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Existence and homogenization for the problem $-\operatorname{div} a(x, Du) = f$ with $a(x, \xi)$ a maximal monotone graph in ξ for every x given

In this lecture I will report on recent joint work [2], [3] with Gilles Francfort and Luc Tartar on the existence of a solution for some nonlinear problem and the homogenization of it.

We consider the problem of finding u such that

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ -\operatorname{div} a(x, Du) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{1}$$

for a given $f \in W^{-1,p'}(\Omega)$. Here Ω is a bounded open set of $\mathbb{R}^N$, p is a real number with $1 < p < +\infty$, and $p' = p/(p-1)$.

The classical setting is the case where $a : (x, \xi) \in \Omega \times \mathbb{R}^N \to a(x, \xi) \in \mathbb{R}^N$ is a Carathéodory function (i.e. a single-valued function, continuous in ξ for almost every $x \in \Omega$, and measurable in x for every $\xi \in \mathbb{R}^N$), which is monotone, i.e. satisfies, for almost every $x \in \Omega$ and for every $\xi_1 \in \mathbb{R}^N$ and $\xi_2 \in \mathbb{R}^N$

$$(a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2) \geq 0, \tag{2}$$

and which further satisfies for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$

$$\begin{cases} a(x, \xi)\xi \geq \alpha |\xi|^p - |a(x)|, \\ a(x, \xi)\xi \geq \beta |a(x, \xi)|^{p'} - |b(x)|, \\ a(x, 0) = 0, \end{cases} \tag{3}$$

for some $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ (the second assertion of (3) is another way to state the classical growth condition $|a(x, \xi)| \leq \gamma |\xi|^{p-1} + |h(x)|$ for some $\gamma > 0$ and some $h \in L^{p'}(\Omega)$). In such a classical setting it is well known that there exists a solution to (1).

In our work we investigate the case where for $x \in \Omega$ given, $\xi \to a(x,\xi)$ is no more a single-valued monotone continuous function from $\mathbb{R}^N$ into $\mathbb{R}^N$, but actually a multi-valued maximal monotone graph of $\mathbb{R}^N \times \mathbb{R}^N$. In such a case, the crucial issues are on the one hand the definition of an adequate setting, in particular for what concerns measurability properties of the graph which will replace $a(x,\xi)$, on the other hand the choice of a proper approximation procedure of this graph allowing one to prove an existence result, and finally the statement and proof of results on maximal monotone graphs allowing one to perform the homogenization of the problem.

The literature on these topics is restricted to the paper [1] by V. Chiadò Piat, G. Dal Maso & A. Defranceschi, in which delicate measurability assumptions are made in the definition of the graph and delicate measurability selection theorems are used in the proofs. In our paper we provide a definition which can be proved to be equivalent to their, but which in our opinion provides a simpler framework, and in the proofs we use only classical theorems of single-valued analysis.

As far as existence is concerned, we prove the existence of a function u and of a vector field d such that

$$(4) \qquad \begin{cases} u \in W_0^{1,p}(\Omega),\ d \in L^{p'}(\Omega)^N, \\ e = Du,\ -\operatorname{div} d = f \quad \text{in } \mathcal{D}'(\Omega), \\ (e,d) \in \mathcal{A}, \end{cases}$$

when the graph $\mathcal{A} \subset L^p(\Omega)^N \times L^{p'}(\Omega)^N$ is such that for every $(e,d) \in \mathcal{A}$ one has for almost every $x \in \Omega$

$$(5) \qquad \begin{cases} d(x)e(x) \geq \alpha |e(x)|^p - |a(x)|, \\ d(x)e(x) \geq \beta |d(x)|^{p'} - |b(x)|, \\ (0,0) \in \mathcal{A}, \end{cases}$$

for some $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$, and when the graph $\mathcal{A}$ further satisfies one of the the two following equivalent conditions.

– *First condition.*
The graph $\mathcal{A}$ is defined by

$$(6) \qquad (e,d) \in \mathcal{A} \quad \Longleftrightarrow \quad d(x) - e(x) = \phi(x, d(x) + e(x)), \ \text{ a.e. } x \in \Omega,$$

where $\phi : (x,\lambda) \in \Omega \times \mathbb{R}^N \to \phi(x,\lambda) \in \mathbb{R}^N$ is a given (single-valued) Carathéodory function which is defined on the whole of $\Omega \times \mathbb{R}^N$ and satisfies

$$(7) \qquad \begin{cases} x \to \phi(x,\lambda) \text{ is measurable on } \Omega \text{ for every } \lambda \in \mathbb{R}^N, \\ |\phi(x,\lambda_1) - \phi_(x,\lambda_2)| \leq |\lambda_1 - \lambda_2|, \quad \text{a.e. } x \in \Omega,\ \forall \lambda_1, \lambda_2 \in \mathbb{R}^N. \end{cases}$$

– *Second condition.*
The graph $\mathcal{A} \subset L^p(\Omega)^N \times L^{p'}(\Omega)^N$ is pointwise monotone, i.e. satisfies for every $(e_1, d_1) \in \mathcal{A}$ and $(e_2, d_2) \in \mathcal{A}$

$$(d_1(x) - d_2(x))(e_1(x) - e_2(x)) \geq 0, \quad \text{a.e. } x \in \Omega, \tag{8}$$

and is such that for every $g \in L^p(\Omega)^N$ and for every $t > 0$, there exists a unique (e, d) such that

$$\begin{cases} (e, d) \in \mathcal{A}, \\ e(x) + tj(d(x)) = g(x), \quad \text{a.e. } x \in \Omega, \end{cases} \tag{9}$$

where $j : \mathbb{R}^N \to \mathbb{R}^N$ is the function defined by $j(\lambda) = |\lambda|^{p'-2}\lambda$.

Assuming that hypothesis (5) holds, we prove that the two conditions above are equivalent in the following sense. If the graph $\mathcal{A}$ is defined by (6) for some function ϕ which is defined on the whole of $\Omega \times \mathbb{R}^N$ and which satisfies (7), then $\mathcal{A}$ satisfies (8) and (9). Conversely, if the graph $\mathcal{A}$ satisfies (8) and (9), then there exists a fonction ϕ which is defined on the whole of $\Omega \times \mathbb{R}^N$ and which satisfies (7), such that $\mathcal{A}$ is defined by (6).

These two equivalent conditions provide a convenient definition of a "multi-valued maximal monotone graph in $\mathbb{R}^N \times \mathbb{R}^N$ depending on x". The justification of this assertion is twofold. On the one hand, if $\mathcal{A}$ satisfies (8) and (9), it is easy to prove that $\mathcal{A}$ is a maximal monotone graph of $L^p(\Omega)^N \times L^{p'}(\Omega)^N$. The converse is also true, but this is a result of abstract maximal monotone graphs theory which is more difficult to prove. On the other hand, if ϕ is independent of x, is defined on the whole of $\mathbb{R}^N$ and satisfies (7), it is easy to prove that $\mathcal{A}$ defined by (6) is a maximal monotone graph of $\mathbb{R}^N \times \mathbb{R}^N$. The converse is also true, but the proof uses Kirzbraun's theorem.

If the graph $\mathcal{A}$ satisfies (5) and one of the two equivalent conditions given above, we prove in [2] the existence of a solution to (4). We give two different proofs of this existence result, using in each proof a different approximation procedure which relies on one of the two conditions above.

For what concerns homogenization of this problem, we prove in [3] the following compactness result with respect to the $H-$convergence (or in other terms an homogenization result) concerning this class of graphs. From every sequence ε of graphs $\mathcal{A}_\varepsilon$ which satisfy (5) uniformly (i.e. for the same $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$), and which are such that one of the two equivalent conditions given above is satisfied, one can extract a subsequence, still denoted by ε, and there exists a graph $\mathcal{A}_0$ in the same class, such that for every $f \in W^{-1,p'}(\Omega)$, any accumulation point (u, d) (in the weak topology of $W_0^{1,p}(\Omega) \times L^{p'}(\Omega)^N$) of the solutions $(u_\varepsilon, d_\varepsilon)$ to

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega), \ d_\varepsilon \in L^{p'}(\Omega)^N, \\ e_\varepsilon = Du_\varepsilon, \ -\mathrm{div}\, d_\varepsilon = f \quad \text{in } \mathcal{D}'(\Omega), \\ (e_\varepsilon, d_\varepsilon) \in \mathcal{A}_\varepsilon, \end{cases} \tag{10_ε}$$

(observe that u_ε is bounded in $W_0^{1,p}(\Omega)$ and that d_ε is bounded in $L^{p'}(\Omega)^N$), is a solution (u_0, d_0) to (10_0). This provides a new (and in our opinion simpler) proof of the analogous result proved by V. Chiadò Piat, G. Dal Maso & A. Defranceschi in [1].

References

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