Parabolic Quasi-Variational Inequality with Gradient Constraints

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Abstract. We know many mechanical phenomena whose dynamics is described by a class of quasi-variational inequalities of the parabolic type. Our system consists of a second-order parabolic variational inequality with gradient constraint depending on the temperature, coupled with the heat equation:

\[
\begin{align*}
  u_t - \nu \Delta u + \partial I_{K_t}(u) + g(u) &\ni f \quad \text{in } Q, \\
  \theta_t - \kappa \Delta \theta + h(x,t,u) & = 0 \quad \text{in } Q, \\
  \frac{\partial u}{\partial n} & = 0, \quad \theta = 0 \quad \text{on } \Sigma, \\
  u(\cdot, 0) & = u_0, \quad \theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega.
\end{align*}
\]

Here, \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) with \( 1 \leq N \leq 3 \), \( \Gamma := \partial \Omega \), \( Q := \Omega \times (0, T) \) with \( 0 < T < \infty \) and \( \Sigma := \Gamma \times (0, T) \); \( \nu \) and \( \kappa \) are positive constants, \( K_t(\cdot) \) is the closed convex set in \( H^1(\Omega) \) defined by

\[
K_t(\cdot) := \{ v \in H^1(\Omega); \quad |\nabla v| \leq \psi(\theta(\cdot, t)) \text{ a.e. on } \Omega \}
\]

for a given positive and smooth continuous function \( \psi(\cdot) \) on \( \mathbb{R} \). Further, \( I_{K_t}(\cdot) \) is the indicator function of \( K_t(\cdot) \) in \( L^2(\Omega) \) and \( \partial I_{K_t}(\cdot) \) is its subdifferential. Also, \( h(x,t,u) \) is a smooth function on \( \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \), \( g(\cdot) \) is a smooth function on \( \mathbb{R} \), and \( f \) is a function given in \( L^2(Q) \) and \( u_0 \in H^1(\Omega) \) with \( |\nabla u_0| \leq \psi(\theta_0) \) as well as \( \theta_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) given as data.

Since the temperature is unknown in our problem and should be determined as a part of the solution, the constraint function is unknown as well. In this sense, our problem includes the quasi-variational structure, and in the mathematical treatment one of main difficulties comes from it. Our approach to the problem is based on the abstract theory of quasi-variational inequalities with non-local constraint which evolved in a paper in Banach Center Pub.(Vol.86,pp.175-194,2009) by Kano-Kenmochi-Murase. In this talk, we prove the existence of weak solutions in higher space dimensions and the existence of strong solutions in one space dimension.