On Local refinement for IGA

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Outline

1. Isogeometric Analysis: Basics

2. IGA: local refinement
   - T-mesh, spline space and dimension
   - spline with hierarchical structure
   - T-splines and Analysis-suitable T-splines

Outline

B-splines

Knot vector $\Xi = \{u_0, u_1, \ldots, u_m\}$;
B-splines:

$$N_i^0 = \begin{cases} 1, & u_i \leq u < n_{i+1}; \\ 0, & \text{otherwise.} \end{cases}$$

$$N_i^p(u) = \frac{u - u_i}{u_{i+p} - u_i} N_i^{p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1}^{p-1}(u).$$

Tensor-product NURBS:

$$R_{i,j}(u,v) = \frac{N_i^p(u) N_j^q(v) \omega_{i,j}}{\sum_{i,j} N_i^p(u) N_j^q(v) \omega_{i,j}}.$$

Geometric map: $F(u,v) = \sum C_{i,j} R_{i,j}(u,v)$,

What is IGA

IGA [T. Hughes etal, 2005]

Use the spline basis $R_{i,j} \circ F^{-1}$ which describes the geometry as a basis for Galerkin projection.

- no meshing;
- exact geometric description at all refinement levels;
- high order smooth functions;
- easy mesh refinement;
- global refinement;

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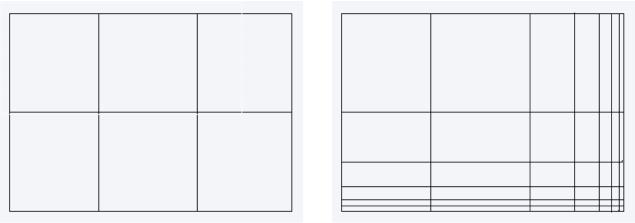
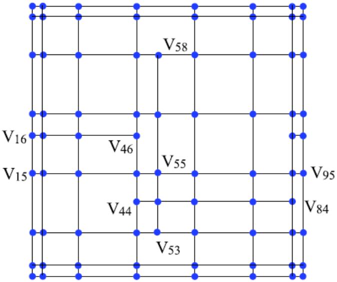
Tensor-produce mesh

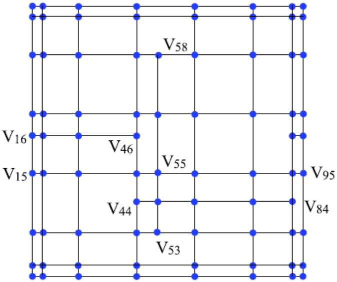
- The knot vectors define a rectangle tiling;
- Refinement where you really don't need it





Bi-degree: (d_1, d_2), Smoothness: α, β:

$$\mathcal{S}(d_1, d_2, \alpha, \beta, \mathcal{T}) := \left\{ f(x, y) \in C^{\alpha,\beta}(\Omega) \middle| f|_\phi \in \mathrm{P}_{d_1 d_2}, \forall \phi \in \mathcal{F} \right\},$$



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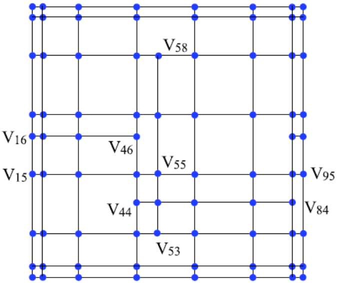
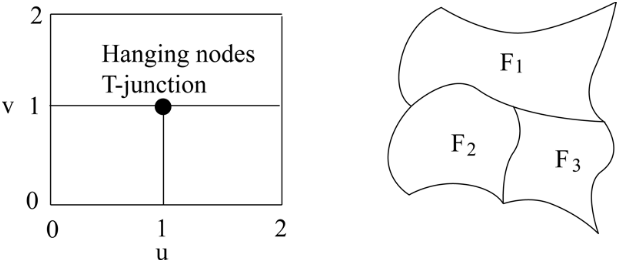
Understand the space



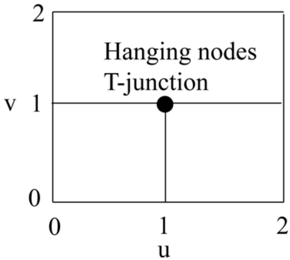
- Linear space;
- Dimension?
- Basis functions?

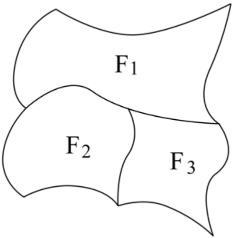
T-junction



- Three maps $F_i(u, v)$ meat at a T-junction (hanging node)

T-junction

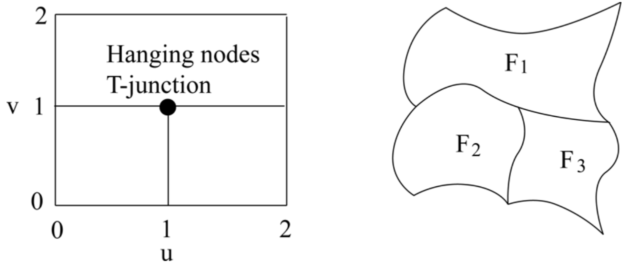




C^0 continuity:

$$F_1(u,1) = \begin{cases} F_2(u,1), & 0 \le u < 1; \\ F_3(u,1), & 1 \le u < 2. \end{cases} \quad (1)$$

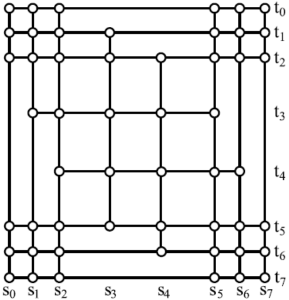
T-junction



C^1 continuity:

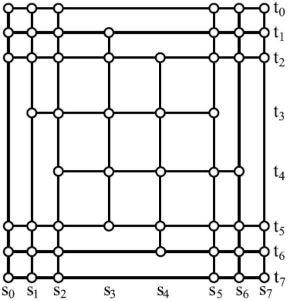
$$F_{1,v}(u,1) = \begin{cases} F_{2,v}(u,1), & 0 \le u < 1; \\ F_{3,v}(u,1), & 1 \le u < 2. \end{cases} \tag{2}$$

Dimension: why is hard?



- Bi-cubic with C^2, knots: $s_i = i$ and $t_i = i$
- Dimension = 65.

Dimension: why is hard?



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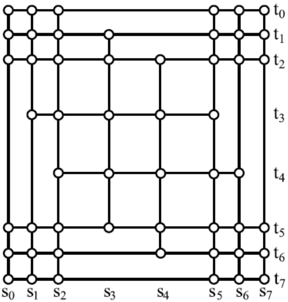
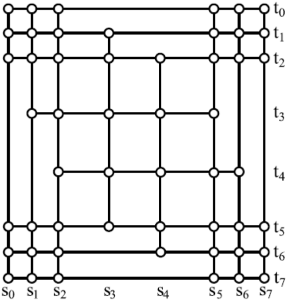
Instability [X. Li and F. Chen, 2011]



- Bi-cubic with C^2, knots: $s_i = i$ and $t_i = i$ except $s_3 = 2.999999$.
- Dimension = 64.

Instability [X. Li and F. Chen, 2011]



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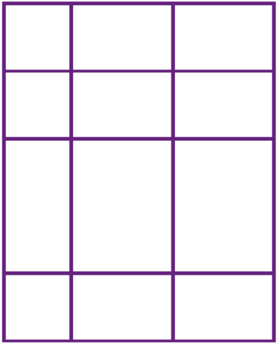
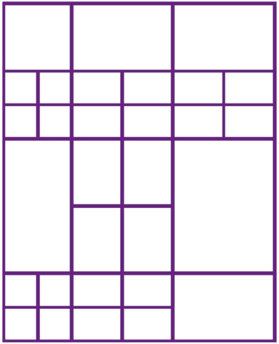
Outline

Hierarchical T-mesh



Hierarchical T-mesh



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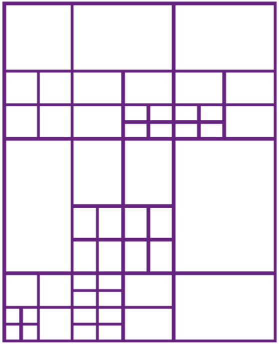
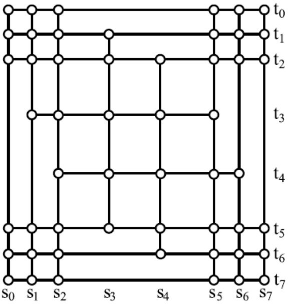


Reduce Regularity [J. Deng etal 2008]



- In case of $d_1 \geq 2\alpha + 1$ and $d_2 \geq 2\beta + 1$,
- Dimension $= 4 \times (V^+ + V^b)$ for $\mathcal{S}(3, 3, 1, 1, \mathcal{T})$.
- 176 in the above example.

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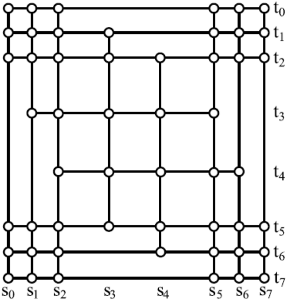

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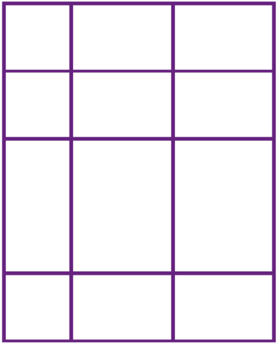
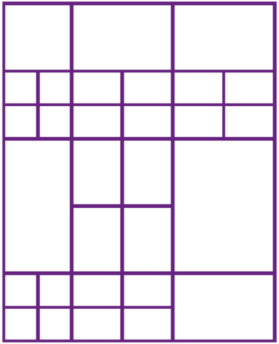
Basis functions



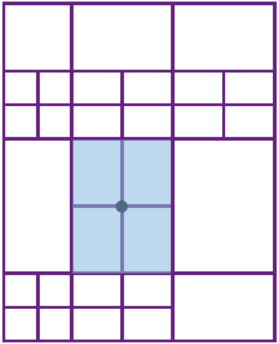
- Each knots has multiple two.
- Each vertex corresponds four basis functions.

Basis functions



- Origin cross vertices are kept;

Basis functions

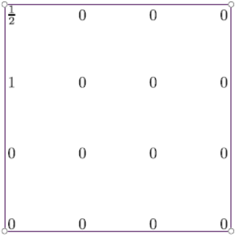


- Origin cross vertices are kept;
- Each vertex is a center of a rectangle and corresponds to four basis function.

Basis functions

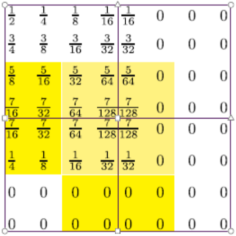
- Linear independency: Trivial;
- Positivity: trivial;
- perfect local refinement;
- Positivity;
- affine invariance: No;

Basis functions



- Origin basis function;

Basis functions



- Redefine the origin basis function in the new T-mesh;

Basis functions

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	0
$\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{16}$	$\frac{3}{32}$	$\frac{3}{32}$	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

- Modify the basis function;

- Linear independency;
- Positivity;
- perfect local refinement;
- Positivity;
- affine invariance;
- reduce regularity: more DOF.

B-splines

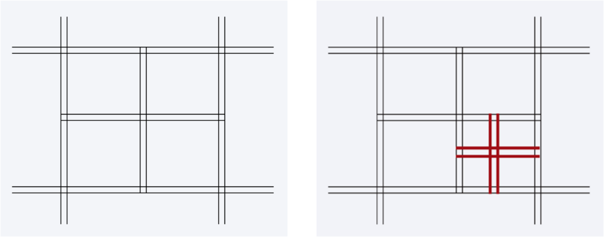
B-splines:

$$N_i^0 = \begin{cases} 1, & u_i \leq u < n_{i+1}; \\ 0, & \text{otherwise.} \end{cases}$$

$$N_i^p(u) = \frac{u - u_i}{u_{i+p} - u_i} N_i^{p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1}^{p-1}(u).$$

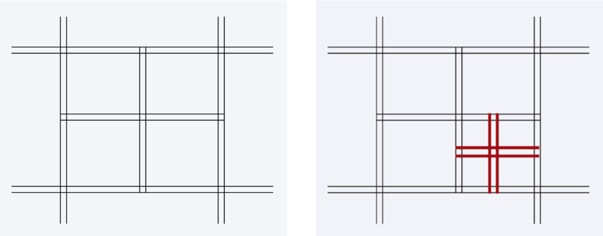
$N_i^p(u)$ is only associated with knots $u_i, u_{i+1}, \ldots, u_{i+p+1}$, i.e., $N[u_i, u_{i+1}, \ldots, u_{i+p+1}]$.
B-spline refinement:

$$N[u_i, u_{i+1}, \ldots, u_{i+p+1}] = \frac{u_* - u_i}{u_{i+p} - u_i} N[u_i, \ldots, u_*, \ldots, u_{i+p}] + \frac{u_{i+p+1} - u_*}{u_{i+p+1} - u_{i+1}} N[u_{i+1}, \ldots, u_*, \ldots, u_{i+p+1}].$$



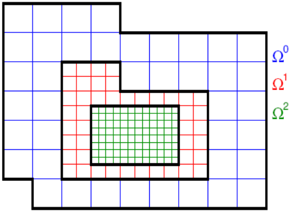
- For one basis function $N = N[0,1,1,2,2] \times N[0,0,1,1,2]$
- Four new basis functions:

$$N_1 = N[1,1,1.5,1.5,2] \times N[0,0,0.5,0.5,1],$$
$$N_2 = N[1,1,1.5,1.5,2] \times N[0,0.5,0.5,1,1],$$
$$N_3 = N[1,1.5,1.5,2,2] \times N[0,0,0.5,0.5,1],$$
$$N_4 = N[1,1.5,1.5,2,2] \times N[0,0.5,0.5,1,1]$$

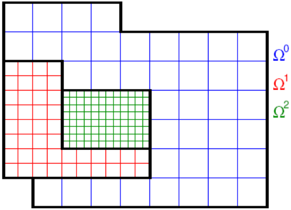


- Origin basis: $N = \frac{5}{32}N_1 + \frac{25}{64}N_2 + \frac{1}{16}N_3 + \frac{5}{32}N_4 + \ldots,$
- Modified basis: $\hat{N} = N - (\frac{5}{32}N_1 + \frac{25}{64}N_2 + \frac{1}{16}N_3 + \frac{5}{32}N_4).$

Hierarchical B-splines [A.-V. Vuong, C. Giannelli, etal. 2012]



- Nested domains;
- Each domain spanned by B-splines;
- Nested spaces;



- Strict nested domains to make linear independency;

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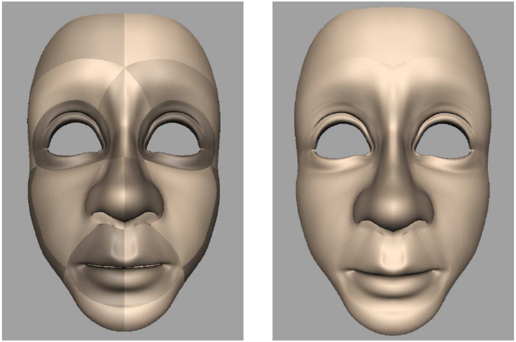
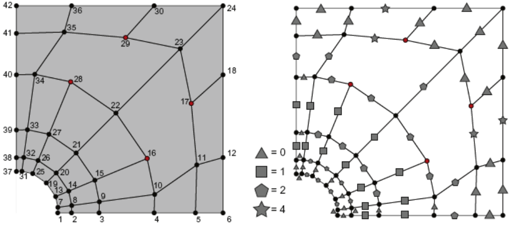
Merging



- T-junction cannot be avoided;
- Using the local information;
- Nested structure is not acceptable.

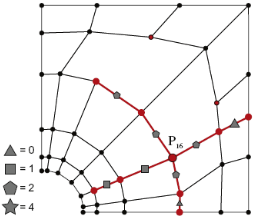
T-splines

Sederberg, et al, 2003, Sederberg, Cardon, Finnigan, North, Zheng, Lyche, 2004



- A T-grid and a valid knot interval configuration;

T-splines



- $T(\xi) = \sum_{i=1}^{n} C_i T_i(\xi),;$
- $C_i = (x_i, y_i, z_i)$ or $C_i = (\omega_i x_i, \omega_i y_i, \omega_i z_i, \omega_i)$.

T-splines

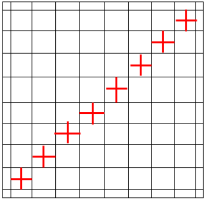
T-splines are attractive:

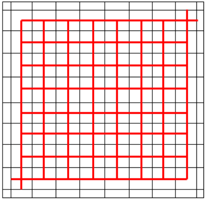
- NURBS compatible;
- efficient local refinement;
- trimmed NURBS to T-splines conversion: watertight;
- patch gluing : an example;

T-splines

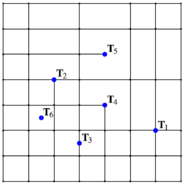
However,

- Linear independence;
- Possible severe fill-in of the T-mesh
- No well characterization

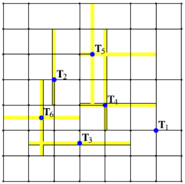




Analysis-suitable T-splines [X. Li etal 2012]



(a) T-mesh.



(b) Extensions.

Definition

- An AS T-mesh is one on which no horizontal T-junction extension intersects a vertical T-junction extension;
- AS T-splines are defined on AS T-meshes

Analysis-suitable T-splines

- Positivity;
- Linear independence;
- Affine invariance: $\sum_{i=1}^{n} T_i(\xi) = 1$;
- Convex hull;
- Relatively local refinement (numerics);
- Characterization in terms of piecewise polynomials;
- Projections and approximation properties.
  1. Dual basis [Veiga, Buffa, and Sangalli, 2012];
  2. Characterization.

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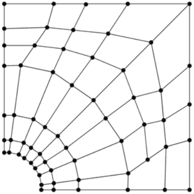
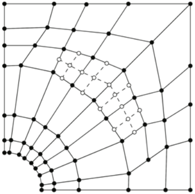
Local refinement [M. A. Scott, X. Li, etal 2012]

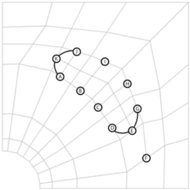
- Direct insert the control points and make the finial T-mesh to be analysis-suitable;
- Use greedy strategy to find analysis-suitable refinement



Initial T-mesh

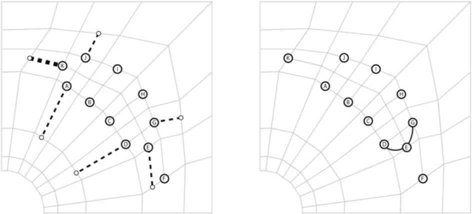


Initial Refinement

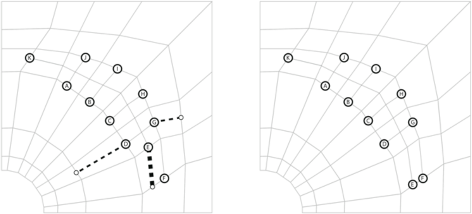


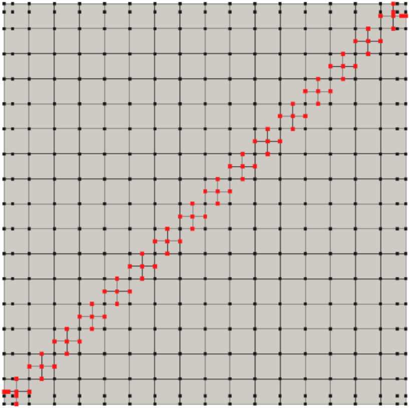
Not analysis-suitable

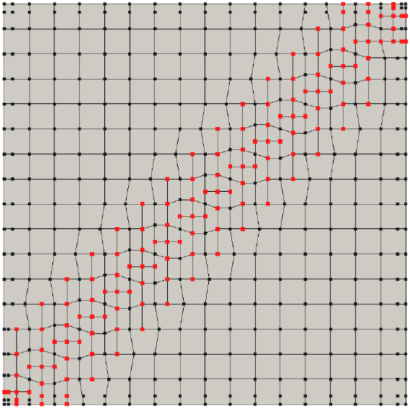
Iteration 1

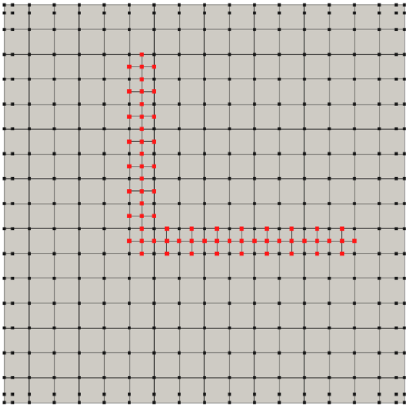


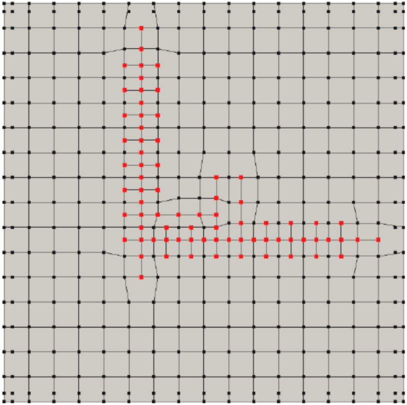
Iteration 2











Characterization

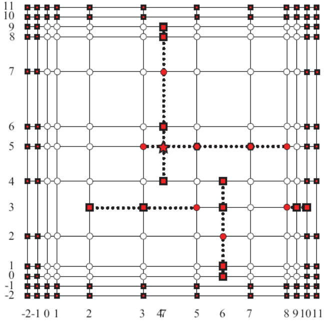
T-spline space

$$\mathcal{T} = \left\{ f(\xi) \in L^2(\Omega) \mid f = \sum c_i T_i(\xi), c_i \in \mathbb{R} \right\} \tag{3}$$

where $L^2(\Omega)$ is the space of square integrable functions over Ω.

Characterization

If the extended T-mesh of an analysis-suitable T-mesh T has **no overlap vertices and no zero knot intervals**, then the associated T-splines space $\mathcal{T}$ and $\mathcal{S}(d_1, d_2, d_1 - 1, d_2 - 1, T_{ext})$ are identical spline spaces.

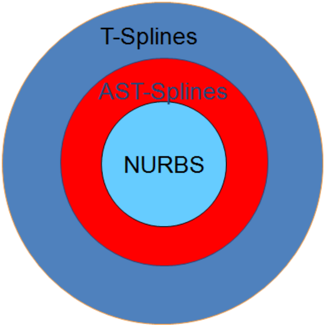


Characterization

The space of bi-degree (d_1, d_2) polynomials lives in the associated T-spline space $\mathcal{T}$ for an analysis-suitable T-mesh T.

Analysis-suitable T-splines

- ✓ Watertight
- ✓ Affine covariance
- ✓ Trimless option
- ✓ Simply implemented in FEA codes
- ✓ Used in design
- ✓ Backwards compatible with NURBS
- ✓ Exact representation of conic sections
- ✓ Higher-order smoothness
- ✓ Linearly independent
- ✓ Partition of unity property
- ✓ Optimized locally refineable
- ✓ Characterization



Summary

- We introduce PHT:
  - perfect local refinement,
  - reduced regularity.
- We introduce analysis-suitable T-splines:
  - NURBS-compatible design-though-analysis technology.
  - Arbitrary topological analysis-suitable T-splines (in progress);
  - Gauss quadrature rules for analysis-suitable T-splines.

Thank you for your attention!

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