① $\sum_{n=1}^{+\infty} \sin\left(\frac{1}{n^3}\right) \frac{(1-x)^n}{7^{n+1}} = \frac{1}{7} \sum_{n=1}^{+\infty} \underbrace{(-1)^n \frac{\sin\left(\frac{1}{n^3}\right)}{7^n}}_{a_n} (x-1)^n$

a)

$a_n \sim \frac{(-1)^n}{n^3 \, 7^n}$ (criterio del confronto asintotico)

$\lim_{n \to +\infty} \sqrt[n]{|a_n|} = \frac{1}{7} \Rightarrow R = 7$

b) • $x = 1-7 = -6 \rightsquigarrow \frac{1}{7} \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3 \cdot 7^n} (-7)^n = \frac{1}{7} \sum_{n=1}^{+\infty} \frac{1}{n^3}$

(converge)

• $x = 1+7 = 8 \rightsquigarrow \frac{1}{7} \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3 7^n} (7)^n = \frac{1}{7} \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3}$

(converge)

Insieme di convergenza: $[-6, 8]$

(2) $f(x,y) = x e^{-\frac{x^2+y^2}{2}}$

a) $\partial_x f = e^{-\frac{x^2+y^2}{2}} (1 - x^2)$

$\partial_y f = e^{-\frac{x^2+y^2}{2}} (-xy)$

b) $\nabla f = (0,0)$ nei punti $P = (1,0)$, $Q = (-1,0)$

c) $\partial^2_{xx} f = e^{-\frac{x^2+y^2}{2}} (-x + x^3 - 2x)$

$\partial^2_{xy} f = e^{-\frac{x^2+y^2}{2}} (-y)(1-x^2)$

$\partial^2_{yy} f = e^{-\frac{x^2+y^2}{2}} (xy^2 - x)$

$Hf(P) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \cdot e^{-\frac{1}{2}}$ def. negativa

$Hf(Q) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \cdot e^{-\frac{1}{2}}$ def. positiva

$\Rightarrow$ P punto di max. relativo stretto

Q punto di min. relativo stretto

3

$$L = \int_\gamma F \circ d\gamma = \int_4^6 F\left(t^2+1, t\right) \circ (2t, 1)\, dt$$

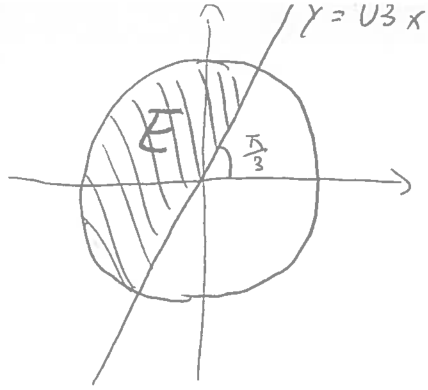
$$= \int_4^6 \left(\frac{1}{t^2+1}, -\frac{1}{t^2+1}\right) \circ (2t, 1)\, dt$$

$$= \int_4^6 \frac{2t}{t^2+1} - \frac{1}{t^2+1}\, dt$$

$$= \Big[\ln\left(t^2+1\right) - \arctan(t) \Big]_{t=4}^{t=6}$$

$$= \ln(37) - \arctan(6) - \ln(17) + \arctan(4)$$

(4) $E = \left\{ x^2+y^2 \leq 25 \ , \ y \geq \sqrt{3}\, x \right\}$



$E = \left\{ \rho \leq 5 \ , \ \frac{\pi}{3} \leq \vartheta \leq \frac{4}{3}\pi \right\}$

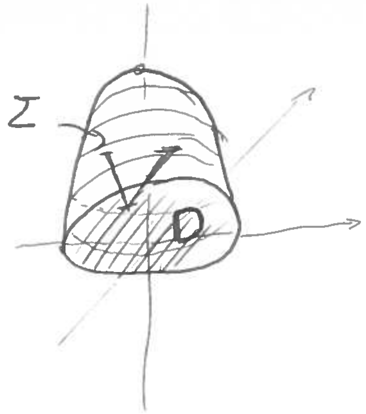
$\left(y = \sqrt{3}\, x \rightsquigarrow \frac{\sin\vartheta}{\cos\vartheta} = \sqrt{3} \right.$

$\left. \rightsquigarrow \vartheta = \frac{\pi}{3} \text{ oppure } \frac{4}{3}\pi \right)$

$$\iint_E e^{x^2+y^2}\, dx\, dy = \int_{\frac{\pi}{3}}^{\frac{4}{3}\pi} \int_0^5 e^{\rho^2} \rho\, d\rho\, d\vartheta = \int_{\frac{\pi}{3}}^{\frac{4}{3}\pi} d\vartheta \cdot \int_0^5 e^{\rho^2} \rho\, d\rho$$

$$= \pi \cdot \left. \frac{e^{\rho^2}}{2} \right|_{\rho=0}^{\rho=5} = \frac{\pi}{2}\left(e^{25} - e^0\right) = \frac{\pi}{2}\left(e^{25} - 1\right)$$

5) Teorema di Gauss



*) $\iiint_V \operatorname{div} F \, dx\,dy\,dz = \iint_\Sigma F \cdot n \, dS + \iint_D F \cdot n \, dS$

1)

$$\iiint_V \operatorname{div} F \, dx\,dy\,dz = \int_0^7 \left(\iint_{\{x^2+y^2 \le 7-z\}} (3x^2 + x^2) \, dx\,dy \right) dz$$

$$= \int_0^7 \left(\int_0^{2\pi} \int_0^{\sqrt{7-z}} 4\rho^2 \cos^2\vartheta \;\rho \, d\rho \, d\vartheta \right) dz$$

$$= \int_0^7 \left(\int_0^{2\pi} \cos^2\vartheta \, d\vartheta \right) \cdot \left(\int_0^{\sqrt{7-z}} 4\rho^3 \, d\rho \right) dz = \int_0^7 \left(2\pi \cdot \frac{1}{2}\right) \left(\sqrt{7-z}\right)^4 dz$$

$$= \pi \int_0^7 (7-z)^2 \, dz = \pi \left. \frac{[z-7]^3}{3} \right|_0^7 = \frac{\pi}{3} \cdot 7^3$$

2) $$\iint_D F \cdot n \, dS = \iint_D F \cdot (0,0,-1) \, dS = \cancel{\int_0^{2\pi} \int_0^{\sqrt{7}}}$$

$$= \iint_{\{x^2+y^2 \le 7\}} -\sqrt{x^2+y^2} \, dx\,dy = -\int_0^{2\pi} \int_0^{\sqrt{7}} \rho \;\rho \, d\rho \, d\vartheta = -2\pi \cdot \left. \frac{\rho^3}{3} \right|_0^{\sqrt{7}} = -\frac{2\pi}{3} \cdot 7\sqrt{7}$$

Flusso attraverso $\Sigma = \iiint_V \operatorname{div} F \, dx\,dy\,dz - \iint_D F \cdot n \, dS = \boxed{\frac{\pi}{3} \cdot 7^3 + \frac{2\pi}{3} \cdot 7\sqrt{7}}$

⑥ estremi di $f(x,y) = y$, vincolati a

$$5(x^2+y^2) + 8xy - 9 = 0$$

Moltiplicatori di Lagrange:

$$\begin{cases} 0 = \lambda(10x + 8y) \\ 1 = \lambda(10y + 8x) \\ 5(x^2+y^2) + 8xy - 9 = 0 \end{cases}$$

$$\Rightarrow 10x + 8y = 0 \quad \Rightarrow \quad y = -\frac{5}{4}x$$

$$\Rightarrow 5\left(x^2 + \frac{25}{16}x^2\right) + 8x\left(-\frac{5}{4}x\right) - 9 = 0$$

$$5\left(\frac{41}{16}x^2\right) - \frac{40}{4}x^2 - 9 = 0$$

$$x^2 \frac{(205 - 160)}{16} = 9$$

$$x^2 = \frac{9 \cdot 16}{45} = \frac{16}{5} \qquad x = \pm\frac{4}{\sqrt{5}}$$

$$\Rightarrow y = \mp\frac{5}{\sqrt{5}} = \mp\sqrt{5}$$

massimo assoluto: $\sqrt{5}$

minimo assoluto: $-\sqrt{5}$

8 Dato che $e^{x+y} \to 1$ per $(x,y) \to (0,0)$,

basta studiare

$$\frac{|x|^\alpha}{(x^2+y^2)^{\frac{3}{2}}} = \frac{|\rho\cos\vartheta|^\alpha}{\rho^3} = \rho^{\alpha-3} \cdot |\cos\vartheta|^\alpha$$

$$\Rightarrow \quad f(x,y) \to 0 \iff \alpha > 3 . \qquad \text{(continuità)}$$

Derivabilità:

$$\partial_x f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{|h|^\alpha \cdot e^h}{|h|^3} - 0}{h}$$

$$\sim \frac{|h|^\alpha}{|h|^3 \cdot h}$$. Il limite esiste, finito se e solo se $\alpha > 4$.

$$\partial_y f(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$\Rightarrow$ f è derivabile in $(0,0)$ $\iff \alpha > 4$.

⑨

$$\mathring{A} = \{x^2+y^2 < 4\} \cap \{y > x\}$$

$$\partial A = \{x^2+y^2 \leq 4,\ y = x\} \cup \{x^2+y^2 = 4,\ y \geq x\}$$

$$\bar{A} = \{x^2+y^2 \leq 4\} \cap \{y \geq x\}$$

10

a) $g\left(\frac{1}{4}, \pi\right) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

b) $\nu\left(\frac{1}{4}, \pi\right) = \partial_t g\left(\frac{1}{4}, \pi\right) \times \partial_\theta g\left(\frac{1}{4}, \pi\right) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

(Dato che g parametrizza mezza sfera, come vettore normale si poteva scegliere P, senza bisogno di altri calcoli)

c) Uso: $(x - x_0) \cdot v_1 + (y - y_0) \cdot v_2 + (z - z_0) \cdot v_3 = 0$

con $(x_0, y_0, z_0) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

e $(v_1, v_2, v_3) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

$\Rightarrow \quad x + \sqrt{2} - z = 0$

11

$$D(\gamma \circ f) = \gamma'(f)\,\nabla f = \begin{pmatrix} \gamma_1'(f) \\ \gamma_2'(f) \end{pmatrix} (\partial_x f, \partial_y f)$$

$$= \begin{pmatrix} \gamma_1'(f)\,\partial_x f & \gamma_1'(f)\,\partial_y f \\ \gamma_2'(f)\,\partial_x f & \gamma_2'(f)\,\partial_y f \end{pmatrix}$$

$$D(\gamma \circ f)\, e_1 = \begin{pmatrix} \gamma_1'(f)\,\partial_x f \\ \gamma_2'(f)\,\partial_x f \end{pmatrix}$$

$$\big(D(\gamma \circ f)\, e_1\big)\, e_2 = \gamma_2'(f)\,\partial_x f$$