On non-periodic homogenization of time-dependent equations

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Abstract: The recently introduced needle problem approach for the homogenization of non-periodic problems was originally designed for the homogenization of elliptic problems. After a short review of the needle problem approach we demonstrate in this note how the stationary results can be transferred to time-dependent problems. The standard parabolic problem of the corresponding heat equation in a heterogeneous material is considered. Furthermore, we include an application to a hysteresis problem which appears in the theory of porous media.

Keywords: non-periodic homogenization; heat equation; hysteresis.

1 Introduction

The aim of this note is to show how general elliptic homogenization results can be transferred to time-dependent problems. More precisely, in [7] we studied the homogenization of the elliptic equation

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= f \qquad \text{in } Q,\\ u^\varepsilon &= \psi \qquad \text{on } \partial Q, \end{aligned} \tag{1.1}$$

where Q is a Lipschitz bounded domain in $\mathbb{R}^n$ and $\psi : Q \to \mathbb{R}$ is affine. Under a weak averaging assumption on the coefficients a^ε, we introduced a new method to prove that solutions u^ε of (1.1) converge to the solution of an elliptic equation with a constant coefficient a^*. In particular, we do not require periodicity or ergodicity of the coefficients a^ε.

We are now interested in how this result can be can be transferred to a parabolic problem of the form

$$\partial_t u^\varepsilon = \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) \qquad \text{in } Q \times (0,T) \tag{1.2}$$

and to the hysteresis problem with a monotone relation H

$$\begin{aligned} \partial_t z^\varepsilon &= \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) \qquad \text{in } Q \times (0,T),\\ \partial_t z^\varepsilon &= -H(z^\varepsilon - u^\varepsilon) \qquad \text{in } Q \times (0,T), \end{aligned} \tag{1.3}$$

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completed with boundary data and initial conditions.

The underlying idea is that *if* existence, uniqueness and suitable a priori estimates are known for the solutions of the time- and ε-dependent problems, then classical compactness arguments allow for a subsequence of solutions to converge to some limit function as $\varepsilon \to 0$. The homogenization of elliptic equations can then be applied to a time-integrated version of the original equations, in order to identify the limit found by compactness to the solution of the homogenized problem.

Regarding the literature on homogenization of elliptic equations, which is immense and still rapidly growing, we refer the reader to the exceptional compendium [10], and to the references therein. We recall the monograph [2], which treats also the case of stochastic coefficients, and [1], regarding the Γ-convergence of stochastic functionals. About homogenization of equations related to the hysteresis problem studied in Section 3.2, we name the analysis done in the periodic case in [6] and in the stochastic case in [5, 11].

In Section 2 we fix the notation, we review the new method of homogenization introduced in [7], and we give the precise statement regarding problem (1.1). In Section 3 we show the application to the time-dependent problems (1.2) and (1.3).

2 The elliptic needle problem in homogenization

We will review in this first section the needle problem approach to the homogenization of elliptic problems. The main feature of this new approach, introduced in [7], is that the assumption on the oscillatory coefficient is very general. We do neither assume periodicity nor a stochastic construction, but only demand that the coefficients satisfy a weak averaging property, namely that *the coefficients allow averaging* in the sense of Definition 2.1.

Notation. In all results below, $Q \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $\mathbb{R}^{n\times n}_{\text{sym}}$ is the vector space of symmetric $n \times n$ matrices on $\mathbb{R}$. We make use of the following spaces of functions on the domain Q. $L^\infty(Q; \mathbb{R}^{n\times n}_{\text{sym}})$ is the space of measurable, essentially bounded, $\mathbb{R}^{n\times n}_{\text{sym}}$-valued functions, $H^1(Q)$ is the Sobolev space of functions $u \in L^2(Q)$ such that $\partial_i u \in L^2(Q)$ for all $i = 1, \dots, n$, and $H^1_0(Q)$ is the closure, with respect to the $H^1(Q)$ norm, of $C^\infty_c(Q)$ (smooth functions with compact support).

For $\varepsilon > 0$, let $(a^\varepsilon)_\varepsilon$, be a given family of coefficients with $a^\varepsilon \in L^\infty(Q; \mathbb{R}^{n\times n}_{\text{sym}})$, satisfying the uniform ellipticity and boundedness condition

$$\alpha_1 |\eta|^2 \le a^\varepsilon(x)\eta \cdot \eta \le \alpha_2 |\eta|^2, \qquad \forall \eta \in \mathbb{R}^n, \text{ for a.e. } x \in \mathbb{R}^n, \tag{2.1}$$

for constants $0 < \alpha_1 < \alpha_2$.

In order to describe the averaging property, we use an arbitrary simplex $S \subset Q$. By Lax-Milgram's Lemma (or F. Riesz's Theorem, since a^ε is symmetric), to given $S \subset Q$, $\xi \in \mathbb{R}^n$, and $b \in \mathbb{R}$, there exists a unique weak solution $u^\varepsilon : S \to \mathbb{R}$ of the problem

$$\begin{aligned} -\nabla\cdot(a^\varepsilon\nabla u^\varepsilon) &= 0 \qquad\qquad \text{in } S,\\ u^\varepsilon(x) &= \xi\cdot x + b \qquad \text{on } \partial S. \end{aligned} \tag{2.2}$$

Definition 2.1 *We say that the coefficients a^ε* allow averaging of the constitutive relation with the matrix $a^* \in \mathbb{R}^{n\times n}_{sym}$ *if the following is satisfied: for every simplex $S\subset Q$, every $b\in\mathbb{R}$, and every vector $\xi\in\mathbb{R}^n$, the solutions u^ε of* (2.2) *satisfy*

$$\lim_{\varepsilon\to 0} \fint_S a^\varepsilon\nabla u^\varepsilon = a^*\xi\,. \tag{2.3}$$

We remark that property (2.3) is satisfied, for example, for periodic coefficients a^ε and for ergodic stochastic coefficients (see, e.g., [7, Appendix A]).

Moreover, if the coefficients a^ε satisfy condition (2.1), then property (2.3) implies that also the homogenized matrix a^* is uniformly elliptic and bounded, with the same constants. Or, in other words, that the eigenvalues of a^* are contained in the interval $[\alpha_1,\alpha_2]$. This fact can be seen as a consequence of H-convergence or Γ-convergence, but for convenience of the reader we provide a short independent proof in the following

Lemma 2.2 *For $\varepsilon>0$, let $(a^\varepsilon)_\varepsilon$, be a given family of coefficients with $a^\varepsilon\in L^\infty(Q;\mathbb{R}^{n\times n}_{\text{sym}})$, satisfying* (2.1)*. If a^ε* allow averaging of the constitutive relation with the matrix $a^*\in\mathbb{R}^{n\times n}_{sym}$ *in the sense of Definition 2.1, then*

$$\alpha_1|\xi|^2 \le a^*\xi\cdot\xi \le \alpha_2|\xi|^2, \qquad \forall\,\xi\in\mathbb{R}^n. \tag{2.4}$$

Proof: Let $u^\varepsilon: S\to\mathbb{R}$ be the unique solution of problem (2.2), for given $S\subset Q$, $b\in\mathbb{R}$, and $\xi\in\mathbb{R}^n$. Let ν be the outward unit normal to ∂S. Then

$$\int_S a^\varepsilon\nabla u^\varepsilon\cdot\nabla u^\varepsilon = \int_{\partial S} a^\varepsilon\nabla u^\varepsilon\cdot\nu\, u^\varepsilon = \int_{\partial S} a^\varepsilon\nabla u^\varepsilon\cdot\nu\,(\xi\cdot x+b) = \int_S a^\varepsilon\nabla u^\varepsilon\cdot\xi.$$

Therefore, by (2.3)

$$\lim_{\varepsilon\to0}\fint_S a^\varepsilon\nabla u^\varepsilon\cdot\nabla u^\varepsilon = \lim_{\varepsilon\to0}\fint_S a^\varepsilon\nabla u^\varepsilon\cdot\xi = a^*\xi\cdot\xi. \tag{2.5}$$

Since a^ε are symmetric, u^ε is also a minimizer of the following problem

$$\int_S a^\varepsilon\nabla u^\varepsilon\cdot\nabla u^\varepsilon = \min\left\{\int_S a^\varepsilon\nabla v\cdot\nabla v \;:\, v\in H^1(S),\ v(x)=\xi\cdot x+b \text{ on } \partial S.\right\}$$

and in particular, choosing $v(x)=\xi\cdot x+b$ and using (2.1),

$$\fint_S a^\varepsilon\nabla u^\varepsilon\cdot\nabla u^\varepsilon \le \fint_S a^\varepsilon\xi\cdot\xi \le \alpha_2|\xi|^2. \tag{2.6}$$

On the other hand, by (2.1), Jensen's inequality and the boundary condition in (2.2),

$$\fint_S a^\varepsilon\nabla u^\varepsilon\cdot\nabla u^\varepsilon \ge \alpha_1\fint_S|\nabla u^\varepsilon|^2 \ge \alpha_1\left|\fint_S\nabla u^\varepsilon\right|^2 = \alpha_1|\xi|^2. \tag{2.7}$$

We conclude the proof of (2.4) by combining (2.6), (2.7), and passing to the limit as $\varepsilon\to0$ by (2.5). $\square$

2.1 The needle problem

Let Q be a Lipschitz bounded domain in $\mathbb{R}^n$ and $\psi : Q \to \mathbb{R}$ be affine. The needle problem is designed as an approximation of the elliptic equation

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= f \qquad \text{in } Q, \\ u^\varepsilon &= \psi \qquad \text{on } \partial Q, \end{aligned}$$

by functions in $H^1(Q)$ which are piecewise linear on the boundary of a triangulation of the domain. In our approach, we use an auxiliary elliptic problem, independent of the parameter ε, to transform the right-hand side f into a jump condition for the fluxes $a^\varepsilon \nabla u^\varepsilon$ on the boundary of the triangulation. Regarding the jump conditions, which are indeed due to the choice of discretization, we note that unlike, e.g., discontinuous Galerkin methods, solutions of the needle problem are continuous on Q, and only their gradients are allowed to jump in order to accommodate an external forcing on the rigid frame constituted by the boundaries of the triangles. We first state the problem, and then detail its construction.

Definition 2.3 (The needle problem) *We are given a bounded Lipschitz domain $Q \subset \mathbb{R}^n$, a triangulation $\mathcal{T}_h$ of $Q_h \subset Q$ with interfaces $\Gamma_h := \bigcup_k \partial T_k$, and a piecewise affine function ψ prescribing a boundary condition. We introduce the function space*

$$\mathcal{N}_h := \left\{ \phi \in H^1_0(Q) \ : \ \phi|_{\partial T_k} \textit{ is affine for all } T_k \in \mathcal{T}_h,\ \phi \equiv 0 \textit{ on } Q \setminus Q_h \right\} .$$

For a given function $g_h : \Gamma_h \to \mathbb{R}$, the needle problem *is to find $u_h^\varepsilon \in \psi + \mathcal{N}_h$ such that*

$$\int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi = \int_{\Gamma_h} g_h \phi \qquad \forall \phi \in \mathcal{N}_h . \tag{2.8}$$

The needle problem depends on a grid of elements with typical size h. We choose a polygonal subset $Q_h \subset Q$ which is large in the sense that $x \in Q$, $\operatorname{dist}(x, \partial Q) \geq h$ implies $x \in Q_h$. The domain Q_h is discretized with open simplices such that

$$\mathcal{T}_h := \{T_k\}_{k \in \Lambda_h} \quad \text{is a subdivision of } Q_h, \quad \operatorname{diam}(T_k) < h,\ \forall k \in \Lambda_h .$$

We consider the corresponding finite element space of continuous and piecewise linear functions with vanishing boundary values,

$$Y_h := \left\{ \phi \in H^1_0(Q) \ : \ \phi|_{T_k} \text{ is affine for all } T_k \in \mathcal{T}_h,\ \phi \equiv 0 \text{ on } Q \setminus Q_h \right\} .$$

This formulation could be regarded as a variational crime, in the sense of [8]. We show in [7] how the errors thus introduced can be estimated and controlled.

With the matrix $a^* \in \mathbb{R}^{n \times n}$ of Definition 2.1, with $f \in L^2(Q)$ and the given affine boundary condition ψ, we consider the following auxiliary problem.

$$\text{Find} \quad U_h \in \psi + Y_h \text{ with } \int_Q (a^* \nabla U_h) \cdot \nabla \phi = \int_Q f \phi, \qquad \forall\, \phi \in Y_h. \tag{2.9}$$

By Lax-Milgram's Lemma and standard finite element approximations there exists a unique solution U_h of (2.9). Moreover, for an affine boundary condition ψ there holds

$$U_h \rightharpoonup u^* \text{ in } H^1(Q)$$

for $h \to 0$, where u^* is the solution of

$$\begin{aligned} -\nabla \cdot (a^* \nabla u^*) &= f \qquad \text{in } Q, \\ u^* &= \psi \qquad \text{on } \partial Q. \end{aligned}$$

Our next aim is to transform the right-hand side f into jump conditions for the fluxes $a^*\nabla U_h$ across edges of the grid $\mathcal{T}_h$. We will extract the relevant information on jumps from the finite element solution U_h of system (2.9). Let Γ_h be the set of interior interfaces and Γ_{kj} be the interface of two simplices T_k and T_j,

$$\Gamma_h := \left(\bigcup_k \partial T_k \right) \setminus \partial Q_h = \bigcup_{k<j} \Gamma_{kj}, \qquad \Gamma_{kj} := \overline{T}_k \cap \overline{T}_j \, .$$

We furthermore use the notation $\nu_{(k)}$ for the outer normal to T_k on ∂T_k. For a function $\varphi \in L^2(Q;\mathbb{R}^n)$, such that $\varphi|_{T_k}$ has a trace on ∂T_k for all k, the jump across Γ_{kj} is defined as

$$[\![\varphi]\!]_{kj} := \varphi|_{T_k} \cdot \nu_{(k)} + \varphi|_{T_j} \cdot \nu_{(j)} = \left(\varphi|_{T_k} - \varphi|_{T_j} \right) \cdot \nu_{(k)}.$$

By definition, there holds $[\![\varphi]\!]_{kj} = [\![\varphi]\!]_{jk}$. We consider the jump as a scalar function on Γ_h. With the solution U_h of (2.9), we define $g_h : \Gamma_h \to \mathbb{R}$ as the function

$$g_h|_{\Gamma_{kj}} := [\![a^* \nabla U_h]\!]_{kj}. \tag{2.10}$$

The gradients ∇U_h are constant in each simplex T_k, hence $g_h : \Gamma_h \to \mathbb{R}$ is constant on each interface Γ_{kj}.

Remark 2.4 The finite element solution U_h was defined in (2.9) with f. We can characterize U_h with g_h as the unique solution of

$$U_h \in \psi + Y_h, \quad \text{with} \quad [\![a^* \nabla U_h]\!]_{kj} = g_h|_{\Gamma_{kj}} \quad \forall k < j. \tag{2.11}$$

Problem (2.11) is equivalent to problem (2.9).

The remark indicates that the right hand side f has been transformed into the jump condition g_h. This is even clearer if we observe that, for all $\phi \in Y_h$,

$$\int_Q f\phi = \int_Q a^* \nabla U_h \cdot \nabla \phi = \sum_k \int_{\partial T_k} (a^* \nabla U_h \cdot \nu_{(k)})\phi = \int_{\Gamma_h} g_h \phi,$$

since $a^*\nabla U_h$ is constant in each T_k. Considering only functions $\phi \in Y_h$, we have therefore equivalently replaced $f \in L^2(Q)$ by g_h.

Let $\mathcal{H}^{n-1}$ denote the $(n-1)$-dimensional Hausdorff measure. Since by the trace theorem any function $u \in H^1(Q)$ has trace $u|_{\Gamma_h} \in H^{1/2}(\Gamma_h)$, we observe that the application

$$u \mapsto \int_{\Gamma_h} u|_{\Gamma_h} \, g_h \, d\mathcal{H}^{n-1}$$

defines a linear and continuous functional on $H^1(Q)$. In this sense, $g_h \mathcal{H}^{n-1}|_{\Gamma_h} \in H^{-1}(Q)$. In particular, in that case, the Lax-Milgram theorem is applicable and yields the unique existence of a solution $u_h^\varepsilon \in \psi + \mathcal{N}_h$ of the needle problem.

A formulation of (2.8) on single simplices is as follows: we search for $u_h^\varepsilon \in \psi + \mathcal{N}_h$ with

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u_h^\varepsilon) &= 0 \qquad \text{in } T_k, \quad \forall\, T_k \in \mathcal{T}_h\,, \\ \int_{\Gamma_h} ([\![a^\varepsilon \nabla u_h^\varepsilon]\!] - g_h)\, \phi &= 0 \qquad \forall\, \phi \in \mathcal{N}_h. \end{aligned}$$

2.2 Homogenization of elliptic equations

We recall the main result of [7] on the non-periodic homogenization of elliptic equations.

Theorem 2.5 *Let $Q \subset \mathbb{R}^n$ be an n-dimensional bounded domain with Lipschitz boundary, with $n = 2$ or $n = 3$. Let $f \in L^2(Q)$ be arbitrary and let $\psi \in H^1(Q)$ be affine. We assume that the coefficients $(a^\varepsilon)_\varepsilon$ satisfy the ellipticity relation* (2.1) *and that they allow averaging of the constitutive relation with the matrix a^* in the sense of Definition 2.1. Then the sequence $(u^\varepsilon)_\varepsilon$ of weak solutions of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= f \qquad \textit{in } Q, \\ u^\varepsilon &= \psi \qquad \textit{on } \partial Q, \end{aligned} \tag{2.12}$$

satisfies

$$\begin{aligned} u^\varepsilon &\rightharpoonup u^* && \textit{weakly in } H^1(Q), \\ a^\varepsilon \nabla u^\varepsilon &\rightharpoonup a^* \nabla u^* && \textit{weakly in } L^2(Q;\mathbb{R}^n), \end{aligned}$$

where u^* is the weak solution of

$$\begin{aligned} -\nabla \cdot (a^* \nabla u^*) &= f \qquad \textit{in } Q, \\ u^* &= \psi \qquad \textit{on } \partial Q. \end{aligned} \tag{2.13}$$

On the proof. The theorem is shown by a comparison of the original problem with the needle problem. More precisely, in [7], we show the convergence estimates

$$\lim_{h\to 0} \lim_{\varepsilon \to 0} \|u_h^\varepsilon - u^\varepsilon\|_{H^1(Q)} = 0, \tag{2.14}$$

$$u_h^\varepsilon \rightharpoonup U_h \qquad \text{weakly in } H^1(Q), \text{ as } \varepsilon \to 0, \tag{2.15}$$

$$a^\varepsilon \nabla u_h^\varepsilon \rightharpoonup a^* \nabla U_h \quad \text{weakly in } L^2(Q;\mathbb{R}^n), \text{ as } \varepsilon \to 0, \tag{2.16}$$

$$U_h \rightharpoonup u^* \qquad \text{weakly in } H^1(Q), \text{ as } h \to 0, \tag{2.17}$$

where u^ε is the solution of (2.12), U_h solves the finite elements approximation (2.9), u_h^ε is the solution of the needle problem (2.8) where the prescribed jump g_h is given by (2.10), and u^* is the solution of (2.13). While convergence (2.17) is a standard fact of finite elements discretization, the other three convergences are more delicate. The limit in (2.14) is obtained combining classical energy estimates with an improved version of *div-curl* lemma on a mesh $\mathcal{T}_h$ which is adapted to the sequence u^ε in order to avoid singularities. The convergence in (2.15)-(2.16) stems from Definition 2.1, which is shown to imply homogenization of elliptic equations with $f = 0$ and affine boundary conditions on simplices. Since h is arbitrary in (2.15)-(2.16), these four estimates provide the desired result.

3 Time-dependent results

We turn now to the application of the elliptic homogenization result in Theorem 2.5 to two time-dependent problems. In Section 3.1 we recall the classical case of the heat equation, and in Section 3.2 we show how elliptic homogenization can be transferred to a time-dependent problem with hysteresis.

In an abstract language, we can describe the results in this subsection as follows: hypothesis (2.3) on coefficients a^ε implies, by Theorem 2.5, the H-convergence of a^ε to a^*, i.e., the weak convergence $(u^\varepsilon, a^\varepsilon \nabla u^\varepsilon) \to (u^*, a^* \nabla u^*)$ in $H^1(Q) \times L^2(Q;\mathbb{R}^n)$, for solutions of the correspondent elliptic problems (see, e.g., [10, Definition 6.4], [9]). This H-convergence is exploited now in order to conclude homogenization results in time-dependent problems.

3.1 The heat equation

It is straightforward to transfer homogenization results for elliptic problems to the corresponding parabolic problems, see e.g. [2]. For the sake of completeness, in order to show that the needle-problem approach provides new results for parabolic problems, we include this conclusion here. A non-standard time-dependent problem will be treated with similar methods in the next subsection.

Theorem 3.1 *Let $Q \subset \mathbb{R}^n$ be bounded, open, with Lipschitz boundary, $n = 2$ or $n = 3$, and let $T > 0$. Let $\psi \in C^1([0,T]; H^1(Q))$ be such that $\psi(t,\cdot)$ is affine for all $t \in [0,T]$, and $u_0 \in H^1(Q)$ be given. We assume that the coefficients $(a^\varepsilon)_\varepsilon$ satisfy* (2.1) *and that they allow averaging of the constitutive relation with the matrix a^* in the sense of Definition 2.1. Let $(u^\varepsilon)_\varepsilon$ be a sequence of weak solutions of*

$$\begin{aligned} \partial_t u^\varepsilon &= \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) && \textit{in } Q \times (0,T), \\ u^\varepsilon &= \psi && \textit{on } \partial Q \times (0,T), \\ u^\varepsilon(t=0) &= u_0 && \textit{on } Q. \end{aligned} \tag{3.1}$$

Then u^ε satisfies

$$\begin{aligned} u^\varepsilon &\rightharpoonup u^*, && \textit{weakly in } L^2(0,T;H^1(Q)),\\ \partial_t u^\varepsilon &\rightharpoonup \partial_t u^*, && \textit{weakly in } L^2(Q\times(0,T)),\\ a^\varepsilon\nabla u^\varepsilon &\rightharpoonup a^*\nabla u^*, && \textit{weakly in } L^2(Q\times(0,T);\mathbb{R}^n), \end{aligned}$$

where u^* is the weak solution of

$$\begin{aligned} \partial_t u^* &= \nabla\cdot(a^*\nabla u^*) && \textit{in } Q\times(0,T),\\ u^* &= \psi && \textit{on } \partial Q\times(0,T),\\ u^*(t=0) &= u_0 && \textit{on } Q. \end{aligned} \tag{3.2}$$

For notational brevity, the theorem is formulated here for a vanishing right-hand side f.

Proof: By standard a priori estimates for parabolic problems, there exists a constant C, depending only on T, Q, u_0, ψ, such that the solutions $(u^\varepsilon)_\varepsilon$ of (3.1) satisfy

$$\|u^\varepsilon\|_{L^2(0,T;H^1(Q))} + \|u^\varepsilon\|_{H^1(0,T;L^2(Q))} \le C.$$

In particular, we can find a subsequence u^{ε_k} and a limit function u such that

$$u^{\varepsilon_k} \to u \quad \text{strongly in } L^2(0,T;L^2(Q)), \tag{3.3}$$
$$\nabla u^{\varepsilon_k} \rightharpoonup \nabla u \quad \text{weakly in } L^2(0,T;L^2(Q;\mathbb{R}^n)), \tag{3.4}$$
$$\partial_t u^{\varepsilon_k} \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;L^2(Q)). \tag{3.5}$$

In order to prove Theorem 3.1, we need to show that every limit function u is a solution of (3.2), that is, $u = u^*$.

As a first step, we note that Theorem 2.5 can be extended to the case of a weakly-L^2 converging right-hand side.

Lemma 3.2 *Let the hypothesis of Theorem 2.5 be satisfied, and let $f_\varepsilon \in L^2(Q)$ be such that $f_\varepsilon \rightharpoonup f$ weakly in $L^2(Q)$. Then the sequence $(u^\varepsilon)_\varepsilon$ of weak solutions of*

$$\begin{aligned} -\nabla\cdot(a^\varepsilon\nabla u^\varepsilon) &= f_\varepsilon && \textit{in } Q,\\ u^\varepsilon &= \psi && \textit{on } \partial Q, \end{aligned}$$

satisfies

$$\begin{aligned} u^\varepsilon &\rightharpoonup u^* && \textit{weakly in } H^1(Q),\\ a^\varepsilon\nabla u^\varepsilon &\rightharpoonup a^*\nabla u^* && \textit{weakly in } L^2(Q;\mathbb{R}^n), \end{aligned}$$

where u^* is the weak solution of

$$\begin{aligned} -\nabla\cdot(a^*\nabla u^*) &= f && \textit{in } Q,\\ u^* &= \psi && \textit{on } \partial Q. \end{aligned} \tag{3.6}$$

Proof: [of Lemma 3.2] We decompose the solution u^ε additively into two parts. To this end, let v^ε and w^ε be the solutions of

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla v^\varepsilon) &= f_\varepsilon - f && \text{in } Q, \\ v^\varepsilon &= 0 && \text{on } \partial Q, \end{aligned}$$

and

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla w^\varepsilon) &= f && \text{in } Q, \\ w^\varepsilon &= \psi && \text{on } \partial Q. \end{aligned}$$

Owing to Theorem 2.5, the solutions w^ε satisfy

$$\begin{aligned} w^\varepsilon &\rightharpoonup u^* && \text{weakly in } H^1(Q), \\ a^\varepsilon \nabla w^\varepsilon &\rightharpoonup a^* \nabla u^* && \text{weakly in } L^2(Q;\mathbb{R}^n), \end{aligned}$$

where u^* is the weak solution of

$$\begin{aligned} -\nabla \cdot (a^* \nabla u^*) &= f && \text{in } Q, \\ u^* &= \psi && \text{on } \partial Q. \end{aligned}$$

By linearity, $u^\varepsilon = v^\varepsilon + w^\varepsilon$. Therefore, equation (3.6) and the lemma are shown once we verify the appropriate convergences to 0 for v^ε.

We have a bound $\|v^\varepsilon\|_{H^1(Q)} \le C$ by Lax-Milgram's Lemma. Therefore, from any subsequence of v^ε we can extract a subsequence v^{ε_k} and find $v \in H^1(Q)$ such that $v^{\varepsilon_k} \to v$ strongly in $L^2(Q)$. By the ellipticity condition (2.1) on a^ε we may compute

$$\alpha_1 \|v^{\varepsilon_k}\|^2_{H^1(Q)} \le \int_Q a^\varepsilon \nabla v^{\varepsilon_k} \cdot \nabla v^{\varepsilon_k} = \int_Q (f_{\varepsilon_k} - f) v^{\varepsilon_k} \to 0,$$

which implies the desired convergences. □

We turn now to the proof of Theorem 3.1. Let $\varphi \in C_c^\infty((0,T);\mathbb{R})$ be a smooth cut-off function. Let u^ε be the solutions of equation (3.1), and define the averages

$$W^\varepsilon(x) := \int_0^T u^\varepsilon(x,s)\,\varphi(s)\,ds, \qquad F^\varepsilon(x) := -\int_0^T \partial_s u^\varepsilon(x,s)\varphi(s)\,ds,$$

so that, by (3.4) and (3.5), it holds

$$W^\varepsilon \rightharpoonup \int_0^T u(x,s)\varphi(s)\,ds \qquad \text{weakly in } H^1(Q), \tag{3.7}$$

$$F^\varepsilon \rightharpoonup -\int_0^T \partial_s u(x,s)\varphi(s)\,ds, \qquad \text{weakly in } L^2(Q). \tag{3.8}$$

Equation (3.1) translates into the following equation for W^ε,

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla W^\varepsilon) &= F^\varepsilon && \text{in } Q, \\ W^\varepsilon &= \Psi && \text{on } \partial Q, \end{aligned}$$

where

$$\Psi(x) := \int_0^T \psi(x,s)\varphi(s)\,ds.$$

Lemma 3.2 and the convergence (3.8) provide the limit of W^ε. We find

$$W^\varepsilon \rightharpoonup W^* \qquad \text{weakly in } H^1(Q), \tag{3.9}$$

$$a^\varepsilon \nabla W^\varepsilon \rightharpoonup a^* \nabla W^* \qquad \text{weakly in } L^2(Q;\mathbb{R}^n), \tag{3.10}$$

where W^* is the weak solution of

$$-\nabla \cdot (a^* \nabla W^*) = -\int_0^T \partial_s u(x,s)\varphi(s)\,ds \qquad \text{in } Q,$$

$$W^* = \Psi \qquad \text{on } \partial Q.$$

With the help of (3.7) and (3.9), we translate this equation for W^* into an equation for u. We find, for an arbitrary test-function $\phi \in H_0^1(Q)$,

$$\int_0^T \int_Q \{\partial_s u(x,s)\varphi(s)\phi(x) + a^* \nabla u(x,s) \cdot \nabla\phi(x)\varphi(s)\}\, dx\, ds = 0.$$

By density of the functions $\varphi(t)\phi(x)$ in $L^2(0,T;H_0^1(Q))$ we conclude that the limit u of u^ε is the weak solution of

$$\begin{aligned} \partial_t u &= \nabla \cdot (a^* \nabla u) && \text{in } Q \times (0,T),\\ u &= \psi && \text{on } \partial Q \times (0,T),\\ u(t=0) &= u_0 && \text{on } Q. \end{aligned}$$

This conclues the proof of Theorem 3.1. □

3.2 A parabolic problem with hysteresis

We discuss now a problem that appears in porous media analysis. Hysteresis plays an important role in wetting processes. If a porous medium is first wetted and then de-wetted, the pressure-saturation relation is very different during the two processes. We consider a related hysteresis model; the variable z has the physical meaning of a saturation, u stands for the pressure.

Related problems were analyzed in [5] and [3]. Both contributions are concerned with the case that the function H of the hysteresis problem is the multi-valued graph $H = \mathrm{sign}^{-1}$ or a regularization thereof. In [3] an existence result is derived, we refer also to the references therein. In [5] the homogenization of the problem in a stochastic setting was analyzed. The homogenization result is involved since it treats not only multi-valued hysteresis functions, but also hysteresis functions H that depend on the spatial position and have a microstructure. In the contribution at hand, we show that the homogenization limit can be performed without any further difficulties if we assume that the monotone hysteresis relation H is smooth and without microscopic oscillations.

In the proof below we use a formulation of the problem with variational inequalities, employing methods of convex analysis. Such methods were also used in [4] for an outflow problem in porous media analysis, in [7] in a periodic homogenization problem in plasticity, and in [11] in an abstract stochastic homogenization problem.

Theorem 3.3 *Let $Q \subset \mathbb{R}^n$ be bounded, open, with Lipschitz boundary, $n = 2$ or $n = 3$, and let $T > 0$. Let $\psi \in C^1([0,T]; H^1(Q))$ be such that $\psi(t,\cdot)$ is affine for all $t \in [0,T]$, and $z_0 \in L^2(Q)$ be given. Let $H : \mathbb{R} \to \mathbb{R}$ be a monotone function of class C^1 with $H(0) = 0$ and with bounded derivative. We assume that the coefficients $(a^\varepsilon)_\varepsilon$ satisfy* (2.1) *and that they allow averaging of the constitutive relation with the matrix a^* in the sense of Definition 2.1. Let $(u^\varepsilon, z^\varepsilon)_\varepsilon$ be a sequence of classical solutions of*

$$\begin{aligned} \partial_t z^\varepsilon &= \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) && \textit{in } Q \times (0,T), \\ \partial_t z^\varepsilon &= -H(z^\varepsilon - u^\varepsilon) && \textit{in } Q \times (0,T), \\ u^\varepsilon &= \psi && \textit{on } \partial Q \times (0,T), \\ z^\varepsilon(t=0) &= z_0 && \textit{on } Q. \end{aligned} \tag{3.11}$$

Then $(u^\varepsilon, z^\varepsilon)$ satisfies

$$\begin{aligned} u^\varepsilon &\rightharpoonup u^*, && \textit{weakly in } L^2(0,T; H^1(Q)), \\ \partial_t z^\varepsilon &\rightharpoonup \partial_t z^*, && \textit{weakly in } L^2(Q \times (0,T)), \\ a^\varepsilon \nabla u^\varepsilon &\rightharpoonup a^* \nabla u^*, && \textit{weakly in } L^2(Q \times (0,T); \mathbb{R}^n), \end{aligned}$$

where (u^*, z^*) is the unique weak solution of

$$\begin{aligned} \partial_t z^* &= \nabla \cdot (a^* \nabla u^*) && \textit{in } Q \times (0,T), \\ \partial_t z^* &= -H(z^* - u^*) && \textit{in } Q \times (0,T), \\ u^* &= \psi && \textit{on } \partial Q \times (0,T), \\ z^*(t=0) &= z_0 && \textit{on } Q. \end{aligned}$$

Note that, since by Lemma 2.2 the homogenized matrix a^* satisfies the same ellipticity and boundedness conditions of a^ε, existence and uniqueness of a solution (u^*, z^*) can be established with exactly the same technique we use for $(u^\varepsilon, z^\varepsilon)$.

Energy estimates. Consider the equations

$$\partial_t z^\varepsilon = \nabla \cdot (a^\varepsilon \nabla u^\varepsilon), \tag{3.12}$$

$$\partial_t z^\varepsilon = -H(z^\varepsilon - u^\varepsilon). \tag{3.13}$$

For brevity of notation, we assume that the boundary condition is $\psi(t) \equiv 0$. We multiply equation (3.12) with u^ε and equation (3.13) with $u^\varepsilon - z^\varepsilon$, and integrate over Q. Integrating by parts and using the boundary condition in (3.11) we obtain

$$\begin{aligned} \int_Q (\partial_t z^\varepsilon) u^\varepsilon &= -\int_Q a^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon, \\ \int_Q (\partial_t z^\varepsilon) u^\varepsilon &= \int_Q (\partial_t z^\varepsilon) z^\varepsilon + \int_Q H(z^\varepsilon - u^\varepsilon)(z^\varepsilon - u^\varepsilon). \end{aligned}$$

We integrate in time to find

$$\int_0^T \int_Q (\partial_t z^\varepsilon) z^\varepsilon + a^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon + H(z^\varepsilon - u^\varepsilon)(z^\varepsilon - u^\varepsilon) = 0.$$

By monotonicity of H and ellipticity of a^ε we conclude the energy estimate

$$\sup_{t\in(0,T)} \frac{1}{2}\int_Q |z^\varepsilon(t)|^2 + \alpha_1 \int_0^T \int_Q |\nabla u^\varepsilon|^2 \le \frac{1}{2}\int_Q |z_0|^2.$$

We may additionally use Poincaré's inequality for u^ε to see that for some $C > 0$ we have the uniform bound

$$\|z^\varepsilon\|^2_{L^\infty(0,T;L^2(Q))} + \|u^\varepsilon\|^2_{L^2(0,T;H^1(Q))} \le C.$$

Higher order estimates. We present here a derivation of higher order a priori estimates, assuming the regularity $z^\varepsilon \in W^{1,\infty}(0,T;L^2(Q))$ and $u^\varepsilon \in H^1(0,T;H^1(Q))$ for every ε. The proof can be performed without the regularity assumption on the solution, see [3]. In this calculation we exploit that the nonlinear relation H is differentiable. We differentiate in time equations (3.12) and (3.13)

$$\partial_t^2 z^\varepsilon = \nabla \cdot (a^\varepsilon \nabla \partial_t u^\varepsilon), \tag{3.14}$$
$$\partial_t^2 z^\varepsilon = -H'(z^\varepsilon - u^\varepsilon)\partial_t(z^\varepsilon - u^\varepsilon). \tag{3.15}$$

We multiply equation (3.14) by $\partial_t u^\varepsilon$ and equation (3.15) by $\partial_t(u^\varepsilon - z^\varepsilon)$. Integrating by parts and employing the boundary condition in (3.11) we obtain

$$\int_Q \partial_t^2 z^\varepsilon \partial_t u^\varepsilon = -\int_Q a^\varepsilon \nabla(\partial_t u^\varepsilon)\cdot\nabla(\partial_t u^\varepsilon),$$
$$\int_Q \partial_t^2 z^\varepsilon \partial_t u^\varepsilon = \int_Q \partial_t^2 z^\varepsilon \partial_t z^\varepsilon + \int_Q H'(z^\varepsilon - u^\varepsilon)(\partial_t u^\varepsilon - \partial_t z^\varepsilon)^2.$$

Integrating in time and using the ellipticity of a^ε and the positivity of H' we obtain

$$\sup_{t\in(0,T)} \frac{1}{2}\int_Q |\partial_t z^\varepsilon(t)|^2 + \alpha_1 \int_0^T \int_Q |\nabla \partial_t u^\varepsilon|^2 \le \frac{1}{2}\int_Q |\partial_t z_0^\varepsilon|^2.$$

In this equation, the expression $\partial_t z_0^\varepsilon$ stands for the following. We can combine the evolution equations in $t = 0$ into $-H(z_0^\varepsilon - u_0^\varepsilon) = \nabla\cdot(a^\varepsilon \nabla u_0^\varepsilon)$. The solution u_0^ε of this elliptic relation is bounded in $H^1(Q)$, independently of ε. The formal time derivative of z in $t = 0$ is then the L^2-function $\partial_t z_0^\varepsilon := -H(z_0^\varepsilon - u_0^\varepsilon)$. In other words, we do not need to prescribe the initial value for $\partial_t z^\varepsilon$, since for classical solutions it is determined by the initial datum z_0^ε, through equations (3.12) and (3.13).

We conclude that there exists $C > 0$ such that

$$\|\partial_t z^\varepsilon\|^2_{L^\infty(0,T;L^2(Q))} + \|\partial_t u^\varepsilon\|^2_{L^2(0,T;H^1(Q))} \le C.$$

In particular, we can find a subsequence (ε_k) and limit functions z and u such that

$$u^{\varepsilon_k} \to u \quad \text{strongly in } L^2(0,T;L^2(Q)), \tag{3.16}$$
$$\nabla u^{\varepsilon_k} \rightharpoonup \nabla u \quad \text{weakly in } L^2(0,T;L^2(Q;\mathbb{R}^n)), \tag{3.17}$$
$$\partial_t z^{\varepsilon_k} \rightharpoonup \partial_t z \quad \text{weakly in } L^2(0,T;L^2(Q)). \tag{3.18}$$

Proof: [Proof of Theorem 3.3] We proceed as in the proof of the heat equation, defining

$$W^\varepsilon(x) := \int_0^T u^\varepsilon(x,s)\varphi(s)\,ds, \qquad F^\varepsilon(x) := -\int_0^T \partial_s z^\varepsilon(x,s)\varphi(s)\,ds,$$

for a smooth cut-off function $\varphi \in C_c^\infty((0,T);\mathbb{R})$. By (3.17), (3.18), it holds

$$W^\varepsilon \rightharpoonup \int_0^T u(x,s)\varphi(s)\,ds \qquad \text{weakly in } H^1(Q), \tag{3.19}$$
$$F^\varepsilon \rightharpoonup -\int_0^T \partial_s z(x,s)\varphi(s)\,ds, \qquad \text{weakly in } L^2(Q).$$

By equation (3.12), W^ε solves

$$\begin{aligned} -\nabla\cdot(a^\varepsilon\nabla W^\varepsilon) &= F^\varepsilon && \text{in } Q,\\ W^\varepsilon &= \Psi && \text{on } \partial Q, \end{aligned}$$

where

$$\Psi(x) := \int_0^T \psi(x,s)\varphi(s)\,ds,$$

and ψ is the given boundary condition in (3.11). By Lemma 3.2 and convergence (3.19), W^ε satisfies

$$W^\varepsilon \rightharpoonup W^* \qquad \text{weakly in } H^1(Q), \tag{3.20}$$
$$a^\varepsilon\nabla W^\varepsilon \rightharpoonup a^*\nabla W^* \qquad \text{weakly in } L^2(Q;\mathbb{R}^n),$$

where W^* is the weak solution of

$$\begin{aligned} -\nabla\cdot(a^*\nabla W^*) &= -\int_0^T \partial_s z(x,s)\varphi(s)\,ds && \text{in } Q,\\ W^* &= \Psi && \text{on } \partial Q. \end{aligned}$$

Proceeding as in Section 3.1, we compare (3.19) and (3.20), obtaining, for an arbitrary test-function $\phi \in H_0^1(Q)$,

$$\int_0^T \int_Q \left\{\partial_s z(x,s)\varphi(s)\phi(x) + a^*\nabla u(x,s)\cdot\nabla\phi(x)\varphi(s)\right\}\,dx\,ds = 0.$$

By density of the test functions $\varphi(t)\phi(x)$ in $L^2(0,T;H_0^1(Q))$ we conclude that every limit (u,z) of $(u^\varepsilon, z^\varepsilon)$ satisfies

$$\begin{aligned} \partial_t z = \nabla \cdot (a^* \nabla u) \qquad & \text{in } Q \times (0,T),\\ u = \psi \qquad & \text{on } \partial Q \times (0,T),\\ z(t=0) = z_0 \qquad & \text{on } Q. \end{aligned}$$

The limit process in the pointwise and nonlinear equation $\partial_t z^\varepsilon = -H(z^\varepsilon - u^\varepsilon)$ is more complicated than the limit process in the linear elliptic equation. The reason is that $(z^\varepsilon - u^\varepsilon)$ converges only weakly in $L^2(0,T;L^2(Q))$; we have no spatial estimates for z^ε. We solve this problem with the help of convex analysis.

We firstly define a primitive of H, the function $G : \mathbb{R} \to \mathbb{R}$ with $G' = H$ and $G(0) = 0$. This function is convex and non-negative by our assumptions on H. We introduce the convex functional on $L^2(0,T;L^2(Q))$

$$\mathcal{G}(\xi) := \int_0^T \int_Q G(\xi(x,t)\, dx\, dt \in \mathbb{R} \quad \text{for} \quad \xi \in L^2(0,T;L^2(Q)).$$

We will use $\xi^\varepsilon := z^\varepsilon - u^\varepsilon$ as an argument for $\mathcal{G}$. The delay relation reads $\partial_t z^\varepsilon = -H(z^\varepsilon - u^\varepsilon) = -G'(z^\varepsilon - u^\varepsilon)$; it is equivalent to the subdifferential inclusion $\partial_t z^\varepsilon \in -\partial\mathcal{G}(\xi^\varepsilon)$, since the subdifferential coincides with the Frechét-derivative. We can now use the defining relation for the subdifferential and conclude: for every comparison argument $\eta \in L^2(0,T;L^2(Q))$ holds

$$\mathcal{G}(\eta) \geq \mathcal{G}(\xi^\varepsilon) + \langle -\partial_t z^\varepsilon, \eta - \xi^\varepsilon \rangle_{L^2} .$$

We recognize one total time derivate and re-write as

$$\mathcal{G}(z^\varepsilon - u^\varepsilon) + \frac{d}{dt}\frac{1}{2}\|z^\varepsilon\|^2 - (\partial_t z^\varepsilon, \eta + u^\varepsilon) \leq \mathcal{G}(\eta).$$

We integrate over the time interval $(0,T)$ to find

$$\int_0^T \mathcal{G}(z^\varepsilon - u^\varepsilon) + \frac{1}{2}\|z^\varepsilon(T)\|^2 - \int_0^T (\partial_t z^\varepsilon, \eta + u^\varepsilon) \leq \int_0^T \mathcal{G}(\eta) + \frac{1}{2}\|z_0\|^2.$$

In this inequality, we can pass to the limit exploiting the convexity (and hence lower semi-continuity) of $\mathcal{G}$ and of the norm, and furthermore exploiting the weak convergence of $\partial_t z^\varepsilon$ and the strong convergence of u^ε. We find that

$$\int_0^T \mathcal{G}(z - u) + \frac{1}{2}\|z(T)\|^2 - \int_0^T (\partial_t z, \eta + u) \leq \int_0^T \mathcal{G}(\eta) + \frac{1}{2}\|z_0\|^2.$$

By writing $\|z(T)\|^2 - \|z_0\|^2$ again as a time integral we find that

$$-\partial_t z \in \partial\mathcal{G}(z - u).$$

Since G is convex and differentiable, this is equivalent to

$$-\partial_t z = G'(z-u) = H(z-u) \qquad \text{a.e. in } (0,T)\times Q.$$

This shows the claim. $\square$

Remark. The convergence proof uses only the property of monotonicity of H. The existence of H' is only used in the derivation of the higher-order estimates. The statement of Theorem 3.3 could therefore be extended to monotone graphs H which are subdifferentials of convex functions. This is the case that is considered in applications.

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