On Local refinement for IGA

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Outline

1. Isogeometric Analysis: Basics

2. IGA: local refinement
   - T-mesh, spline space and dimension
   - spline with hierarchical structure
   - T-splines and Analysis-suitable T-splines
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B-splines

Knot vector $\Xi = \{u_0, u_1, \ldots, u_m\}$;

B-splines:

$$N_i^0(u) = \begin{cases} 1, & u_i \leq u < n_i + 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$N_i^p(u) = \frac{u - u_i}{u_{i+p} - u_i} N_i^{p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1}^{p-1}(u).$$

Tensor-product NURBS:

$$R_{i,j}(u, v) = \frac{N_i^p(u)N_j^q(v)\omega_{i,j}}{\sum_{i,j} N_i^p(u)N_j^q(v)\omega_{i,j}}.$$

Geometric map: $F(u, v) = \sum C_{i,j}R_{i,j}(u, v)$,
What is IGA

IGA [T. Hughes et al., 2005]

Use the spline basis $R_{i,j} \circ F^{-1}$ which describes the geometry as a basis for Galerkin projection.

- no meshing;
- exact geometric description at all refinement levels;
- high order smooth functions;
- easy mesh refinement;
- global refinement;
What is IGA

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- no meshing;
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- high order smooth functions;
- easy mesh refinement;
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On Local refinement for IGA
Tensor-produce mesh

- The knot vectors define a rectangle tiling;
- Refinement where you really don’t need it
Bi-degree: \((d_1, d_2)\), Smoothness: \(\alpha, \beta\):

\[
S(d_1, d_2, \alpha, \beta, T) := \left\{ f(x, y) \in C^{\alpha, \beta}(\Omega) \mid f|_\phi \in P_{d_1 d_2}, \forall \phi \in \mathcal{F} \right\},
\]
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\]
Understand the space

- Linear space;
- Dimension?
- Basis functions?
T-junction

Three maps $F_i(u, v)$ meet at a T-junction (hanging node)
$C^0$ continuity:

$$F_1(u, 1) = \begin{cases} F_2(u, 1), & 0 \leq u < 1; \\ F_3(u, 1), & 1 \leq u < 2. \end{cases}$$ (1)
T-junction

\[ F_{1,v}(u, 1) = \begin{cases} 
  F_{2,v}(u, 1), & 0 \leq u < 1; \\
  F_{3,v}(u, 1), & 1 \leq u < 2.
\end{cases} \]
Dimension: why is hard?

- Bi-cubic with $C^2$, knots: $s_i = i$ and $t_i = i$
- Dimension = 65.
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Instability [X. Li and F. Chen, 2011]

- Bi-cubic with $C^2$, knots: $s_i = i$ and $t_i = i$ except $s_3 = 2.999999$.
- Dimension = 64.
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Hierarchical T-mesh
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Reduce Regularity [J. Deng et al 2008]

In case of $d_1 \geq 2\alpha + 1$ and $d_2 \geq 2\beta + 1$,

- Dimension $= 4 \times (V^+ + V^b)$ for $S(3, 3, 1, 1, T)$.
- 176 in the above example.
In case of $d_1 \geq 2\alpha + 1$ and $d_2 \geq 2\beta + 1$,

Dimension = $4 \times (V^+ + V^b)$ for $S(3, 3, 1, 1, T)$.

176 in the above example.
Basis functions

- Each knots has multiple two.
- Each vertex corresponds four basis functions.
Origin cross vertices are kept;
Basis functions

• Origin cross vertices are kept;
• Each vertex is a center of a rectangle and corresponds to four basis function.
Basis functions

- Linear independency: Trivial;
- Positivity: trivial;
- perfect local refinement;
- Positivity;
- affine invariance: No;
Basis functions

- Origin basis function;
Redefine the origin basis function in the new T-mesh;
Basis functions

- Modify the basis function;

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- Linear independency;
- Positivity;
- perfect local refinement;
- Positivity;
- affine invariance;
- reduce regularity: more DOF.
B-splines:

\[
N_i^0 = \begin{cases} 
1, & u_i \leq u < n_{i+1}; \\
0, & \text{otherwise.}
\end{cases}
\]

\[
N_i^p(u) = \frac{u - u_i}{u_{i+p} - u_i} N_i^{p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1}^{p-1}(u).
\]

\(N_i^p(u)\) is only associated with knots \(u_i, u_{i+1}, \ldots, u_{i+p+1}\), i.e., \(N[u_i, u_{i+1}, \ldots, u_{i+p+1}]\).

B-spline refinement:

\[
N[u_i, u_{i+1}, \ldots, u_{i+p+1}] = \frac{u_* - u_i}{u_{i+p} - u_i} N[u_i, \ldots, u_*, \ldots, u_{i+p}] + \frac{u_{i+p+1} - u_*}{u_{i+p+1} - u_{i+1}} N[u_{i+1}, \ldots, u_*, \ldots, u_{i+p+1}].
\]
For one basis function $N = N[0, 1, 1, 2, 2] \times N[0, 0, 1, 1, 2]$

Four new basis functions:

- $N_1 = N[1, 1, 1.5, 1.5, 2] \times N[0, 0, 0.5, 0.5, 1]$
- $N_2 = N[1, 1, 1.5, 1.5, 2] \times N[0, 0.5, 0.5, 1, 1]$
- $N_3 = N[1, 1.5, 1.5, 2, 2] \times N[0, 0, 0.5, 0.5, 1]$
- $N_4 = N[1, 1.5, 1.5, 2, 2] \times N[0, 0.5, 0.5, 1, 1]$
Origin basis: $N = \frac{5}{32} N_1 + \frac{25}{64} N_2 + \frac{1}{16} N_3 + \frac{5}{32} N_4 + \ldots$,

Modified basis: $\hat{N} = N - (\frac{5}{32} N_1 + \frac{25}{64} N_2 + \frac{1}{16} N_3 + \frac{5}{32} N_4)$. 
Hierarchical B-splines [A.-V. Vuong, C. Giannelli, etal. 2012]

- Nested domains;
- Each domain spanned by B-splines;
- Nested spaces;
Strict nested domains to make linear independency;
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Merging

- T-junction cannot be avoided;
- Using the local information;
- Nested structure is not acceptable.

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T-splines


- A T-grid and a valid knot interval configuration;
T-splines

\[ T(\xi) = \sum_{i=1}^{n} C_i T_i(\xi), \]
\[ C_i = (x_i, y_i, z_i) \text{ or } C_i = (\omega_i x_i, \omega_i y_i, \omega_i z_i, \omega_i). \]
T-splines

T-splines are attractive:

- NURBS compatible;
- efficient local refinement;
- trimmed NURBS to T-splines conversion: watertight;
- patch gluing: an example;
T-splines

However,

- Linear independence;
- Possible severe fill-in of the T-mesh
- No well characterization
Analysis-suitable T-splines [X. Li et al. 2012]

Definition

- An AS T-mesh is one on which no horizontal T-junction extension intersects a vertical T-junction extension;
- AS T-splines are defined on AS T-meshes
Analysis-suitable T-splines

- Positivity;
- Linear independence;
- Affine invariance: $\sum_{i=1}^{n} T_i(\xi) = 1$;
- Convex hull;
- Relatively local refinement (numerics);
- Characterization in terms of piecewise polynomials;
- Projections and approximation properties.
  1. Dual basis [Veiga, Buffa, and Sangalli, 2012];
  2. Characterization.
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Local refinement [M. A. Scott, X. Li, et al. 2012]

- Direct insert the control points and make the final T-mesh to be analysis-suitable;
- Use greedy strategy to find analysis-suitable refinement

Initial T-mesh  |  Initial Refinement  |  Not analysis-suitable
Isogeometric Analysis: Basics
IGA: local refinement
Summary
T-mesh, spline space and dimension
spline with hierarchical structure
T-splines and Analysis-suitable T-splines

Iteration 1

Iteration 2
Isogeometric Analysis: Basics
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Characterization

T-spline space

\[ T = \left\{ f(\xi) \in L^2(\Omega) \mid f = \sum c_i T_i(\xi), c_i \in \mathbb{R} \right\} \]  \hspace{1cm} (3)

where \( L^2(\Omega) \) is the space of square integrable functions over \( \Omega \).

Characterization

If the extended T-mesh of an analysis-suitable T-mesh \( T \) has no overlap vertices and no zero knot intervals, then the associated T-splines space \( T \) and \( S(d_1, d_2, d_1 - 1, d_2 - 1, T_{ext}) \) are identical spline spaces.
Characterization

The space of bi-degree \((d_1, d_2)\) polynomials lives in the associated T-spline space \(\mathcal{T}\) for an analysis-suitable T-mesh \(T\).
Analysis-suitable T-splines

- Watertight
- Affine covariance
- Trimless option
- Simply implemented in FEA codes
- Used in design
- Backwards compatible with NURBS
- Exact representation of conic sections
- Higher-order smoothness
- Linearly independent
- Partition of unity property
- Optimized locally refineable
- Characterization
Summary

- We introduce PHT:
  - perfect local refinement,
  - reduced regularity.
- We introduce analysis-suitable T-splines:
  - NURBS-compatible design-though-analysis technology.
  - Arbitrary topological analysis-suitable T-splines (in progress);
  - Gauss quadrature rules for analysis-suitable T-splines.
Thank you for your attention!
X. Li, F. Chen, On the instability in the dimension of spline space over particular t-meshes, CAGD, 28, 420-426, 2011.


T. J. R. Hughes, J. A. Cottrell, Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry, and mesh refinement, CMAME, 194, 4135-4195, 2005.


