi}_D^{p-1} s = s \qquad \forall s \in S^{p-1}$$

$$|\Pi_D v|_{H^\ell(I)} \leq C|v|_{H^\ell(\widetilde{I})}, \qquad \forall u \in H^\ell(0,1)\,, \quad 0 \leq \ell \leq p-1\,.$$

The Simple Complex, 3D

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\nabla} & H(\mathbf{curl}\,,\Omega) & \xrightarrow{\mathbf{curl}} & H(\operatorname{div}\,,\Omega) & \xrightarrow{\operatorname{div}} & L^2(\Omega) \\
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\end{array}$$

- $\Pi^0 = \Pi_S \otimes \Pi_S \otimes \Pi_S$
- $\Pi^1 = (\Pi_D \otimes \Pi_S \otimes \Pi_S,$

The Simple Complex, 3D

$$
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- ...
- $\Pi^3 = \Pi_D \otimes \Pi_D \otimes \Pi_D$

$\Pi_i : L^2(\Omega)^d \to L^2(\Omega)$ are bounded operators

Finite Element Exterior Calculus applies!

Well-posedness for IGA discretizations of various problems.

Darcy and Maxwell problems

- $\boldsymbol{\sigma} = \mathbb{K}\nabla u - \mathbf{g}\,, \qquad \operatorname{div}(\boldsymbol{\sigma}) = f\,, \qquad \boldsymbol{\sigma}\cdot\mathbf{n} = 0$ at $\partial\Omega$.

  Find $\boldsymbol{\sigma} \in H_0(\operatorname{div},\Omega)$, $u \in L^2_0(\Omega)$ s.t.
  $(\mathbb{K}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau}) - (\operatorname{div}\,\boldsymbol{\tau}, u) = (\mathbf{g},\boldsymbol{\tau}) \quad (\operatorname{div}\,\boldsymbol{\sigma}, q) = (f, q)$
  for all $\boldsymbol{\tau} \in H_0(\operatorname{div},\Omega)$, $q \in L^2_0(\Omega)$

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- $\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u} - \omega^2\varepsilon\,\mathbf{u} = \mathbf{f}, \qquad \mathbf{u}\times\mathbf{n} = 0$ at $\partial\Omega$

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  Find $\mathbf{u} \in H_0(\mathbf{curl}\,,\Omega)$, $xs\omega \in \mathbb{R}$ such that
  $(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u},\,\mathbf{curl}\,\mathbf{v}) = \omega^2(\varepsilon\mathbf{u},\mathbf{v})$ for all $\mathbf{v} \in H_0(\mathbf{curl}\,,\Omega)$.

Darcy and Maxwell problems

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  Find $\boldsymbol{\sigma} \in \Lambda^2_{0,h}$, $u \in \Lambda^3_{0,h}$ s.t.

  $(\mathbb{K}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\,\mathrm{div}\ \boldsymbol{\tau}, u) = (\mathbf{g}, \boldsymbol{\tau}) \quad (\,\mathrm{div}\ \boldsymbol{\sigma}, q) = (f, q)$

  for all $\boldsymbol{\tau} \in \Lambda^2_{0,h}$, $q \in \Lambda^3_{0,h}$

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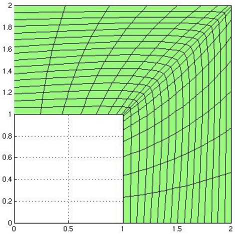
  Find $\mathbf{u} \in \Lambda^1_{0,h}$, $\omega \in \mathbb{R}$ such that

  $(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \omega^2(\varepsilon\mathbf{u}, \mathbf{v})$ for all $\mathbf{v} \in \Lambda^1_{0,h}$.

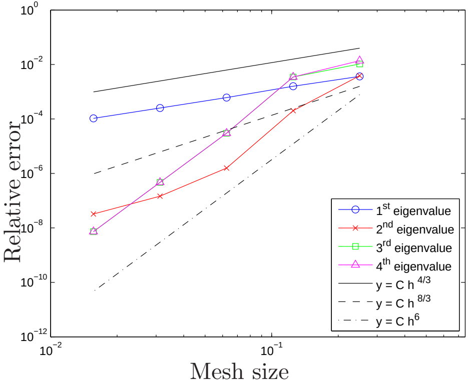
Maxwell eigenproblem

B. , Sangalli, Vazquez 2009

$$\mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = k^2\mathbf{u} \qquad k \neq 0\,,\ \mathbf{u} \neq 0$$



$\mathbf{F}^{-1}$ singular!



$p = 3,\ C^1$ on coarse mesh

The Spline Complex

$$\mathbf{curl}\ \mathbf{curl}\,\mathbf{u} = k^2\mathbf{u} \qquad k \neq 0\,,\ \mathbf{u} \neq 0$$

on the Fichera corner domain.

Eigensolutions are singular and going to infinity at reetrant edges

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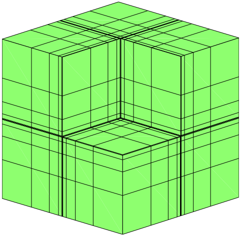

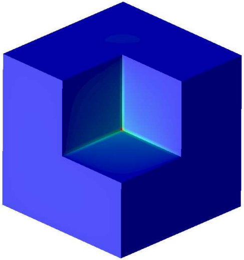

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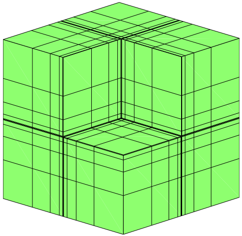

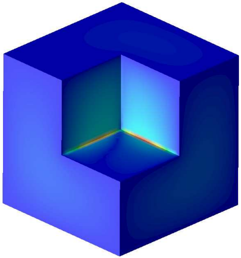

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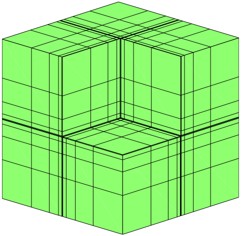

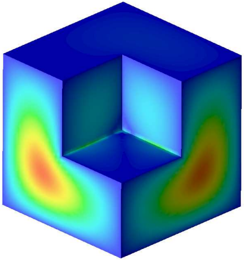

The Spline Complex

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = k^2\mathbf{u} \qquad k \neq 0\,,\ \mathbf{u} \neq 0$$

on the Fichera corner domain.

Eigensolutions are singular and going to infinity at reetrant edges

	Eigenvalues computation				
CODE by	Schoberl	Dauge et al.	IGA, p = 3	IGA, p = 6	Reliable digits
d.o.f.	53982	41691	8421	5436	
Eig. 1.	3.2199939	3.3138052	3.2194306	3.2111746	3.2???e+00
Eig. 2.	5.8804425	5.8863499	5.8804604	5.8809472	5.88??e+00
Eig. 3.	5.8804553	5.8863499	5.8804604	5.8809472	5.88??e+00
Eig. 4.	10.6856632	10.6945143	10.6866214	10.6938099	1.0694e+01
Eig. 5.	10.6936955	10.6945143	10.6949643	10.7069155	1.0694e+01
Eig. 6.	10.6937289	10.7005804	10.6949643	10.7069155	1.07??e+01

Other applications

Since fields are **smooth**, we can also solve other 2nd (or more) order problems!

For example: B., de Falco, Sangalli, 2010

- **Stokes/Linear elasticity** : $-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \ \mathrm{div}\,(\mathbf{u}) = 0$

seek for a $\mathbf{u}_h \in \Lambda_h^2$ such that $\mathrm{div}\,(\mathbf{u}_h) = 0$!

John's Lecture tomorrow!

Solving the Stokes problem with IGA

B. De Falco, Sangalli 2010

Find $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$, such that

$$
\begin{aligned}
(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad && \forall \mathbf{v} \in H^1(\Omega)^2 \\
(q, \operatorname{div} \mathbf{u}) &= 0 && \forall q \in L^2(\Omega)
\end{aligned}
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Use Λ_h^2 for fields and Λ_h^3 for pressures:

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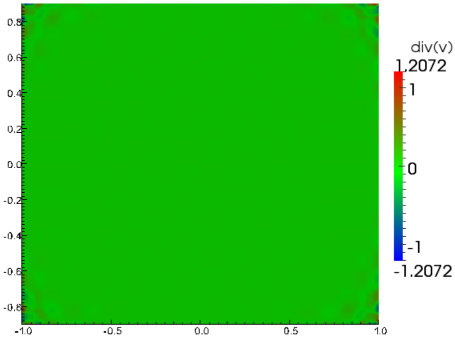
$\operatorname{div} : \Lambda_h^2 \to \Lambda_h^3$ is onto: wellposedness (up to constants) and

$$\operatorname{div} \mathbf{u_h} = \mathbf{0}$$

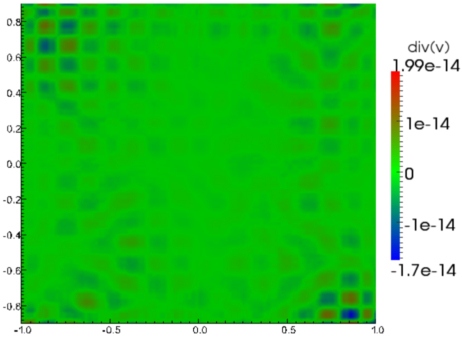
Numerical validation: Lid Driven cavity

B. , De Falco, Sangalli 2010

Divergence for Taylor-Hood



Divergence for IGA



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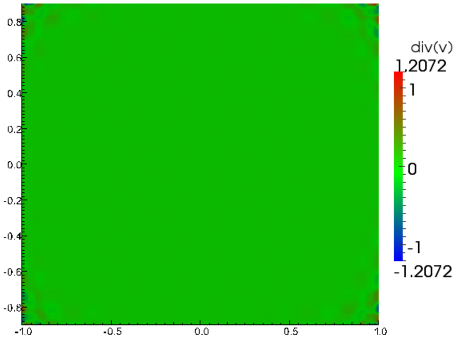
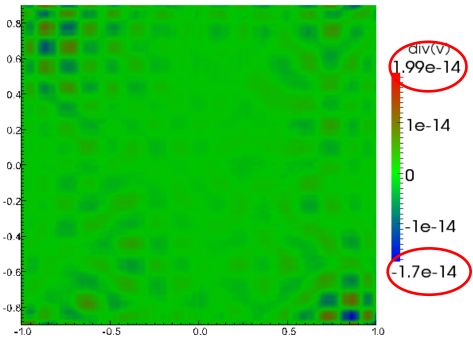


Divergence for IGA



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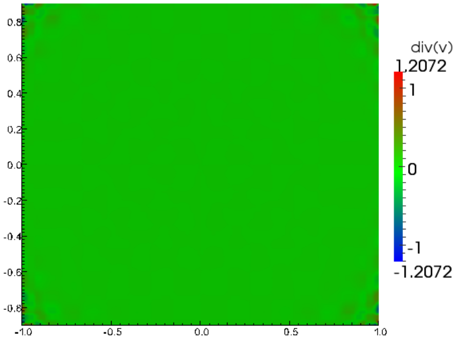
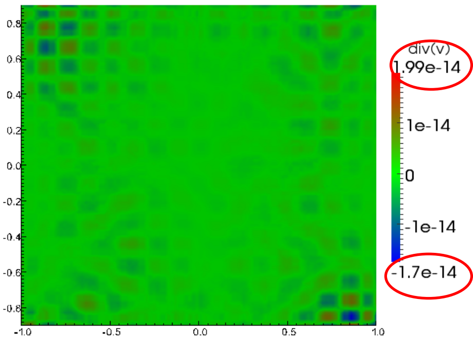


Divergence for IGA



Evans-Hughes 2011-2012, and J. Evans PhD Thesis:

weakly imposed BCs, steady and unsteady Navier Stokes

IGA techniques for Reissner Mindlin plates

Beirao da Veiga, B., Lovadina, Martinelli, Sangalli 2011

- $\Omega \subset \mathbf{R}^2$ = midsurface of the plate, t = the tickness
- w = deflection, $\boldsymbol{\theta}$ = rotation of the normal fibers
  f = applied scaled normal load.

Reissner Mindlin plate blending problem:
Find $\boldsymbol{\theta} \in H_0^1(\Omega)^2, w \in H_0^1(\Omega)$ s.t. for all $\boldsymbol{\eta} \in H_0^1(\Omega)^2, v \in H_0^1(\Omega)$

$$(\mathbb{C}\varepsilon(\boldsymbol{\theta}), \varepsilon(\boldsymbol{\eta})) + \mu k\, t^{-2}(\boldsymbol{\theta} - \nabla w, \boldsymbol{\eta} - \nabla v) = (f, v)$$

Locking

When $t \to 0$, R-M $\to$ Kirchoff which means: $\boldsymbol{\theta} = \nabla w$.
Gradients needs to be represented in the space of rotations

IGA techniques for Reissner Mindlin plates

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On the physical domain, by the transformation ι^1:

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\nabla} & H(\mathbf{curl},\Omega) & \longrightarrow & \dots \\ \Pi^0 \big\downarrow & & \Pi^1 \big\downarrow & & \\ \Lambda_h^0 & \xrightarrow{\nabla} & \Lambda_h^1 & \longrightarrow & \dots \end{array}$$

$\Lambda_h^1 \subset H^1(\Omega)^2$ as soon as $\Lambda_h^0 \subset C^1$

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$$\mathbf{\Theta}_h = \Lambda_h^1 \cap H_0^1(\Omega)^2 \; , \; W_h = \Lambda_h^0 \cap H_0^1(\Omega)$$

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No stabilization, no reduced integration.. built-in **stable and locking free**

Convergence analysis

We can obtain a number of estimates.. and use a number of techniques..

Let $\gamma = t^{-2}(\boldsymbol{\theta} - \nabla w)$ and $\gamma_h = t^{-2}(\boldsymbol{\theta}_h - \nabla w_h)$ (shear stresses) then, for regular solutions (and q.u. meshes):

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{H^1} + h^{-1}\|w - w_h\|_{H^1} + (t+h)\|\gamma - \gamma_h\|_{L^2} \leq Ch^{p-1}$$

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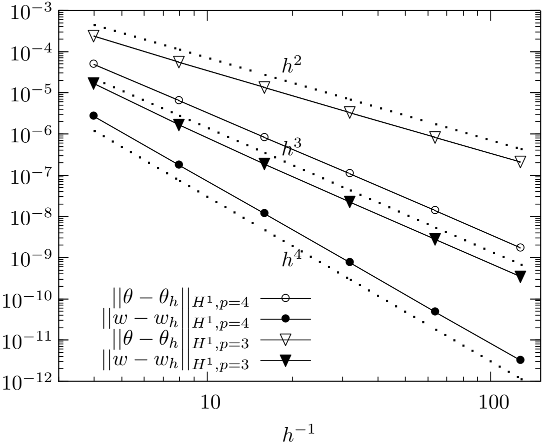
1. it works with all possible BCs.
2. the quasi uniformity can be relaxed.

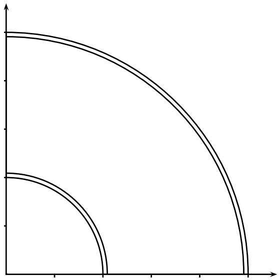
Numerics for R-M

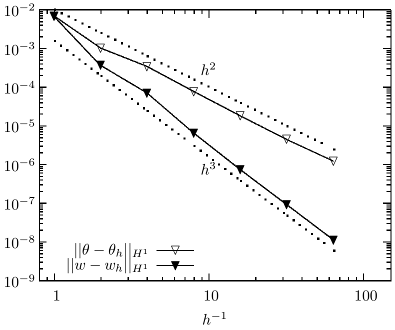
On a square, fully clamped, $t = 10^{-3}$, regular solution



Numerics for R-M

Solution with boundary layer at curved sides (simply supported BCs), $t = 10^{-2}$, $p = 3$





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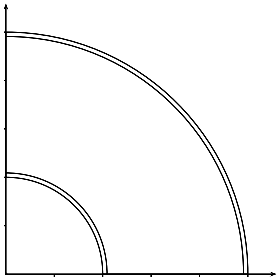
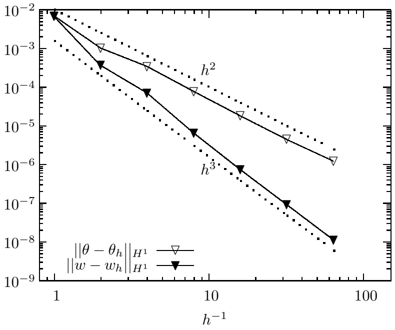
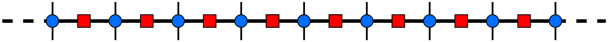




It works very well...

The Spline Complex on non-tensor product mesh

The Spline Complex on non-tensor product mesh



- Cubic B-spline anchors, or odd B-splines anchors;
- Quadratic B-spline anchors, or even B-splines anchors.

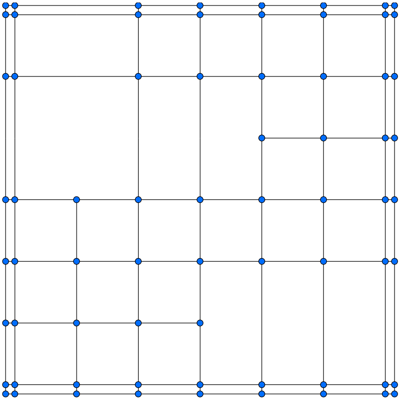
The Spline Complex on non-tensor product mesh

- Cubic B-spline anchors, or odd B-splines anchors;
- Quadratic B-spline anchors, or even B-splines anchors.

$$B'_{i,3}(\zeta) = \frac{p}{\xi_{i+3} - \xi_i} B_{i,2}(\zeta) - \frac{p}{\xi_{i+3+1} - \xi_{i+1}} B_{i+1,2}(\zeta)$$

The Spline Complex on non-tensor product mesh

Cubic T-splines



The Spline Complex on non-tensor product mesh

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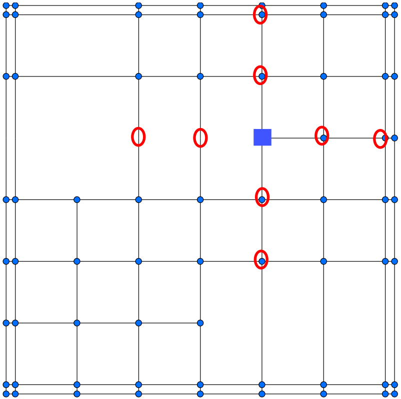
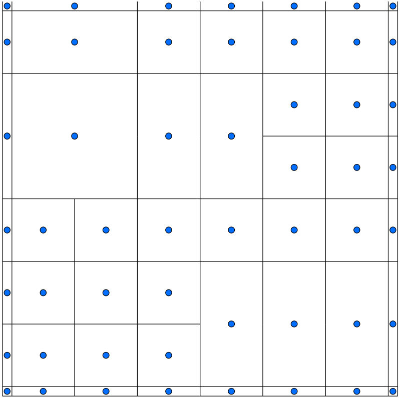


The Spline Complex on non-tensor product mesh

Quadratic T-splines



The Spline Complex in 2D

$$
\begin{array}{ccccc}
H^1(\widehat{\Omega}) & \xrightarrow{\nabla} & H(\mathrm{curl}, \widehat{\Omega}) & \xrightarrow{\mathrm{curl}} & L^2(\widehat{\Omega}) \\
\Pi^0 \downarrow & & \Pi^1 \downarrow & & \Pi^2 \downarrow \\
\widehat{\Lambda}_h^0 & \xrightarrow{\nabla} & \widehat{\Lambda}_h^1 & \xrightarrow{\mathrm{curl}} & \widehat{\Lambda}_h^2
\end{array}
$$

- $\widehat{\Lambda}_h^0 = \widehat{S}^{p,p}$
- $\widehat{\Lambda}_h^1 = (\widehat{S}^{p-1,p}, \widehat{S}^{p,p-1})$
- $\widehat{\Lambda}_h^2 = \widehat{S}^{p-1,p-1}$

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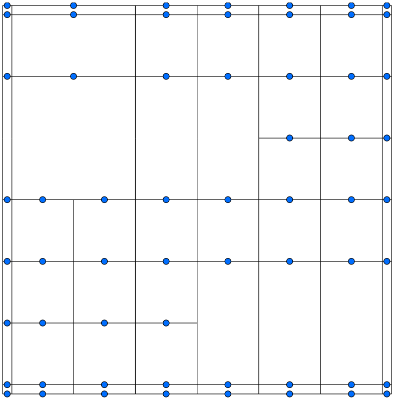
- $\widehat{\Lambda}_h^2 = \widehat{S}^{p-1,p-1}$

Let us fix, $p = 3$.
We need to know details for $S^{3,3}$, $S^{2,2}$... but also $S^{2,3}$ and $S^{3,2}$!

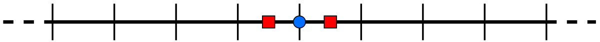
The Spline Complex on non-tensor product mesh

Quadratic-cubic T-splines

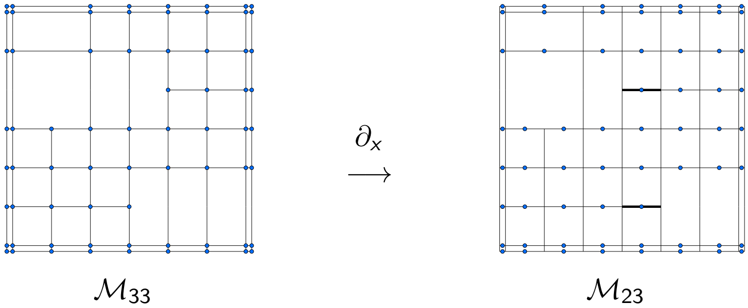


Derivatives of B-splines

$$B'_{i,3}(\zeta) = \frac{p}{\xi_{i+3} - \xi_i} B_{i,2}(\zeta) - \frac{p}{\xi_{i+3+1} - \xi_{i+1}} B_{i+1,2}(\zeta)$$



Implies the following:



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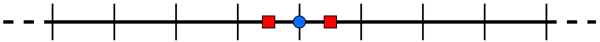

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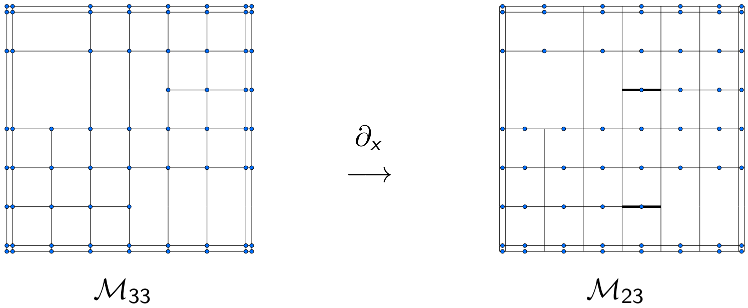
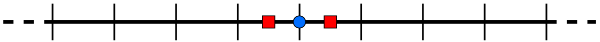


A tangent vector field basis function associated to each horizontal edges

Derivatives of B-splines

$$B'_{i,3}(\zeta) = \frac{p}{\xi_{i+2} - \xi_i} B_{i,2}(\zeta) - \frac{p}{\xi_{i+2+1} - \xi_{i+1}} B_{i+1,2}(\zeta)$$



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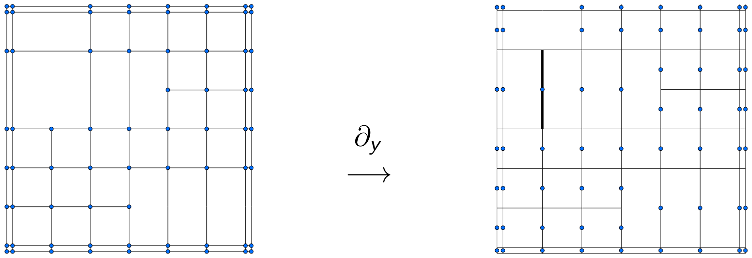

$\mathcal{M}_{33}$ $\xrightarrow{\partial_y}$ $\mathcal{M}_{32}$

Derivatives of B-splines

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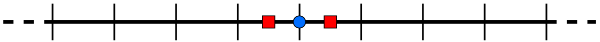

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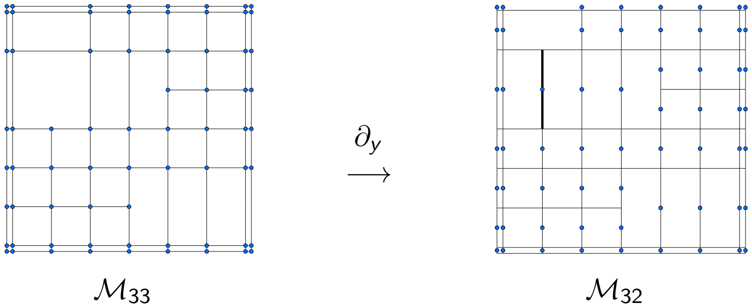

A tangent vector field basis function associated to each vertical edges

Derivatives of B-splines

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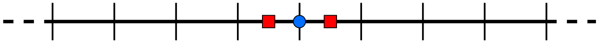

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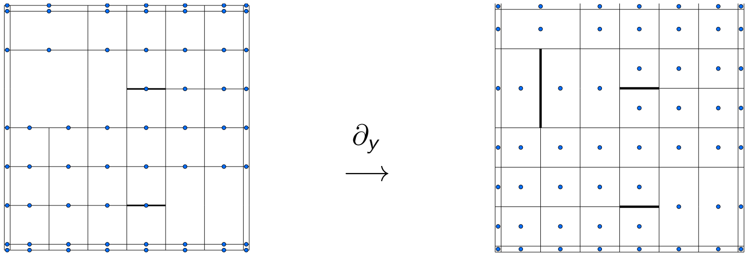
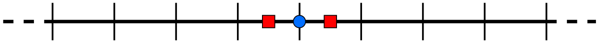


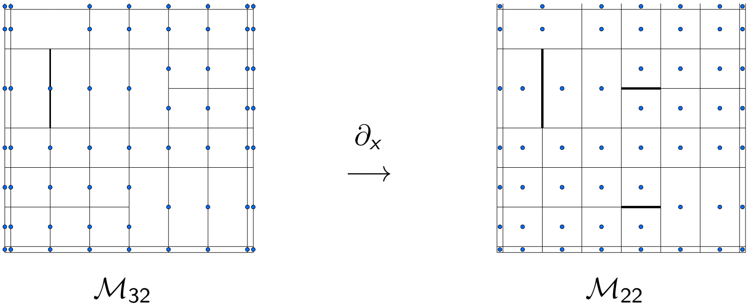
$\mathcal{M}_{23}$ $\xrightarrow{\partial_y}$ $\mathcal{M}_{22}$

Derivatives of B-splines

$$B'_{i,3}(\zeta) = \frac{p}{\xi_{i+2} - \xi_i} B_{i,2}(\zeta) - \frac{p}{\xi_{i+2+1} - \xi_{i+1}} B_{i+1,2}(\zeta)$$



Implies the following:



T-simple complex

B., G. Sangalli, R. Vazquez, 2012

$$\mathbb{R} \rightarrow \widehat{\Lambda}_h^0(\mathcal{M}_{33}) \xrightarrow{\nabla} \widehat{\Lambda}_h^1(\mathcal{M}_{23}, \mathcal{M}_{32}) \xrightarrow{\text{curl}} \widehat{\Lambda}_h^2(\mathcal{M}_{22}) \rightarrow 0$$

- $\widehat{\Lambda}_h^0 = \widehat{S}^{p,p}(\mathcal{M}_{33})$
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Theorem

The sequence is exact for all **Analysis Suitable** T-meshes

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Theorem

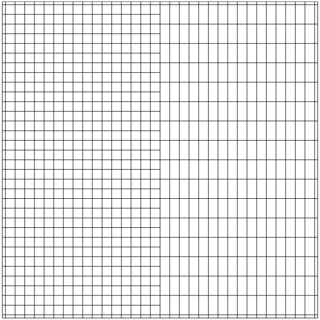
The sequence is exact for all **Analysis Suitable** T-meshes

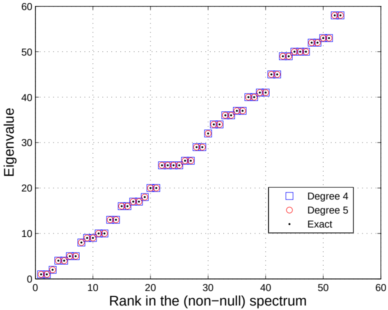
- Exactness of the discrete diagram relies upon polynomial characterization.
- Construction of suitable **commuting** projectors is an open question.
- Practical side: the spaces are all defined **on the same Bézier mesh** !

The T-spline complex: numerics

B Sangalli Vazquez 2012

$$\mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = k^2\mathbf{u} \qquad k \neq 0\,,\ \mathbf{u} \neq 0$$





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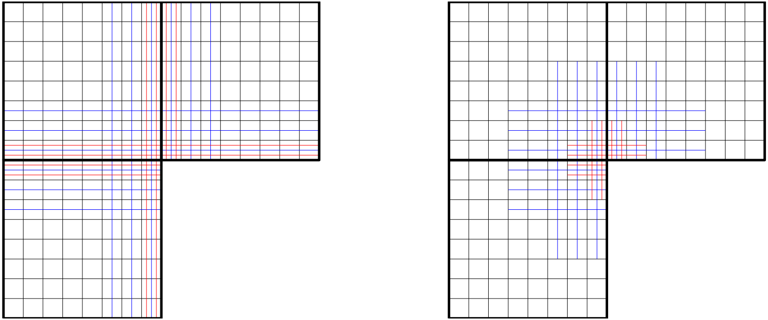

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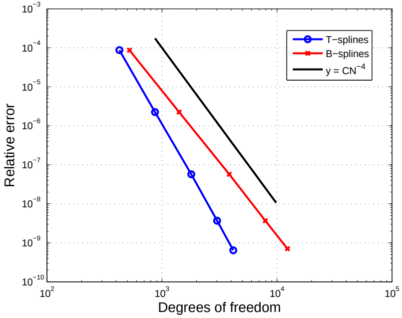

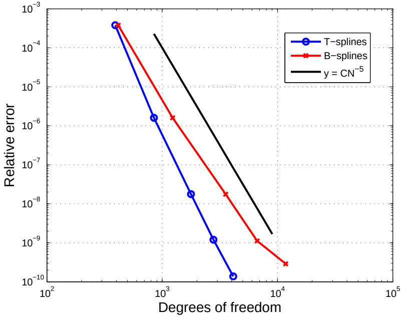

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Ask to either me or Giancarlo for further information