Tutorato 8 Aprile

per info e chiarimenti
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① $f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{per } (x,y) \neq (0,0) \\ 0 & \text{per } (x,y) = (0,0) \end{cases}$

calcolare le derivate direzionali in $(0,0)$ rispetto a $v = (\cos\theta ; \sin\theta)$

$$D_v f(\bar{z}_0) = \lim_{t \to 0} \frac{f(\bar{z}_0 + t\bar{v}) - f(\bar{z}_0)}{t} \qquad \text{con} \quad \bar{z}_0 = (0,0)$$

$$f(\bar{z}_0 + t\bar{v}) = f(0 + t\cos\theta ; 0 + t\sin\theta) = \frac{t^2\cos^2\theta\, t\sin\theta}{t^4\cos^4\theta + t^2\sin^2\theta}$$

$$D_v f(0,0) = \lim_{t\to 0} \frac{1}{t} \frac{t^2\cos^2\theta\, t\sin\theta}{t^4\cos^4\theta + t^2\sin^2\theta} = \lim_{t\to 0} \frac{t^2\cos^2\theta\sin\theta}{t^4\cos^4\theta + t^2\sin^2\theta}$$

$$\sim \lim_{t\to 0} \frac{t^2\cos^2\theta\sin\theta}{t^2\sin^2\theta} = \frac{\cos^2\theta}{\sin\theta}$$

Per $\sin\theta = 0$ → asse x ($y=0$): $f(x,0) \equiv 0$ → $D_{(x,0)} f = 0$

Per $\sin\theta \neq 0$

② $f(x,y) = x^2 - 2x + \frac{y^2}{4} - y + 2$, $\gamma(t) = (1 + \cos t ; 2 + 2\sin t)$ $t \in \mathbb{R}$

a) calcolare $\frac{d}{dt} f(\gamma(t))$

oss $f(\gamma)$: $\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$, $f(\gamma)$

$$f(\gamma) = (1+\cos t)^2 - 2(1+\cos t) + \frac{1}{4}(2+2\sin t)^2 - (2+2\sin t) + 2$$

$$= 1 + 2\cos t + \cos^2 t - 2 - 2\cos t + 1 + 2\sin t + \sin^2 t - 2 - 2\sin t + 2 = 1$$

$$\frac{d}{dt} f(\gamma) = 0$$

b) calcolare e rappresentare ∇f in $(-1;2)$ $(1;6)$ $(1;-2)$ $(3;2)$

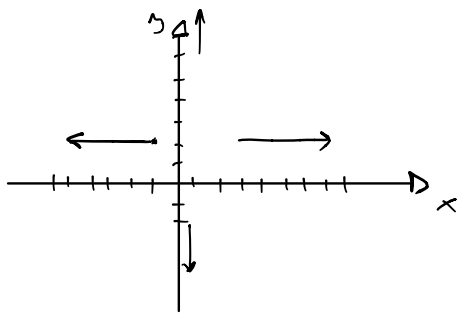
$$\nabla f = \left(2x - 2 ; \tfrac{1}{2}y - 1\right)$$

$\nabla f(-1;2) = (-4 ; 0)$

$\nabla f(1;6) = (0 ; 2)$

$\nabla f(1;-2) = (0 ; -2)$

$\nabla f(3;2) = (4 ; 0)$



c) Disegnare insiemi di livello $L_0(f)$, $L_1(f)$, $L_2(f)$

oss scrivo $f(x;y)$ come il completamento dei quadrati

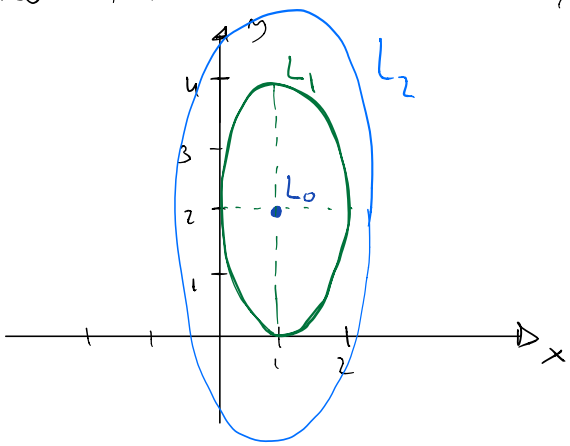
$$f(x;y) = \underbrace{(x-1)^2 - 1}_{x^2 - 2x} + \underbrace{\left(\tfrac{1}{2}y - 1\right)^2 - 1}_{y^2/4 \; - \; y} + 2 \quad \rightarrow \quad f(x;y) = (x-1)^2 + \left(\tfrac{1}{2}y - 1\right)^2$$

a) $L_0(f)$: $(x-1)^2 + \left(\frac{1}{2}y - 1\right)^2 = 0$ equazione di un punto nel piano $x_c = 1$, $y_c = 2$

b) $L_1(f)$: $(x-1)^2 + \frac{(y-2)^2}{4} = 1$ ellisse di centro $(1;2)$ con semiassi $a = 1$, $b = 2$

c) $L_2(f)$: $(x-1)^2 + \frac{(y-2)^2}{4} = 2$

$\frac{(x-1)^2}{(\sqrt{2})^2} + \frac{(y-2)^2}{(2\sqrt{2})^2} = 1$ ellisse di centro $(1;2)$ e semiassi $a = \sqrt{2}$, $b = 2\sqrt{2}$



③ Dominio e Hessiana di $f(x,y,z) = z\ln(x^2y)$

$$D(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x^2y > 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x \neq 0 \, , \, y > 0 \right\}$$

$\partial_x f = \frac{1}{x^2y} 2xy \cdot z = \frac{2z}{x}$, $\partial_y f = \frac{1}{x^2y} x^2 z = \frac{z}{y}$, $\partial_z f = \ln(x^2y)$

$\partial^2_{xx} f = -\frac{2z}{x^2}$ $\quad \partial^2_{xy} f = 0$ $\quad \partial^2_{xz} f = \frac{2}{x}$ $\quad \partial^2_{yz} f = \frac{1}{y}$ $\quad \partial^2_{zz} f = 0$

$$\rightarrow \mathcal{H}_f(x,y,z) = \begin{pmatrix} -\frac{2z}{x^2} & 0 & \frac{2}{x} \\ 0 & -\frac{z}{y^2} & \frac{1}{y} \\ \frac{2}{x} & \frac{1}{y} & 0 \end{pmatrix}$$

④ Classificare i punti stazionari di $f(x,y) = 3x^2 + y^2 - x^3y$

$\nabla f(x,y) = \left(6x - 3x^2y \, ; \, 2y - x^3\right) \rightarrow \begin{cases} 6x - 3x^2y = 0 \\ 2y - x^3 = 0 \end{cases} \rightarrow y = \frac{x^3}{2}$

$6x - 3x^2\left(\frac{x^3}{2}\right) = 0$

$12x - 3x^5 = 0$

$3x(4 - x^4) = 0 \rightarrow x = 0$, $(2 - x^2)(2 + x^2) = 0 \rightarrow x = \pm\sqrt{2}$ (non scomponibile in $\mathbb{R}$)

P.ti STAZ $\quad P_1(0;0)$, $P_2(\sqrt{2}\,;\sqrt{2})$, $P_3(-\sqrt{2}\,;-\sqrt{2})$

$\partial^2_{xx} f = 6 - 6xy$

$\partial^2_{yy} f = 2$ $\rightarrow \mathcal{H}_f(x,y) = \begin{pmatrix} 6 - 6xy & -3x^2 \\ -3x^2 & 2 \end{pmatrix}$

$\partial^2_{xy} f = -3x^2$

P_1 $\mathcal{H}_f(0;0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{cases} \det(\mathcal{H}_f) = 12 > 0 \\ \partial^2_{xx} f > 0 \end{cases}$ → P_1 Pto di minimo relativo

P_2 $\mathcal{H}_f(\sqrt{2};\sqrt{2}) = \begin{pmatrix} -6 & -6 \\ -6 & 2 \end{pmatrix}$, $\det(\mathcal{H}_f) = -12 - (36) = -48 < 0$

→ P_2 Punto di sella

P_3 $\mathcal{H}_f(-\sqrt{2};-\sqrt{2}) = \begin{pmatrix} -6 & -6 \\ -6 & 2 \end{pmatrix}$ → P_3 Punto di sella

⑤ $f(x;y) = 5(4x-3)^2 + \cos y$ trovare Punti stazionari e classificarli

$\partial_x f = 40(4x-3)$
$\partial_y f = -\sin y$

$\nabla f = \bar{0}$ → $\begin{cases} x = \frac{3}{4} \\ y = k\pi \end{cases}$ P.ti stazionari

$\partial^2_{xx} f = 40 \cdot 4 = 160$
$\partial^2_{xy} f = 0$
$\partial^2_{yy} f = -\cos y$

→ $\mathcal{H}_f(x;y) = \begin{pmatrix} 160 & 0 \\ 0 & -\cos y \end{pmatrix}$

$\cos(k\pi) = (-1)^k$
→ $-\cos(k\pi) = (-1)^{k+1}$

classificazione Punti: $\mathcal{H}_f\left(\frac{3}{4};k\pi\right) = \begin{pmatrix} 160 & 0 \\ 0 & (-1)^{k+1} \end{pmatrix}$

Per k+1 Pari → $\det(\mathcal{H}) > 0$ → minimo Relativo

Per k+1 dispari → $\det(\mathcal{H}) < 0$ → Sella